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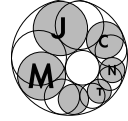
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On the large values of the Riemann zeta-function on the critical line II

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Abstract: We prove some new bounds for the maximum of $|\zeta(0.5 + it)|$ in the segments $T \leq t \leq T + H$ with $H \geq (\ln \ln \ln T)^{1+\varepsilon}$. All the theorems are based on the Riemann hypothesis.

Keywords: Riemann zeta-function, critical line, Selberg's formula, short intervals.

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To professor Kai-Man Tsang in occasion of his 60-th birthday.

Introduction

We continue the investigation of the lower bound estimates for the maximum of modulus of the Riemann zeta-function $\zeta(s)$ in the short segments of the critical line $\operatorname{Re} s = 0.5$.

The theorem of R. Balasubramanian [1] states that the function

$$F(T; H) = \max_{|t-T| \leq H} |\zeta(0.5 + it)|$$

satisfies the inequality

$$F(T; H) \gg \exp\left(\frac{3}{4} \sqrt{\frac{\ln H}{\ln \ln H}}\right) \quad (1.1)$$

for $\ln \ln T \ll H \leq 0.1T$. It is supposed that this bound is close to the best possible (at least, for $H \asymp T$; see [2, 3]). In the case of “very small” H , $0 < H \ll \ln \ln T$, there is a series of lower bound estimates for $F(T; H)$, but all of them differ essentially from (1.1), because their right hand side decreases when T grows (see [4]–[11]).

In particular, it was proved in [7] that

$$F(T; H) \geq \frac{1}{16} \exp \left\{ - \frac{5 \ln T}{6(\pi/\alpha - 1)(\cosh(\alpha H) - 1)} \right\} \tag{1.2}$$

for any fixed α , $1 \leq \alpha < \pi$, $2 \leq \alpha H \leq \ln \ln T - c_1$, where $c_1 > 0$ is some absolute constant. Given $\varepsilon > 0$, it follows from (1.2) that for any $T \geq T_0(\varepsilon) > 0$ and for $H \geq \pi^{-1}(1 + \varepsilon) \ln \ln T - c_1$, the function $F(T; H)$ is bounded from below by some constant:

$$F(T; H) > c_2 = \frac{1}{16} \exp(-1.7 \varepsilon^{-1} e^{c_1}) > 0.$$

In [7], A.A. Karatsuba posed the problem of proving $F(T; H) \geq 1$ for the values of H essentially smaller than $\ln \ln T$, namely, for $H \geq \ln \ln \ln T$. The conditional solution of this problem was obtained in [12]. Namely, it was proved that for an arbitrary large but fixed constant $A > 1$ there exist (non-effective) constants c_0, T_0 such that

$$F(T; H) \geq A \quad \text{for any } T \geq T_0 \quad \text{and} \quad H \geq \pi^{-1} \ln \ln \ln T + c_0. \tag{1.3}$$

The comparison of (1.1) and (1.3) leads us to the following questions:

- 1) for what size of H , $H \ll \ln \ln T$, the inequality $F(T; H) \gg \exp((\ln H)^{0.5-\varepsilon})$ holds?
- 2) for what size of H , $H \gg \ln \ln \ln T$, the inequality $F(T; H) \gg f(H)$ holds for some unbounded function $f(u)$?

The main goal of the present paper is to prove the following assertions based on the Riemann hypothesis (RH).

THEOREM 1. *Suppose that RH is true, and let $m \geq 1$ is any fixed integer. Then*

$$F(T; H) \geq \exp \left(\frac{0.05\sqrt{\ln H}}{(2m \ln \ln H)^m} \right)$$

for $T \geq T_0$ and $(\ln \ln T)^{\frac{1}{2m}} \leq H \leq \ln \ln T$.

THEOREM 2. *Suppose that RH is true, and let $0 < \varepsilon < 0.1$ be any fixed number. Then*

$$F(T; H) \geq \exp \left(\sqrt{\ln H} e^{-c (\ln \ln H)^{1-0.5\varepsilon}} \right)$$

for any $T \geq T_1(\varepsilon)$, $H \geq (\ln \ln \ln T)^{2+\varepsilon}$ and for some constant $c = c(\varepsilon) > 0$.

THEOREM 3. *Suppose that RH is true, and let $0 < \varepsilon < 0.1$ be any fixed number. Then*

$$F(T; H) \geq \exp \left((\ln H)^{\gamma - \varepsilon} \right)$$

for any $T \geq T_1(\varepsilon)$, $H \geq (\ln \ln \ln T)^2$,

$$\gamma = \frac{1}{2 + (\pi \varrho)^{-1}} = 0.46862145 \dots,$$

where $\varrho = 2.37689234 \dots$ stands for the least positive root of the function

$$h(\lambda) = \int_0^{+\infty} e^{-(\cosh \sqrt{u} + \cos \sqrt{u})} \cos(\lambda u) du.$$

THEOREM 4. *Suppose that RH is true, and let $0 < \varepsilon < 0.1$ be any fixed number. Then*

$$F(T; H) \geq \exp \left(0.5 e^{(\ln \ln H)^{0.5\varepsilon}} \right)$$

for any $T \geq T_1(\varepsilon)$, $H \geq (\ln \ln \ln T)^{1+\varepsilon}$.

The proof of all the above assertions is based on a general Theorem A below. Its particular cases are used in [13]–[20]. At the same time, the proof of Theorem A is based on the convolution formula (Lemma 1 of present paper) going back to A. Selberg (see [13] and [15]) and on the lemma of prof. K.-M. Tsang (see lemma 2 below). The original parts of paper are the upper bound estimates for the rate of decreasing for Fourier transforms of some rapidly decreasing functions (Lemma 4). The idea of varying of the function $f(u)$ in convolution formula for minimization of H belongs to R.N. Boyarinov [18], [19].

1. Auxilliary assertions

In this section, we give some auxilliary assertions needed for the proof of Theorem A.

LEMMA 1. *Suppose that the function $f(z)$ is analytical in the strip $|\operatorname{Im} z| \leq 0.5 + \alpha$, where it satisfies the inequality $|f(z)| \leq c(|z| + 1)^{-(1+\beta)}$ with some positive α, β and c . Then the identity*

$$\int_{-\infty}^{+\infty} f(u) \ln \zeta(0.5 + i(t + u)) du = \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} n^{-it} \widehat{f}(\ln n) + 2\pi \left(\sum_{\beta > 0.5}^{\beta - 0.5} \int_0^{\beta - 0.5} f(\gamma - t - iv) dv - \int_0^{0.5} f(-t - iv) dv \right),$$

holds for any t , where $\rho = \beta + i\gamma$ in the last sum runs through all complex zeros of $\zeta(s)$ to the right from the critical line, $\Lambda_1(n) = \Lambda(n)/\ln n$.

This assertion goes back to A. Selberg (see for example [13, Lemma 16]). In [14, Ch. II, §11], [21, Ch. II, §2], [15], there are some variants of this lemma, where $f(z)$ satisfies slightly different conditions. These proofs can be easily adopted to the case under considering.

LEMMA 2. *Let $H > 0, M > 0, k \geq 1$, and suppose that the real function $W(t)$ satisfies to the following conditions:*

$$\int_T^{T+H} W^{2k}(t) dt > HM^{2k}, \quad \left| \int_T^{T+H} W^{2k+1}(t) dt \right| \leq 0.5HM^{2k+1}.$$

Then

$$\max_{T \leq t \leq T+H} (\pm W(t)) \geq 0.5M.$$

This is a small modification of Lemma 11.3 from [14, Ch. II, §11] (see also Lemma 4 from [15]).

LEMMA 3. *Let $\varpi_1 = 2, \varpi_2 = 3, \varpi_3 = 5, \dots, \dots$ are all the primes indexed in ascending order. Then $\varpi_n < n(\ln n + \ln \ln n)$ for any $n \geq 6$. Next,*

$$\sum_{\varpi_n \leq x} \frac{1}{\varpi_n} = \ln \ln x + m + \frac{\theta}{\ln^2 x},$$

where $m = 0.261497 \dots$ is Mertens' constant and $-0.5 < \theta < 1$ for any $x > 1$.

These assertions follows from Theorems 3, 5 and 6 of [22].

2. General theorem

This section is devoted entirely to the proof of one general assertion, which implies all Theorems 1–4.

