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Defect of an admissible octahedron in a centering obtained by adding rational vectors to an integer lattice

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Abstract: In this paper, we study a problem in the geometry of numbers. Namely, we consider a centering Λ of the integer lattice $\mathbb{Z}^n$ and find new upper bounds for the smallest number of vectors from the standard basis of $\mathbb{Z}^n$ that must be replaced by some vectors from Λ in order that the resulting system of vectors forms a basis in Λ.

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1. Definitions and notation

This article is devoted to a problem in the geometry of numbers that originates from 1995 (see [5]). Let us introduce the necessary notation and state this problem formally.

Let $\Gamma \subset \mathbb{R}^n$ be an arbitrary lattice in an n-dimensional Euclidean space, and let $O = (0, 0, \dots, 0) \in \Gamma$ be the point of origin. If Γ is a sublattice of a lattice Λ, then Λ is called a *centering* of the lattice Γ. We are going to investigate the difference between the basis of a lattice and the basis of its centering.

Let us consider a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of Γ. The set of vectors $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ will be called a *frame*. The *defect of the frame $\mathcal{E}$ with respect to the lattice* Λ is defined as the smallest integer d such that certain $(n-d)$ vectors from $\mathcal{E}$ together with some d vectors from the lattice Λ form a basis of Λ. It is denoted as $d(\mathcal{E}, \Lambda) = d$.

An *octahedron* corresponding to the frame $\mathcal{E}$ is defined as the set

$$O^n_{\mathcal{E}} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} = \lambda_1 \mathbf{e}_1 + \ldots + \lambda_n \mathbf{e}_n; \ |\lambda_1| + \ldots + |\lambda_n| \leqslant 1\} .$$

The octahedron $O^n_{\mathcal{E}}$ is called *admissible* with respect to the lattice Λ if its interior contains no points of the lattice Λ, except for O and $\pm\mathbf{e}_i$:

$$O^n_{\mathcal{E}} \cap \Lambda = \{O, \mathbf{e}_1, -\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_n\}.$$

If the octahedron $O^n_{\mathcal{E}}$ corresponding to the frame $\mathcal{E}$ is admissible with respect to the centering Λ, then the quantity $d(\mathcal{E}, \Lambda)$ is denoted as $d(O^n_{\mathcal{E}}, \Lambda)$ and is called the *defect of the admissible octahedron $O^n_{\mathcal{E}}$ in the lattice* Λ.

Note that without loss of generality we can take Γ to be $\mathbb{Z}^n$ and the frame $\mathcal{E}$ to represent the standard basis (n unit vectors going in the directions of the coordinate axes).

Our results are strongly connected to the work [5] (as well as several related papers cited below), where the following quantities depending only on the dimension n have been introduced. The first of them is defined as follows:

$$d_n = \max_{\Lambda} d(O^n_{\mathcal{E}}, \Lambda),$$

where the maximum is taken over all centerings such that the defect $d(O^n_{\mathcal{E}}, \Lambda)$ is correctly defined. The second quantity is defined as

$$d^*_n = \max_{\Lambda} d(O^n_{\mathcal{E}}, \Lambda),$$

where the maximum is taken over *cyclic* centerings, meaning that Λ can be obtained from $\mathbb{Z}^n$ by adding a single rational vector $\mathbf{a}$: $\Lambda = \langle \mathbb{Z}^n, \mathbf{a} \rangle_{\mathbb{Z}}$. In other words, $\Lambda / \mathbb{Z}^n = \langle \mathbf{a} \rangle$.

This paper is going to be devoted to studying the quantity

$$d^m_n = \max_{\Lambda \in \mathcal{A}_m} d(O^n_{\mathcal{E}}, \Lambda),$$

where $\mathcal{A}_m$ is the set of all centerings of the integer lattice $\mathbb{Z}^n$ that can be obtained by adding exactly m rational vectors:

$$\Lambda = \langle \mathbb{Z}^n, \mathbf{a}_1, \dots, \mathbf{a}_m \rangle_{\mathbb{Z}}, \quad \mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Q}^n.$$

In other words, $d_n^* = d_n^1$ and $d_n = \max\limits_m d_n^m$.

The next section will introduce several known results and the new results obtained by the authors.

2. Formulation of the results

The asymptotic order of growth of d_n^1 was found in the papers [5], [6].

THEOREM 1. *There exist positive constants c_1, c_2 such that*

$$c_1 \frac{n}{\ln n} (\ln \ln n)^2 \le d_n^1 \le c_2 \frac{n}{\ln n} (\ln \ln n)^2.$$

In 2001, S. V. Konyagin gave a simple proof (see [10], [11]) of the following lower bound:

$$d_n \ge n - c \ln n,$$

where $c > 0$ is a constant. We also have an upper bound $d_n \le n$.