THEOREM A. *Suppose RH is true, and let the function $\Phi(u)$ satisfies the following conditions:*

- 1) $\Phi(u) \geq 0$ for real u and $f(z) = \Phi(\tau z)$ is analytic in the strip $|\operatorname{Im} z| \leq 0.5 + \delta$ for any $\tau > 0$ and satisfies the inequality $|f(z)| \ll c(1 + |z|)^{-(1+\beta)}$ for some positive c , β and δ (both β and δ may depend on τ);
- 2) $|\Phi(u)| \leq e^{-G(|u|)}$ for any real u , $|u| \geq u_0$, where the functions $G(u)$, $G'(u)$ are positive and unboundedly increasing and such that the functions $g'(v)$, $g'(v) \ln g(v)$ are positive and decreasing for $v \geq v_0 > 0$ (here $g(v)$ stands for the inverse function to $G(u)$);
- 3) $\widehat{\Phi}(\lambda)$ is real for real λ , strictly positive and monotonically decreasing on $[0, \alpha]$ for some $\alpha > 0$; moreover,

$$|\widehat{\Phi}(\lambda)| \leq e^{-|\lambda|F(|\lambda|)} \quad (2.1)$$

for some increasing function $F(u)$ and for any real λ , $|\lambda| \geq \lambda_0$;

- 4) the function $\varphi(v)$, which is inverse to $F(u)$, is increasing for $v \geq v_0$ and satisfies the inequalities

$$\ln v \leq \ln \varphi(v) \leq e^{0.5\alpha v}; \quad (2.2)$$

Suppose also that τ_0 is a root of the transcendental equation

$$\alpha\tau_0 + \ln \varphi\left(\frac{\tau_0}{2} + 1\right) = \ln \ln H, \quad (2.3)$$

which is unique when H is sufficiently large. Finally, let $T \geq T_0(\Phi; \alpha) > 0$ and $H\tau_0 \geq g(\ln \ln T)$. Then

$$\max_{T-H \leq t \leq T+2H} \ln |\zeta(0.5 + it)| \geq \mu^*, \quad (2.4)$$

where

$$\mu^* = \frac{1}{10\alpha} \frac{\widehat{\Phi}(\alpha)}{\widehat{\Phi}(0)} \sqrt{\frac{\ln \kappa}{\kappa}} \frac{e^{0.5\alpha\tau_0}}{\tau_0}, \quad \kappa = \max(61, 4\alpha^{-1}).$$

If, in addition, the function $\varphi(v)$ satisfies the condition

$$\ln \varphi(v) \leq 0.5v, \tag{2.5}$$

and τ_1 denotes the root of the equation

$$\frac{\alpha\tau_1}{2} + \ln \varphi\left(\frac{\tau_1}{2} + 1\right) = \ln \ln H, \tag{2.6}$$

then the inequality

$$\max_{T-H \leq t \leq T+2H} \ln |\zeta(0.5 + it)| \geq \mu^{**} \tag{2.7}$$

holds for $T_1(\Phi; \alpha) > 0$, $H\tau_1 \geq g(\ln \ln T)$ with

$$\mu^{**} = \frac{1}{6\sqrt{\alpha\kappa}} \frac{\widehat{\Phi}(\alpha)}{\widehat{\Phi}(0)} \frac{e^{0.25\alpha\tau_1}}{\sqrt{\tau_1}}, \quad \kappa = \max(0.5, 4\alpha^{-1}).$$

COROLLARY. Suppose that all the conditions are satisfied and let $\varrho > 0$ be the least positive root of $\widehat{\Phi}(\lambda)$. If $\widehat{\Phi}(\lambda)$ decreases on $[0, \varrho]$, then the inequality (2.4) holds with

$$\mu^* = \frac{1}{5e\varrho} \sqrt{\frac{\ln \kappa}{\kappa}} \frac{|\widehat{\Phi}'(\varrho)|}{\widehat{\Phi}(0)} \frac{e^{0.5\varrho\tau_0}}{\tau_0^2}, \quad \kappa = \max(32, 5\varrho^{-1}),$$

where τ_0 denotes the root of (2.3) corresponding to $\alpha = \varrho$. If, in addition, the condition (2.5) holds true, then the inequality (2.7) is true for

$$\mu^{**} = \frac{1}{5\sqrt{e\kappa}} \frac{|\widehat{\Phi}'(\varrho)|}{\widehat{\Phi}(0)} \frac{e^{0.25\varrho\tau_1}}{\tau_1^{1.5}}, \quad \kappa = \max(4, 0.5\varrho),$$

where τ_1 denotes the root of (2.6) corresponding to $\alpha = \varrho$.

PROOF. Let τ_0 be a root of (2.3), and suppose that $H\tau_0 \geq g(\ln \ln T)$, $T \geq T_0(\Phi; \alpha)$. By lemma 1, $I(t) = A(t) - B(t)$, where

$$I(t) = \int_{-\infty}^{+\infty} \Phi(\tau_0 u) \ln |\zeta(0.5 + i(t + u))| du,$$

$$\begin{aligned}
 A(t) &= \frac{1}{\tau_0} \sum_{n=2}^{+\infty} \frac{\Lambda_1(n)}{\sqrt{n}} \widehat{\Phi} \left(\frac{\ln n}{\tau_0} \right) \cos(t \ln n), \\
 B(t) &= 2\pi \int_0^{0.5} \operatorname{Re} \Phi(-(t+iu)\tau_0) du.
 \end{aligned} \tag{2.8}$$

Transforming $I(t)$, we get

$$I(t) = \left(\int_{-H}^H + \int_H^{+\infty} + \int_{-\infty}^{-H} \right) \dots du = I_0(t) + I_1(t) + I_2(t).$$

Estimating I_1, I_2 from above, we note that if $|\zeta(0.5 + i(t+u))| < 1$ for every u , $u_1 \leq u \leq u_2$, then the integral over (u_1, u_2) is negative. Hence, it is sufficient to estimate the integrals over the set of u such that $|\zeta(0.5 + i(t+u))| \geq 1$. Thus, the trivial bound

$$|\zeta(0.5 + i(t+u))| \leq |t+u| + 3$$

yields

$$I_1(t) = \int_H^{+\infty} \Phi(\tau_0 u) \ln(t+u+3) du \leq \ln(2t+3) \int_H^t \Phi(\tau_0 u) du + 2 \int_t^{+\infty} \Phi(\tau_0 u) (\ln u) du.$$

Standing j_1, j_2 for the last integrals, we get

$$\begin{aligned}
 j_1 &= \frac{1}{\tau_0} \int_{\tau_0 H}^{+\infty} \Phi(v) dv \leq \frac{1}{\tau_0} \int_{\tau_0 H}^{+\infty} e^{-G(v)} dv = \frac{1}{\tau_0} \int_{G(\tau_0 H)}^{+\infty} e^{-u} g'(u) du \leq \\
 &\leq \frac{g'(G(\tau_0 H))}{\tau_0} \int_{G(\tau_0 H)}^{+\infty} e^{-u} du = \frac{g'(G(\tau_0 H))}{\tau_0} e^{-G(\tau_0 H)}.
 \end{aligned}$$

Since $g'(G(u))G'(u) = 1$ then

$$j_1 \leq \frac{e^{-G(\tau_0 H)}}{\tau_0 G'(\tau_0 H)}.$$

Similarly we have

$$\begin{aligned}
 j_2 &\leq \frac{1}{\tau_0} \int_{\tau_0 t}^{+\infty} \Phi(v) \left(\ln \frac{v}{\tau_0} \right) dv \leq \frac{1}{\tau_0} \int_{\tau_0 t}^{+\infty} e^{-G(v)} \left(\ln \frac{v}{\tau_0} \right) dv = \\
 &= \frac{1}{\tau_0} \int_{G(\tau_0 t)}^{+\infty} e^{-u} g'(u) \ln \left(\frac{g(u)}{\tau_0} \right) du \leq \\
 &\leq \frac{g'(G(\tau_0 t))}{\tau_0} \ln \left(\frac{1}{\tau_0} g(G(\tau_0 t)) \right) e^{-G(\tau_0 t)} = \frac{e^{-G(\tau_0 t)}}{\tau_0} \frac{\ln t}{G'(\tau_0 t)}.
 \end{aligned}$$