The main result of this article is the following theorem.

THEOREM 2. *There exists a positive constant C such that*

$$d_n^m \le C \frac{n \ln(m+1)}{\ln \frac{n}{m}} \left(\ln \ln \left(\frac{n}{m} \right)^m \right)^2.$$

Note that for $m = 1$ Theorem 2 provides the same asymptotic upper bound for d_n^m as Theorem 1. At the same time, the result of Theorem 2 is non-trivial only for

$$m \le e^{(\ln n)^{1/3}},$$

which implies

$$\ln(m+1) \le (1 + o(1))(\ln n)^{1/3}, \quad \ln \frac{n}{m} \sim \ln n, \quad \ln \ln \left(\frac{n}{m} \right)^m \le (1 + o(1))(\ln n)^{1/3},$$

and thus

$$C\frac{n\ln(m+1)}{\ln\frac{n}{m}}\left(\ln\ln\left(\frac{n}{m}\right)^m\right)^2 \le C'\frac{n(\ln n)^{1/3}}{\ln n}(\ln n)^{2/3} = C'n.$$

It should be mentioned that studies of the defect have not been restricted to the case of admissible octahedrons. Several other sets have been considered in [8], [9].

The remaining part of the paper is organized as follows. In Section 3, we are going to make several combinatorial constructions and quantify their relations to the defect. Section 4 will be devoted to the proof of Theorem 2.

3. Auxiliary combinatorial constructions

3.1. A system of families of sets $\mathfrak{M}$

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{Q}^n$ be given vectors. Let us reduce their coordinates to a common denominator:

$$\mathbf{a}_1 = \left(\frac{a_1^1}{q}, \frac{a_2^1}{q}, \ldots, \frac{a_n^1}{q}\right), \ldots, \mathbf{a}_m = \left(\frac{a_1^m}{q}, \frac{a_2^m}{q}, \ldots, \frac{a_n^m}{q}\right),$$

$$\left(a_1^1, \ldots, a_n^1, a_1^2, \ldots, a_n^m, q\right) = 1.$$

Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$ be the factorization of q. Define A as the matrix formed by writing vectors $q\mathbf{a}_1, \ldots, q\mathbf{a}_m$ as its rows. For each j, the rank of the matrix A in the ring $\mathbb{Z}_{p_j^{\alpha_j}}$ will be denoted as m_j.

Let $\mathcal{R}_n = \{1, \ldots, n\}$ be the set of all coordinate indexes. For each $j \in \{1, \ldots, s\}$ let M_j^i denote m_j-element subsets $\mathcal{R}_n$ such that for an arbitrary i the columns of the matrix A with numbers from M_j^i are linearly independent over the ring $\mathbb{Z}_{p_j^{\alpha_j}}$. For a fixed j, the family of sets M_j^i will be denoted as $\mathcal{M}_j$. Finally, the system of families of sets $\mathfrak{M}$ is defined as $\mathfrak{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_s\}$.

3.2. The relation between the defect and the system $\mathfrak{M}$

Let M be a subset of $\mathcal{R}_n$ such that for any $j \in \{1, \ldots, s\}$ there exists $i \in \left\{1, \ldots, |\mathcal{M}_j|\right\}$ for which $M_j^i \subseteq M$.

THEOREM 3. *Let $\Lambda = \langle \mathbb{Z}^n, \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle_{\mathbb{Z}}$. Then the following inequality is satisfied:* $d(\mathcal{E}, \Lambda) \le |M|$.

Proof. Any point of the lattice Λ can be represented as $\frac{1}{q}\mathbf{k}A + \mathbf{b}$, where $\mathbf{k} = (k_1, \dots, k_m)$ is a row of m integer numbers, A is the matrix defined in the previous section, and $\mathbf{b}$ is a vector in $\mathbb{Z}^n$.