Hence,

$$I_1 \leq \frac{2(\ln t)e^{-G(\tau_0 H)}}{\tau_0 G'(\tau_0 H)}.$$

The same bound holds for I_2 . Thus,

$$I(t) \leq I_0 + \frac{4(\ln t)}{\tau_0} \frac{e^{-G(\tau_0 H)}}{G'(\tau_0 H)}. \tag{2.9}$$

Further we have

$$|B(t)| \leq 2\pi \int_0^{0.5} \frac{c \, du}{(1 + |t + iu|)^{1+\beta}} \leq \frac{\pi c}{t}. \tag{2.10}$$

We split the sum $A(t)$ to the parts A_1, A_2 and A_3 according to the conditions $p \leq X, n = p^k \leq X, k \geq 2$ (p is prime) and $n > X$, where

$$X = \exp \left(\tau_0 \varphi \left(\frac{\tau_0}{2} + 1 \right) \right).$$

First we have

$$\begin{aligned}
 |A_3| &\leq \frac{1}{\tau_0} \sum_{n>X} \frac{1}{\sqrt{n}} \left| \widehat{\Phi} \left(\frac{\ln n}{\tau_0} \right) \right| \leq \frac{1}{\tau_0} \sum_{n>X} \frac{1}{\sqrt{n}} \exp \left(-\frac{\ln n}{\tau_0} F \left(\frac{\ln n}{\tau_0} \right) \right) = \\
 &= \frac{1}{\tau_0} \sum_{m=0}^{+\infty} \sum_{X e^{m\tau_0} < n \leq X e^{(m+1)\tau_0}} \frac{1}{\sqrt{n}} \exp \left(-\frac{\ln n}{\tau_0} F \left(\frac{\ln n}{\tau_0} \right) \right). \tag{2.11}
 \end{aligned}$$

Since F is monotonic, we have

$$F\left(\frac{\ln n}{\tau_0}\right) \geq F\left(\frac{1}{\tau_0} \ln(Xe^{m\tau_0})\right) = F\left(\varphi\left(\frac{\tau_0}{2}+1\right) + m\right) \geq F\left(\varphi\left(\frac{\tau_0}{2}+1\right)\right) = \frac{\tau_0}{2} + 1$$

for any $m \geq 0$ and for $Xe^{m\tau_0} < n \leq Xe^{(m+1)\tau_0}$. Hence, the sum over n in (2.11) does not exceed

$$\begin{aligned} \frac{1}{\tau_0} \sum_{Xe^{m\tau_0} < n \leq Xe^{(m+1)\tau_0}} \frac{1}{\sqrt{n}} \exp\left\{-\left(\varphi\left(\frac{\tau_0}{2}+1\right) + m\right)\left(\frac{\tau_0}{2}+1\right)\right\} &\leq \\ &\leq \frac{3}{\tau_0} (Xe^{(m+1)\tau_0})^{0.5} \exp\left\{-\left(\varphi\left(\frac{\tau_0}{2}+1\right) + m\right)\left(\frac{\tau_0}{2}+1\right)\right\} = \\ &= \frac{3}{\tau_0} \exp\left\{\frac{\tau_0}{2} \varphi\left(\frac{\tau_0}{2}+1\right) + \frac{\tau_0}{2}(m+1) - \left(\varphi\left(\frac{\tau_0}{2}+1\right) + m\right)\left(\frac{\tau_0}{2}+1\right)\right\} = \\ &= \frac{3}{\tau_0} \exp\left\{\frac{\tau_0}{2} - \varphi\left(\frac{\tau_0}{2}+1\right)\right\} e^{-m} < \frac{3e^{-m}}{\tau_0}. \end{aligned} \quad (2.12)$$

Finally we get

$$|A_3| \leq \frac{3}{\tau_0} \sum_{m=0}^{+\infty} e^{-m} < \frac{5}{\tau_0}. \quad (2.13)$$

Since $\Phi(u)$ is non-negative, $|\widehat{\Phi}(\lambda)| \leq \widehat{\Phi}(0)$ for real λ . Hence, Lemma 3 implies

$$\begin{aligned} |A_2| &\leq \frac{1}{\tau_0} \sum_{k \geq 2} \sum_{p^k \leq X} \frac{\widehat{\Phi}(0)}{kp^{0.5k}} \leq \frac{\widehat{\Phi}(0)}{\tau_0} \left(\frac{1}{2} \sum_{p \leq \sqrt{X}} \frac{1}{p} + \frac{1}{3} \sum_{k \geq 3} \sum_p p^{-0.5k} \right) < \\ &< \frac{\widehat{\Phi}(0)}{2\tau_0} \left(\ln \ln X - \ln 2 + m + \frac{2}{3} \sum_p \frac{1}{p(\sqrt{p}-1)} + \frac{4}{\ln^2 X} \right) < \\ &< \frac{\widehat{\Phi}(0)}{2\tau_0} \left(\ln \left(\tau_0 \varphi\left(\frac{\tau_0}{2}+1\right) \right) + 2 \right) \end{aligned} \quad (2.14)$$

Summation of (2.10), (2.13), (2.14) yields:

$$|A_2| + |A_3| + |B| \leq \frac{\widehat{\Phi}(0)}{\tau_0} \ln \left(\tau_0 \varphi\left(\frac{\tau_0}{2}+1\right) \right). \quad (2.15)$$

Now we set

$$k = \left\lceil \frac{e^{\alpha\tau_0}}{\alpha\kappa\tau_0} \right\rceil \geq 7,$$

where $\kappa > 0$ will be chosen later. Denote

$$a(p) = \widehat{\Phi}\left(\frac{\ln p}{\tau}\right), \quad V(t) = \sum_{p \leq X} \frac{a(p)}{\sqrt{p}} p^{it}, \quad A_0(t) = 0.5(V(t) + \overline{V}(t)),$$

and define the integrals

$$I(k) = \int_T^{T+H} A_0^{2k}(t) dt, \quad J(k) = \int_T^{T+H} A_0^{2k+1}(t) dt.$$

Thus we find that

$$I(k) = 2^{-2k} \sum_{\nu=0}^{2k} \binom{2k}{\nu} j(\nu), \quad j(\nu) = \int_T^{T+H} V^\nu(t) \overline{V}^\mu(t) dt, \quad \mu = 2k - \nu.$$

Setting for brevity $P = p_1 \dots p_\nu$, $Q = q_1 \dots q_\mu$, in the case $\mu \neq \nu$ we get

$$\begin{aligned} |j(\nu)| &= \left| \sum_{\substack{p_1, \dots, p_\nu \leq X \\ q_1, \dots, q_\mu \leq X}} \frac{a(p_1) \dots a(p_\nu)}{\sqrt{p_1 \dots p_\nu}} \int_T^{T+H} \left(\frac{p_1 \dots p_\nu}{q_1 \dots q_\mu} \right)^{it} dt \right| \leq \\ &\leq 2 \sum_{P, Q} \frac{|a(p_1)| \dots |a(p_\nu)|}{\sqrt{PQ}} \left| \ln \frac{P}{Q} \right|^{-1} \leq 2\widehat{\Phi}^{2k}(0) \sum_{P, Q} \frac{1}{\sqrt{PQ}} \left| \ln \frac{P}{Q} \right|^{-1}. \end{aligned}$$

If $P < Q$ then

$$\left| \ln \frac{P}{Q} \right| = \ln \frac{Q}{P} \geq \ln \frac{P+1}{P} \geq \frac{1}{2P}; \quad \text{otherwise,} \quad \left| \ln \frac{P}{Q} \right| \geq \frac{1}{2Q}.$$

Hence,

$$|j(\nu)| \leq 2\widehat{\Phi}^{2k}(0) \left(\sum_{P < Q} \frac{2P}{\sqrt{PQ}} + \sum_{P > Q} \frac{2Q}{\sqrt{PQ}} \right) \leq$$

$$\begin{aligned}
&\leq 4\widehat{\Phi}^{2k}(0) \left(\sum_{p_1, \dots, p_\nu \leq X} (p_1 \dots p_\nu)^{0.5} \sum_{q_1, \dots, q_\mu \leq X} (q_1 \dots q_\mu)^{-0.5} + \right. \\
&\quad \left. + \sum_{p_1, \dots, p_\nu \leq X} (p_1 \dots p_\nu)^{-0.5} \sum_{q_1, \dots, q_\mu \leq X} (q_1 \dots q_\mu)^{0.5} \right) = \\
&= 4\widehat{\Phi}^{2k}(0) (S^\nu C^{2k-\nu} + S^{2k-\nu} C^\nu), \tag{2.16}
\end{aligned}$$

where

$$S = \sum_{p \leq X} p^{0.5}, \quad C = \sum_{q \leq X} q^{-0.5}.$$

The same bound is true for the non-diagonal terms in the case $\nu = k$. Summing (2.16) over $0 \leq \nu \leq 2k$ we get

$$I(k) = 2^{-2k} \binom{2k}{k} H \mathfrak{S}_k + 8\theta \widehat{\Phi}^{2k}(0) S^{2k}, \tag{2.17}$$

where

$$\mathfrak{S}_k = \sum_{p_1 \dots p_k = q_1 \dots q_k} \frac{a^2(p_1) \dots a^2(p_k)}{p_1 \dots p_k}, \quad |\theta| \leq 1.$$

Since

$$S = \left(\frac{2}{3} + o(1) \right) \frac{X^{1.5}}{\ln X},$$

then the last term in (2.17) is less than $(X^{1.5}(\ln X)^{-0.5})^{2k}$ in modulus.

Estimating \mathfrak{S}_k from below, we retain in \mathfrak{S}_k all the terms corresponding to the tuples (p_1, \dots, p_k) without repetitions. Thus we get

$$\mathfrak{S}_k \geq k! \sum_{\substack{p_1 \dots p_k = q_1 \dots q_k \\ p_1, \dots, p_k \text{ are distinct}}} \frac{a^2(p_1) \dots a^2(p_k)}{p_1 \dots p_k}. \tag{2.18}$$

Since $\varphi(v)$ is monotonic, we have $X \geq e^{\alpha\tau_0}$ for sufficiently large H . Replacing the upper limit for p_1, \dots, p_k in (2.18) by $e^{\alpha\tau_0}$ and noting that

$$\widehat{\Phi}\left(\frac{\ln p}{\tau_0}\right) \geq \widehat{\Phi}(\alpha) > 0$$

for $2 \leq p \leq e^{\alpha\tau_0}$, we have

$$\begin{aligned} \mathfrak{S}_k &\geq k! \widehat{\Phi}^{2k}(\alpha) \sum_{\substack{p_1, \dots, p_k \leq e^{\alpha\tau_0} \\ p_1, \dots, p_k \text{ are distinct}}} (p_1 \dots p_k)^{-1} \geq \\ &\geq k! \widehat{\Phi}^{2k}(\alpha) \sum_{p_1 \leq e^{\alpha\tau_0}} \frac{1}{p_1} \sum_{\substack{p_2 \leq e^{\alpha\tau_0} \\ p_2 \neq p_1}} \frac{1}{p_2} \dots \sum_{\substack{p_k \leq e^{\alpha\tau_0} \\ p_k \neq p_1, \dots, p_{k-1}}} \frac{1}{p_k} \geq \\ &\geq k! \widehat{\Phi}^{2k}(\alpha) \sum_{p_1 \leq e^{\alpha\tau_0}} \frac{1}{p_1} \sum_{\substack{p_2 \leq e^{\alpha\tau_0} \\ p_2 \neq \varpi_1}} \frac{1}{p_2} \dots \sum_{\substack{p_k \leq e^{\alpha\tau_0} \\ p_k \neq \varpi_1, \dots, \varpi_{k-1}}} \frac{1}{p_k} \geq \\ &\geq k! \widehat{\Phi}^{2k}(\alpha) \left(\sum_{\varpi_{k-1} < p \leq e^{\alpha\tau_0}} \frac{1}{p} \right)^k. \end{aligned}$$

Let us take $\kappa = \max(61, 4\kappa^{-1})$. Then, by Lemma 3 we get

$$\varpi_{k-1} < k(\ln k + \ln \ln k) \leq \frac{e^{\alpha\tau_0}}{\alpha\kappa\tau_0} (\alpha\tau_0 - \ln(\alpha\kappa\tau_0) + \ln(\alpha\tau_0)) < \frac{e^{\alpha\tau_0}}{\kappa},$$

$$\begin{aligned} k \ln k &> \left(\frac{e^{\alpha\tau_0}}{\alpha\kappa\tau_0} - 1 \right) \ln \left(\frac{e^{\alpha\tau_0}}{\alpha\kappa\tau_0} - 1 \right) > \frac{e^{\alpha\tau_0}}{2\alpha\kappa\tau_0} \ln \left(\frac{e^{\alpha\tau_0}}{2\alpha\kappa\tau_0} \right) = \\ &= \frac{e^{\alpha\tau_0}}{2\kappa} \left(1 - \frac{\ln(2\alpha\kappa\tau_0)}{\alpha\tau_0} \right), \end{aligned}$$

and hence

$$\ln(k \ln k) > \alpha\tau_0 \left(1 - \frac{\ln \kappa + 1}{\alpha\tau_0} \right), \quad \frac{1}{\ln^2(k \ln k)} < \frac{1}{(\alpha\tau_0)^2} \left(1 + \frac{3(\ln \kappa + 1)}{\alpha\tau_0} \right).$$

Using Lemma 3 again, we obtain

$$\begin{aligned} \sum_{\varpi_{k-1} < p \leq e^{\alpha\tau_0}} \frac{1}{p} &> \sum_{k \ln k < p \leq e^{\alpha\tau_0}} \frac{1}{p} > \ln \ln e^{\alpha\tau_0} - \ln \ln \frac{e^{\alpha\tau_0}}{\kappa} - \frac{1.5}{(\alpha\tau_0)^2} \left(1 + \frac{3(\ln \kappa + 1)}{\alpha\tau_0} \right) = \\ &= -\ln \left(1 - \frac{\ln \kappa}{\alpha\tau_0} \right) - \frac{1.5}{(\alpha\tau_0)^2} \left(1 + \frac{3(\ln \kappa + 1)}{\alpha\tau_0} \right) = \end{aligned}$$

$$= \frac{\ln \kappa}{\alpha \tau_0} + \frac{1}{2(\alpha \tau_0)^2} ((\ln \kappa)^2 - 3) + \frac{1}{3(\alpha \tau_0)^3} \left((\ln \kappa)^3 - \frac{27}{2} (\ln \kappa + 1) \right) > \frac{\ln \kappa}{\alpha \tau_0}.$$

Passing to the estimation of $I(k)$ and noting that $k = \left\lceil \frac{1}{\alpha \kappa} \frac{\ln H}{\ln X} \right\rceil$, we find:

$$\begin{aligned} \left(\frac{X^{1.5}}{\sqrt{\ln X}} \right)^{2k} &< X^{3k} \leq \exp \left(\frac{3 \ln H}{\alpha \kappa} \right) \leq H^{0.75}, \\ I(k) &> 2^{-2k} \binom{2k}{k} H k! \widehat{\Phi}^{2k}(\alpha) \left(\frac{\ln \kappa}{\alpha \tau_0} \right)^k - \left(\frac{X^{1.5}}{\sqrt{\ln X}} \right)^{2k} > \\ &> \frac{(2k)!}{k!} H \left(\frac{\widehat{\Phi}(\alpha)}{4} \sqrt{\frac{\ln \kappa}{\alpha \tau_0}} \right)^{2k} - H^{0.75} > \\ &> \frac{e}{2} \left(\frac{4k}{e} \right)^k H \left(\frac{\widehat{\Phi}(\alpha)}{4} \sqrt{\frac{\ln \kappa}{\alpha \tau_0}} \right)^{2k} - H^{0.75} > HM^{2k}, \end{aligned}$$

where

$$M = \frac{\widehat{\Phi}(\alpha)}{2} \sqrt{\frac{k \ln \kappa}{e \alpha \tau_0}} > 2.$$

Repeating word-by-word the estimation of the non-diagonal terms of $I(k)$, we get

$$|J(k)| < \left(\frac{X^{1.5}}{\sqrt{\ln X}} \right)^{2k+1} < X^{3k(1+1/(2k))} \leq X^{4k} \leq H^{4/(\alpha \kappa)} \leq H,$$

and hence $|J(k)| < 0.5HM^{2k+1}$. By Lemma 2, there exists t_0 such that $T \leq t_0 \leq T + H$ and $A_0(t_0) > 0.5M$. Setting $t = t_0$ in (2.8) and taking into account (2.9), (2.15) we find that

$$\begin{aligned} I_0(t_0) &= A(t_0) - B(t_0) - I_1(t_0) - I_2(t_0) \geq \\ &\geq \frac{M}{2\tau_0} - \frac{\widehat{\Phi}(0)}{\tau_0} \ln \left(\tau_0 \varphi \left(\frac{\tau_0}{2} + 1 \right) \right) - \frac{4(\ln t_0)}{\tau_0} \frac{e^{-G(\tau_0 H)}}{G'(\tau_0 H)} \geq \\ &\geq \frac{\widehat{\Phi}(\alpha)}{4\tau_0} \left(\frac{e^{\alpha \tau_0}}{2\alpha \kappa \tau_0} \frac{\ln \kappa}{e \alpha \tau_0} \right)^{0.5} - \frac{\widehat{\Phi}(0)}{\tau_0} (\ln \tau_0 + e^{0.25\alpha \tau_0}) - \frac{4(\ln t_0) e^{-\ln \ln T}}{\tau_0 G'(\tau_0 H)} > \\ &> \frac{\widehat{\Phi}(\alpha)}{10\alpha} \sqrt{\frac{\ln \kappa}{\kappa}} \frac{e^{0.5\alpha \tau_0}}{\tau_0^2}. \end{aligned} \tag{2.19}$$

The inequality (2.19) and the definition of I_0 implies that the maximum M_1 of the function $\ln |\zeta(0.5 + i(t_0 + u))|$ on the segment $|u| \leq H$ is strictly positive. Hence,

$$I_0(t_0) \leq M_1 \int_{-H}^H \Phi(\tau_0 H) du \leq M_1 \int_{-\infty}^{+\infty} \Phi(\tau_0 H) du = \frac{\widehat{\Phi}(0)}{\tau_0} M_1. \tag{2.20}$$

Comparing (2.19) with (2.20) and noting that the point $t_0 + u$ of maximum is contained in $[T - H, T + 2H]$, we find that

$$\max_{T-H \leq t \leq T+2H} \ln |\zeta(0.5 + it)| \geq M_1 > \frac{1}{10\alpha} \sqrt{\frac{\ln \kappa}{\kappa}} \frac{\widehat{\Phi}(\alpha)}{\widehat{\Phi}(0)} \frac{e^{0.5\alpha\tau_0}}{\tau_0}. \tag{2.21}$$

Thus, (2.4) is proved.

Suppose now that $\varphi(v)$ satisfies (2.5). Then, setting

$$X = e^{\tau_1 \varphi\left(\frac{\tau_1}{2} + 1\right)}, \quad k = \left\lceil \frac{e^{0.5\alpha\tau_1}}{\alpha\kappa\tau_1} \right\rceil = \left\lceil \frac{1}{\alpha\kappa} \frac{\ln H}{\ln X} \right\rceil, \quad \kappa = \max(0.5, 4\alpha^{-1})$$

and repeating word-by-word the above arguments, we find that

$$\begin{aligned} \varpi_{k-1} &< e^{0.5\alpha\tau_1}, \quad \sum_{\varpi_{k-1} < p \leq e^{\alpha\tau_1}} \geq \ln \ln e^{\alpha\tau_1} - \ln \ln e^{0.5\alpha\tau_1} - \frac{4.5}{(\alpha\tau_1)^2} > \frac{2}{3}, \\ \mathfrak{S}_k &> 2^{-2k} \frac{(2k)!}{k!} H \widehat{\Phi}^{2k}(\alpha) \left(\frac{2}{3}\right)^k - H^{0.75} > HM^{2k}, \end{aligned}$$

where $M = \widehat{\Phi}(\alpha) \sqrt{\frac{2k}{3e}}$, and, similarly,

$$|J(k)| < X^{15k/4} \leq H^{15/16} < 0.5HM^{2k+1}.$$

By Lemma 2, $A(t_0) > 0.5M$ for some t_0 , $T \leq t_0 \leq T + H$. Since (2.5) and (2.15) imply the bound

$$|A_2| + |A_3| + |B| \leq \frac{\widehat{\Phi}(0)}{\tau_1} \ln \left(\tau_1 \varphi\left(\frac{\tau_1}{2} + 1\right) \right) < 0.5\widehat{\Phi}(0),$$

we get

$$\max_{T-H \leq t \leq T+2H} \ln |\zeta(0.5 + it)| > \frac{\widehat{\Phi}(\alpha)}{\sqrt{6e}} \sqrt{k} > \frac{\widehat{\Phi}(\alpha)}{6\sqrt{\alpha\kappa}} \frac{e^{0.25\alpha\tau_1}}{\sqrt{\tau_1}}.$$

Theorem is proved.

To prove the Corollary, we use (2.21) with $\alpha = \varrho - \varepsilon$, $\varepsilon = 2\tau_0^{-1}$. For sufficiently large H we have $\widehat{\Phi}(\alpha) = -\widehat{\Phi}'(\varrho - \theta\varepsilon)\varepsilon \geq 0.5\varepsilon |\widehat{\Phi}'(\varrho)| = |\widehat{\Phi}'(\varrho)|\tau_0^{-1}$,

$$\begin{aligned} \max_{T-H \leq t \leq T+2H} \ln |\zeta(0.5 + it)| &> \frac{1}{10\varrho} \sqrt{\frac{\ln \kappa}{\kappa}} \frac{2}{\tau_0} \frac{|\widehat{\Phi}'(\varrho)|}{\widehat{\Phi}(0)} \frac{1}{\tau_0} e^{0.5\tau_0(\varrho-2/\tau_0)} > \\ &> \frac{1}{5e\varrho} \sqrt{\frac{\ln \kappa_1}{\kappa_1}} \frac{|\widehat{\Phi}'(\varrho)|}{\widehat{\Phi}(0)} \frac{e^{0.5\tau_0\varrho}}{\tau_0^2}, \end{aligned}$$

where $\kappa_1 = \max(62, 5\varrho^{-1})$. The second assertion of the Corollary can be proved similarly. \square

3. The rate of decreasing of some Fourier transforms

In order to apply Theorem A for given function $\Phi(u)$, we need an estimate of type (2.1) for the rate of decreasing of $\widehat{\Phi}(\lambda)$ when $\lambda \rightarrow \pm\infty$. In what follows, we obtain some bounds of such type.

LEMMA 4. *Suppose $m \geq 1$ is any fixed integer, $\Phi(u) = \exp\left(-\frac{u^{2m}}{2m}\right)$. Then the inequality*

$$|\widehat{\Phi}(\lambda)| < \frac{5}{\sqrt{m}} |\lambda|^{-\beta} \exp\left(-\frac{|\lambda|^\alpha}{\alpha} \sin \pi\kappa\right)$$

holds for any real λ , $|\lambda| \geq \lambda_0$, with

$$\alpha = \frac{2m}{2m-1}, \quad \beta = \frac{m-1}{2m-1}, \quad \kappa = \frac{1}{2(2m-1)}.$$

PROOF. The case $m = 1$ is obvious. If $m \geq 2$, this assertion follows from the asymptotic formula for $\widehat{\Phi}(\lambda)$ from [23, Ch. IV, §7]. \square

LEMMA 5. Let p, q be integers, $1 \leq p < q$, $(p, q) = 1$, $r = p/q$, $\varepsilon = e^{\pi i/q}$, and let

$$G_r(z) = \sum_{k=0}^{q-1} \cosh(\varepsilon^k z^{p/q}), \quad \Phi_r(z) = \exp(-G_r(z)).$$

Then the estimate $|\widehat{\Phi}_r(\lambda)| < \exp(-|\lambda| F_r(|\lambda|))$ holds for any real λ , $|\lambda| > \lambda_0$, with

$$F_r(u) = \frac{3}{5} \left(\ln \frac{\lambda}{q} \right)^{\frac{q}{p}-1}.$$

PROOF. Suppose that $\lambda > \lambda_0 > 0$ (the case of negative λ is treated in the same way). Since the function

$$G_r(z) = q \sum_{n=0}^{+\infty} \frac{z^{2np}}{(2nq)!}$$

is entire function of order r , then, for any $y > 0$, we have

$$\widehat{\Phi}_r(\lambda) = \int_{-\infty}^{+\infty} e^{-G_r(z) - i\lambda z} dz = \int_{-\infty}^{+\infty} e^{-G_r(x-iy) - i\lambda(x-iy)} dx = e^{-\lambda y} \int_{-\infty}^{+\infty} e^{-G_r(x-iy) - i\lambda x} dx$$

and hence

$$|\widehat{\Phi}_r(\lambda)| \leq e^{-\lambda y} \int_{-\infty}^{+\infty} e^{-\operatorname{Re} G_r(x-iy)} dx.$$

In what follows, we suppose $y > y_0(p, q)$ to be sufficiently large and set

$$x - iy = \rho e^{-i\varphi}, \quad \text{where } \rho = \sqrt{x^2 + y^2}, \quad \varphi = \arctan \frac{y}{x}.$$

Then

$$\begin{aligned} \operatorname{Re} G_r(x - iy) &= \operatorname{Re} G_r(\rho e^{-i\varphi}) = \operatorname{Re} \sum_{k=0}^{q-1} \cosh\left(\rho^{\frac{p}{q}} e^{\frac{i}{q}(\pi k - p\varphi)}\right) = \\ &= \sum_{k=0}^{q-1} \cosh\left(\rho^{\frac{p}{q}} \cos \frac{\pi k - p\varphi}{q}\right) \cos\left(\rho^{\frac{p}{q}} \sin \frac{\pi k - p\varphi}{q}\right). \end{aligned} \quad (3.1)$$

Let

$$x_0 = \left(\frac{3\sqrt{2}}{\pi} y \right)^{\frac{q}{q-p}}.$$

Then, for $x \geq x_0$, we have

$$\begin{aligned} x^{1-\frac{p}{q}} &\geq \frac{3\sqrt{2}}{\pi} y, & 0 \leq \varphi \leq \frac{y}{x} &\leq \frac{\pi}{3\sqrt{2}} x^{-\frac{p}{q}}, \\ \rho^{\frac{p}{q}} \sin \frac{p\varphi}{q} &\leq (x^2 + y^2)^{\frac{p}{2q}} \frac{py}{qx} \leq (2x^2)^{\frac{p}{2q}} \frac{\pi}{3\sqrt{2}} x^{-\frac{p}{q}} < \frac{\pi}{3}, \end{aligned}$$

and hence

$$\begin{aligned} \cos \left(\rho^{\frac{p}{q}} \sin \frac{p\varphi}{q} \right) &> \cos \frac{\pi}{3} = \frac{1}{2}, \\ \rho^{\frac{p}{q}} \cos \frac{p\varphi}{q} &\geq \rho^{\frac{p}{q}} \left(1 - \frac{1}{2} \left(\frac{p\varphi}{q} \right)^2 \right) > x^{\frac{p}{q}} \left(1 - \frac{\varphi^2}{2} \right) > \\ &> x^{\frac{p}{q}} \left(1 - \frac{1}{2} x^{-\frac{p}{2q}} \right) > x^{\frac{p}{q}} - x^{-\frac{p}{q}}. \end{aligned}$$

Denote by A_r the term with $k = 0$ in (3.1). Then

$$A_r = \cosh \left(\rho^{\frac{p}{q}} \cos \frac{p\varphi}{q} \right) \cos \left(\rho^{\frac{p}{q}} \sin \frac{p\varphi}{q} \right) > \frac{1}{2} \cosh \left(x^{\frac{p}{q}} - x^{-\frac{p}{q}} \right) > \frac{1}{5} \exp \left(x^{\frac{p}{q}} \right).$$

Suppose now that $1 \leq k \leq q-1$. Then

$$\begin{aligned} \left| \cos \frac{\pi k - p\varphi}{q} \right| &= \left| \cos \frac{\pi k}{q} \cos \frac{p\varphi}{q} + \sin \frac{\pi k}{q} \sin \frac{p\varphi}{q} \right| \leq \\ &\leq \left| \cos \frac{\pi k}{q} \right| + \varphi < \left| \cos \frac{\pi k}{q} \right| + \frac{\pi}{3\sqrt{2}} x^{-\frac{p}{q}}. \end{aligned}$$

Since

$$\rho^{\frac{p}{q}} = x^{\frac{p}{q}} \left(1 + \frac{y^2}{x^2} \right)^{\frac{p}{2q}} < x^{\frac{p}{q}} (1 + x^{-\frac{p}{2q}})^{\frac{p}{2q}} < x^{\frac{p}{q}} \left(1 + \frac{p}{2q} x^{-\frac{p}{2q}} \right) < x^{\frac{p}{q}} + 0.5x^{-\frac{p}{q}},$$

we get

$$\left| \rho^{\frac{p}{q}} \cos \frac{\pi k - p\varphi}{q} \right| < (x^{\frac{p}{q}} + 0.5x^{-\frac{p}{q}}) \left(\left| \cos \frac{\pi k}{q} \right| + \frac{\pi}{3\sqrt{2}} x^{-\frac{p}{q}} \right) < x^{\frac{p}{q}} \left| \cos \frac{\pi k}{q} \right| + \frac{3}{4}.$$

Thus,

$$\cosh \left(\rho^{\frac{p}{q}} \cos \frac{p \varphi}{q} \right) \leq \cosh \left(x^{\frac{p}{q}} \left| \cos \frac{\pi k}{q} \right| + \frac{3}{4} \right) < 1.06 \exp \left(x^{\frac{p}{q}} \left| \cos \frac{\pi k}{q} \right| \right).$$

Denote by B_r the sum in (3.1) of the terms with $k > 0$. Then

$$|B_r| < 1.06 \sum_{k=1}^{q-1} \exp \left(x^{\frac{p}{q}} \left| \cos \frac{\pi k}{q} \right| \right) < 2.2 \exp \left(x^{\frac{p}{q}} \cos \frac{\pi}{q} \right).$$

Hence,

$$\operatorname{Re} G_r(x - iy) \geq A_r - |B_r| > \frac{1}{5} \exp \left(x^{\frac{p}{q}} \right) - \frac{11}{5} \exp \left(x^{\frac{p}{q}} \cos \frac{\pi}{q} \right) > \frac{1}{6} \exp \left(x^{\frac{p}{q}} \right)$$

for any $x \geq x_0$. The similar bound (with $|x|$ instead of x) holds for $x \leq -x_0$. If $|x| \leq x_0$ then

$$\begin{aligned} \rho &\leq (x_0^2 + y^2)^{0.5} = \left\{ \left(\frac{3\sqrt{2}}{\pi} y \right)^{\frac{2p}{q-p}} + y^2 \right\}^{0.5} = \\ &= y^{\frac{q}{q-p}} \left\{ \left(\frac{3\sqrt{2}}{\pi} \right)^{\frac{2q}{q-p}} + y^{-\frac{2p}{q-p}} \right\}^{0.5} < (y\sqrt{2})^{\frac{q}{q-p}}, \end{aligned}$$

so we have

$$|\operatorname{Re} G_r(x - iy)| \leq q \cosh \left(\rho^{\frac{p}{q}} \right) < q \cosh \left((y\sqrt{2})^{\frac{p}{q-p}} \right).$$

Passing to the estimate of $\widehat{\Phi}(\lambda)$, we obtain

$$\begin{aligned} |\widehat{\Phi}(\lambda)| &\leq e^{-\lambda y} \left(\int_{-x_0}^{x_0} \exp \left\{ q \cosh \left((y\sqrt{2})^{\frac{p}{q-p}} \right) \right\} dx + \right. \\ &\quad \left. + \left(\int_{-\infty}^{-x_0} + \int_{x_0}^{+\infty} \right) \exp \left(-\frac{1}{6} \exp \left(x_0^{\frac{p}{q}} \right) \right) dx < \right. \\ &\quad \left. < e^{-\lambda y} \left(2x_0 \exp \left\{ q \cosh \left((y\sqrt{2})^{\frac{p}{q-p}} \right) \right\} + \exp \left\{ -\frac{1}{6} \exp \left(x_0^{\frac{p}{q}} \right) \right\} \right) \right). \end{aligned}$$

Since $\sqrt{2} < 1.5$, we have

$$|\widehat{\Phi}(\lambda)| < \exp\left(-\lambda y + 0.5q e^{(1.5y)^{\frac{p}{q-p}}}\right).$$

Setting $y = \frac{2}{3}(\ln(\lambda/q))^{\frac{q}{p-1}}$, we finally get

$$|\widehat{\Phi}(\lambda)| < \exp\left(-\lambda y + 0.5q e^{\ln \frac{\lambda}{q}}\right) = \exp(-\lambda(y-0.5)) < \exp\left(-\frac{3}{5}\lambda \left(\ln \frac{\lambda}{q}\right)^{\frac{q}{p-1}}\right).$$

Lemma is proved. □

LEMMA 6. Let $G(u) = G_{1/2}(u)$, $\Phi(u) = \Phi_{1/2}(u)$ (in the notation of Lemma 5). Then the inequality

$$|\widehat{\Phi}(\lambda)| \leq \exp\left(-\frac{\pi}{1+\delta} |\lambda| \ln |\lambda|\right)$$

holds for any fixed δ , $0 < \delta < \delta_0 < 0.5$, and for any real λ , $|\lambda| > \lambda_0(\delta)$.

PROOF. Applying the same argument as above, we get

$$|\widehat{\Phi}(\lambda)| \leq e^{-\lambda y} \int_{-\infty}^{+\infty} e^{-\operatorname{Re} G(x-iy)} dx.$$

Let $\delta_1, \delta_2, \dots$ denote some positive constants depending on δ and such that $\delta_j \rightarrow 0$ when $\delta \rightarrow 0$. Taking $\rho = \sqrt{x^2 + y^2}$, $\varphi = \arctan(y/x)$, $x_0 = (1 + \delta_1)(y/\pi)^2$, we get for $x \geq x_0$:

$$\begin{aligned} 0 < \sqrt{\rho} \sin \frac{\varphi}{2} &\leq \sqrt{x} \left(1 + \left(\frac{y}{x}\right)^2\right)^{0.25} \frac{y}{2x} \leq \frac{y}{2\sqrt{x}} (1 + \delta_2) < \frac{\pi}{2} (1 - \delta), \\ 0 < \varphi &< \frac{y}{x} < \frac{\pi}{\sqrt{x}}. \end{aligned}$$

Hence,

$$\sqrt{\rho} \cos \frac{\varphi}{2} > \sqrt{x} \left(1 - \frac{\varphi^2}{8}\right) > \sqrt{x} \left(1 - \frac{\pi^2}{8x}\right) > \sqrt{x} - \frac{5}{4\sqrt{x}},$$

$$\begin{aligned} \cosh\left(\sqrt{\rho}\cos\frac{\varphi}{2}\right)\cos\left(\sqrt{\rho}\sin\frac{\varphi}{2}\right) &> \frac{1}{2}e^{\sqrt{x}-\frac{5}{4\sqrt{x}}}\cos\frac{\pi}{2}(1-\delta) = \frac{1}{2}e^{\sqrt{x}-\frac{5}{4\sqrt{x}}}\sin\frac{\pi\delta}{2}, \\ \left|\cosh\left(\sqrt{\rho}\sin\frac{\varphi}{2}\right)\cos\left(\sqrt{\rho}\cos\frac{\varphi}{2}\right)\right| &\leq \cosh\left(\sqrt{\rho}\sin\frac{\varphi}{2}\right) < \cosh\frac{\pi}{2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \operatorname{Re} G(x-iy) &= \cosh\left(\sqrt{\rho}\cos\frac{\varphi}{2}\right)\cos\left(\sqrt{\rho}\sin\frac{\varphi}{2}\right) + \\ &\quad + \cosh\left(\sqrt{\rho}\sin\frac{\varphi}{2}\right)\cos\left(\sqrt{\rho}\cos\frac{\varphi}{2}\right) > \\ &> \cosh\left(\sqrt{\rho}\cos\frac{\varphi}{2}\right)\cos\left(\sqrt{\rho}\sin\frac{\varphi}{2}\right) - \cosh\left(\sqrt{\rho}\sin\frac{\varphi}{2}\right) > \\ &> \frac{1}{2}\left(\sin\frac{\pi\delta}{2}\right)e^{\sqrt{x}-\frac{5}{4\sqrt{x}}} - \cosh\frac{\pi}{2} > \frac{3\delta}{4}e^{\sqrt{x}}. \end{aligned}$$

for x under considering. The same bound is true for $x \leq -x_0$ with $|x|$ instead of x . In the case $|x| \leq x_0$, we have

$$\sqrt{\rho} \leq (x_0^2 + y^2)^{0.25} \leq \left((1 + \delta_1)^2 \left(\frac{y}{\pi}\right)^4 + y^2 \right)^{0.25} < (1 + \delta_3) \frac{y}{\pi},$$

and hence

$$|\operatorname{Re} G(x-iy)| \leq 2 \cosh(\sqrt{\rho}) < \exp\left((1 + \delta_4) \frac{y}{\pi} \right).$$

Thus we obtain

$$\begin{aligned} |\widehat{\Phi}(\lambda)| &\leq e^{-\lambda y} \left(\int_{-x_0}^{x_0} \exp\left\{ \exp\left((1 + \delta_4) \frac{y}{\pi} \right) \right\} dx + \right. \\ &\quad \left. + 2 \int_{x_0}^{+\infty} \exp\left\{ -\frac{3\delta}{4} e^{\sqrt{x}} \right\} dx \right) < 3x_0 \exp\left\{ -\lambda y + e^{(1+\delta_4)\frac{y}{\pi}} \right\}. \end{aligned}$$

Now let $y = \frac{\pi \ln \lambda}{1 + \delta_4}$. Then

$$|\widehat{\Phi}(\lambda)| < \frac{3(1 + \delta_1)}{(1 + \delta_4)^2} (\ln \lambda)^2 \exp \left\{ -\frac{\pi \lambda \ln \lambda}{1 + \delta_4} + \lambda \right\}.$$

If δ is sufficiently small, then

$$|\widehat{\Phi}(\lambda)| < \exp \left\{ -\frac{\pi \lambda \ln \lambda}{1 + \delta} \right\}.$$

The case of negative λ can be treated in the same way. Lemma is proved. \square

4. Basic assertions

Here we prove Theorems 1–4. In what follows, we use the notation of §2, 3 without any comments.

PROOF OF THEOREM 1. Let $m \geq 2$, $\Phi(u) = \exp \left(-\frac{u^{2m}}{2m} \right)$. By Lemma 4, the estimate (2.1) holds for

$$F(\lambda) = c_0 \lambda^{\frac{1}{2m-1}}, \quad c_0 = \frac{\sin \pi \kappa}{1 + 2\kappa}, \quad \kappa = \frac{1}{2(2m-1)}$$

and for sufficiently large $|\lambda|$. Obviously, we have $\varphi(v) = (cv)^{2m-1}$, $c = c_0^{-1}$. Hence, the equation (2.6) takes the form

$$\frac{\alpha \tau_1}{2} + (2m-1) \left(\ln \left(\frac{\tau_1}{2} + 1 \right) + \ln c \right) = \ln \ln H.$$

For fixed m, α and $H \rightarrow +\infty$, we have

$$\frac{\alpha \tau_1}{2} = \ln \ln H - (2m-1) \ln \ln \ln H + (2m-1) \ln \alpha c_0 + O \left(\frac{\ln \ln \ln H}{\ln \ln H} \right),$$

so hence

$$\frac{e^{0.25\alpha\tau_1}}{\sqrt{\tau_1}} > (\alpha c_0)^m \frac{\sqrt{\ln H}}{(\ln \ln H)^m} \geq \left(\frac{2\alpha\kappa}{2\kappa+1} \right)^m \frac{\sqrt{\ln H}}{(\ln \ln H)^m} = \frac{\alpha^m \sqrt{\ln H}}{(2m \ln \ln H)^m}.$$

Since $g(v) = (2mv)^{1/(2m)}$, then for any $H \geq (1/3)(2m \ln \ln T)^{1/(2m)}$ and some $\alpha > 0$ we obtain from (2.7) that

$$F(T; H) > \exp \left(\frac{1}{6\sqrt{\alpha\kappa}} \frac{\widehat{\Phi}(\alpha)}{\widehat{\Phi}(0)} \frac{\alpha^m \sqrt{\ln H}}{(2m \ln \ln H)^m} \right), \quad \text{where } \kappa = \max(0.5, 4\alpha^{-1}).$$

One can check (see [20]) that $\widehat{\Phi}(u)$ is positive and monotonically decreasing for $0 \leq u \leq 1$ and

$$\widehat{\Phi}(0) = 2(2m)^{\frac{1}{2m}} \Gamma \left(1 + \frac{1}{2m} \right), \quad \widehat{\Phi}(1) > \frac{5}{4} \exp \left(-\frac{1}{2m} \left(\frac{\pi}{4} \right)^{2m} \right).$$

Finally we get

$$\begin{aligned} F(T; H) &> \exp \left\{ \frac{5}{96} \exp \left(-\frac{1}{2m} \left(\frac{\pi}{4} \right)^{2m} \right) (2m)^{-\frac{1}{2m}} \Gamma^{-1} \left(1 + \frac{1}{2m} \right) \frac{\sqrt{\ln H}}{(2m \ln \ln H)^m} \right\} > \\ &> \exp \left(\frac{0.05 \sqrt{\ln H}}{(2m \ln \ln H)^m} \right). \end{aligned}$$

Theorem 1 is proved. □

PROOF OF THEOREM 2. Let $r = p/q < 0.5$, $\Phi(u) = \Phi_r(u)$. By Lemma 5, one can take

$$F(\lambda) = \frac{3}{5} \left(\ln \frac{\lambda}{q} \right)^{\frac{q}{p}-1}, \quad \varphi(v) = q \exp \left(\left(\frac{5v}{3} \right)^{\frac{p}{q-p}} \right)$$

to satisfy (2.1). Thus, (2.3) takes the form

$$\alpha \tau_0 + \left(\frac{5}{3} \left(\frac{\tau_0}{2} + 1 \right) \right)^{\frac{p}{q-p}} + \ln q = \ln \ln H. \tag{4.1}$$

Since $0 < c = \frac{p}{q-p} < 1$, the solution τ_0 satisfies the relation

$$\alpha \tau_0 = \ln \ln H - \left(\frac{5}{6\alpha} \ln \ln H \right)^c - \ln q + O((\ln \ln H)^{2c-1}).$$

One can check that

$$\frac{e^{0.5\alpha\tau_0}}{\tau_0} > \alpha \sqrt{\frac{\ln H}{q}} \exp \left\{ -\frac{1}{2} \left(\frac{5}{6\alpha} \ln \ln H \right)^c + O((\ln \ln H)^{2c-1}) \right\} (\ln \ln H)^{-1}.$$

Since $g(v) = (\ln(3v))^{q/p}$, we have for $H \geq (1/3)(\ln \ln \ln T)^{q/p}$:

$$F(T; H) > \exp \left(\sqrt{\ln H} e^{-(c_0 \ln \ln H)^{\frac{p}{q-p}}} \right), \quad c_0 = \frac{1}{\alpha}.$$

In particular, for $q = 2m + 1$, $p = m$ and $H \geq (\ln \ln \ln T)^{2+1/m}$ we have

$$F(T; H) > \exp \left(\sqrt{\ln H} e^{-c_1 (\ln \ln H)^{1-\frac{1}{m+1}}} \right), \quad c_1 = c_1(m).$$

Given ε , we define m by the conditions $\frac{1}{m} \leq \varepsilon < \frac{1}{m-1}$. Then for any H ,

$$H > \frac{1}{3} (\ln \ln \ln T)^{2+\varepsilon} \geq \frac{1}{3} (\ln \ln \ln T)^{2+\frac{1}{m}}$$

we obtain:

$$F(T; H) > \exp \left(\sqrt{\ln H} e^{-c_1 (\ln \ln H)^{1-\frac{1}{m+1}}} \right) > \exp \left(\sqrt{\ln H} e^{-c_2 (\ln \ln H)^{1-0.5\varepsilon}} \right)$$

for some $c_2 = c_2(\varepsilon) > 0$. Theorem 2 is proved. □

PROOF OF THEOREM3. Let $\Phi(u) = \Phi_{1/2}(u) = e^{-(\cosh \sqrt{u} + \cos \sqrt{u})}$. Since $|\Phi(u)| \leq e^{1-0.5e\sqrt{|u|}}$ for real u , one can take

$$G(u) = \frac{1}{2} e^{\sqrt{u}} - 1, \quad g(v) = \ln^2(2v + 2).$$

Given $\delta > 0$, Lemma 5 implies that

$$|\widehat{\Phi}(\lambda)| \leq \exp \left(-\frac{\pi|\lambda| \ln |\lambda|}{1 + \delta} \right)$$

for any real λ , $|\lambda| > \lambda_0(\delta)$. Therefore,

$$\varphi(v) = \exp\left((1 + \delta) \frac{v}{\pi}\right),$$

and (2.3) takes the form

$$\alpha\tau_0 + \frac{1 + \delta}{\pi} \left(\frac{\tau_0}{2} + 1\right) = \ln \ln H.$$

Hence,

$$\tau_0 = \frac{\ln \ln H - \frac{1 + \delta}{\pi}}{\alpha + \frac{1 + \delta}{2\pi}}.$$

Let $\alpha = \varrho_1$ be the least positive root of the function $\widehat{\Phi}(\lambda)$. Then

$$\begin{aligned} e^{0.5\varrho_1\tau_0} &= \exp\left\{\frac{0.5\varrho_1}{\varrho_1 + (1 + \delta)/(2\pi)} \ln \ln H - \frac{\varrho_1(1 + \delta)}{2\pi\varrho_1 + 1 + \delta}\right\} > \\ &> \exp\left\{\frac{1 - \delta}{2 + (\pi\varrho_1)^{-1}} \ln \ln H\right\} = (\ln H)^{\frac{1 - \delta}{2 + (\pi\varrho_1)^{-1}}}. \end{aligned}$$

Given $\varepsilon > 0$, we can choose δ to satisfy the inequalities

$$F(T; H) > \exp\left\{\frac{1}{5e\varrho_1} \sqrt{\frac{5 \ln 2}{8e}} \frac{|\widehat{\Phi}'(\varrho_1)|}{\widehat{\Phi}(0)} \frac{e^{0.5\varrho_1\tau_0}}{\tau_0^2}\right\} > \exp((\ln H)^{\gamma - \varepsilon})$$

for any $H \geq (\ln \ln \ln T)^2$ and $\gamma = \frac{1}{2 + (\pi\varrho_1)^{-1}}$. The approximate calculations in “Wolfram Mathematica 7.0” show that $2.37689234 < \varrho_1 < 2.37689235$. Hence, $\gamma = 0.46862145 \dots$ Theorem 3 is proved. \square

PROOF OF THEOREM4. Let $0.5 < r = p/q < 1$, $\Phi(u) = \Phi_r(u)$. Similarly to the proof of Theorem 2, one can check that the equation (2.3) has the form (4.1). Since $c = \frac{p}{q - p} > 1$, its solution satisfies the relation

$$\tau_0 = \frac{6}{5}(\ln \ln H)^{\frac{1}{c}} - \frac{36\alpha}{25c}(\ln \ln H)^{\frac{2}{c} - 1} - 2 + O((\ln \ln H)^\eta),$$

where $\eta = \min\left(\frac{3}{c} - 2, \frac{1}{c} - 1\right)$. Hence, for $H \geq (\ln \ln \ln T)^{\frac{q}{p}}$, we have

$$\frac{e^{0.5\alpha\tau_0}}{\tau_0} > \exp\left\{\frac{3\alpha}{5}(\ln \ln H)^{\frac{1}{c}} - \frac{18\alpha^2}{25c}(\ln \ln H)^{\frac{2}{c}-1} - \alpha + O((\ln \ln H)^\eta)\right\}(\ln \ln H)^{-\frac{1}{c}},$$

$$F(T; H) > \exp\left\{\exp\left(0.5\alpha(\ln \ln H)^{\frac{q}{p}-1}\right)\right\}.$$

Given ε , we define m by the inequalities $\frac{1}{m} \leq \varepsilon < \frac{1}{m-1}$ and set $q = m+1, p = m$.

Taking $H \geq (\ln \ln \ln T)^{1+\varepsilon} \geq (\ln \ln \ln T)^{1+\frac{1}{m}}$, we obtain:

$$F(T; H) > \exp\left\{\exp\left(0.5\alpha(\ln \ln H)^{\frac{1}{m}}\right)\right\} > \exp\left\{\exp\left(0.5(\ln \ln H)^{0.5\varepsilon}\right)\right\}.$$

Theorem 4 is proved. □

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