Consider a subspace of $\mathbb{R}^n$ spanned by the coordinate axes with indexes that do not belong to M. Assume that a point $\mathbf{x} = \frac{1}{q}\mathbf{k}A + \mathbf{b}$ of the lattice Λ lies in this subspace. Then its coordinates with numbers from M are equal to zero. Let us fix a number $j \in \{1, \dots, s\}$. By definition of M, there exists a set $M_j^i = \{v_1, \dots, v_{m_j}\}$ which is fully embedded in M. Thus the coordinates of $\mathbf{x}$ numbered as $v_1, \dots, v_{m_j}$ are also equal to zero. In other words, coordinates of the vector $\mathbf{k}A$ numbered as $v_1, \dots, v_{m_j}$ are divisible by q, and thus also by $p_j^{\alpha_j}$. However, columns of the matrix A numbered as $v_1, \dots, v_{m_j}$ form a maximal linearly independent set of vectors of the matrix A over the ring $\mathbb{Z}_{p_j^{\alpha_j}}$ (by the definition of the set M_j^i). Then all other columns of A can be expressed in the ring $\mathbb{Z}_{p_j^{\alpha_j}}$ as linear combinations of these m_j columns. Therefore, all coordinates of the vector $\mathbf{k}A$ are divisible by $p_j^{\alpha_j}$. Since this applies for any $j \in \{1, \dots, s\}$, all coordinates of the vector $\mathbf{k}A$ are therefore divisible by q. Finally, $\mathbf{x} \in \mathbb{Z}^n$, meaning that vectors of the frame $\mathcal{E}$ with numbers from $\mathcal{R}_n \setminus M$ can be completed to form a basis of the lattice Λ, and thus we have $d(\mathcal{E}, \Lambda) \le |M|$. □

Theorem 3 holds for any M, allowing us to write $d(\mathcal{E}, \Lambda) \le \theta(\mathfrak{M})$, where $\theta(\mathfrak{M})$ is the cardinality of the smallest set M. In the next subsection we are going to recall a problem similar to approximation of θ.

3.3. A covering problem

Let $\mathcal{L} = \{L_1, \dots, L_t\}$ be an arbitrary family of subsets of the set $\mathcal{R}_n$. Its *system of common representatives* (*SCR*) is defined as a set $S \subseteq \mathcal{R}_n$ that includes at least one element from each L_i. The size of the minimal SCR for $\mathcal{L}$ is denoted as $\tau(\mathcal{L})$. Clearly, the setting in the previous subsection is more general: instead of a family of sets we consider the system of families of sets $\mathfrak{M}$. If we assume that the size of all sets in every family from $\mathfrak{M}$ equals one, then the set M defined in the previous subsection is, as a matter of fact, an SCR. Theorem 4 below provides an upper bound on the size of a minimal SCR which will later help us to obtain a bound for $\theta(\mathfrak{M})$. A proof and a discussion of this theorem can be found in [11], [7], [4].

THEOREM 4. *Assume that $|L_i| \geq k$ for each $i \in \{1, \ldots, t\}$. Then there exists a constant c such that*

$$\tau(\mathcal{L}) \leq c\frac{n}{k} \cdot \max\left\{1, \ln\frac{tk}{n}\right\}.$$

4. Proof of Theorem 2

4.1. Outline of the proof

Consider vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{Q}^n$. Let us construct a system of families of sets $\mathfrak{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_s\}$ using the method from Subsection 3.1. We would like to prove the inequality

$$\theta(\mathfrak{M}) \leq C\frac{n\ln(m+1)}{\ln\frac{n}{m}}\left(\ln\ln\left(\frac{n}{m}\right)^m\right)^2.$$

by applying Theorem 3. Subsection 4.3 is going to contain this proof, and the auxiliary lemmas used in the proof are presented in the following subsection.

4.2. Auxiliary Lemmas

LEMMA 1. *Consider a prime number p, a positive integer α and a homogeneous system of l independent linear equations in m variables over the ring $\mathbb{Z}_{p^\alpha}$. Then this system has at least p^{m-l} different solutions.*

PROOF OF LEMMA 1. Consider the original system of equations over the field $\mathbb{Z}_p$. This transition can lead to a linear dependence between the equations, therefore let us remove the equations in the system one by one until we are left with a system of l' linearly independent equations over $\mathbb{Z}_p$. The resulting system of equations has exactly $p^{m-l'}$ solutions. Multiplying every solution by $p^{\alpha-1}$ yields a solution of the original system, and this correspondence is injective. Hence the original system has at least $p^{m-l'} \geq p^{m-l}$ solutions, which concludes the proof. □

LEMMA 2. *Let $j \in \{1, \ldots, s\}$ and let $v_1, \ldots, v_l$ be l integers, $0 \leq l < m_j$, $1 \leq v_i \leq n$, such that columns of the matrix A (see Subsection 3.1) numbered as $v_1, \ldots, v_l$ are independent over the ring $\mathbb{Z}_{p_j^{\alpha_j}}$. Consider all possible subsets $\widetilde{M}_j^i$ of the set $\mathcal{R}_n = \{1, \ldots, n\}$ of cardinality $m_j - l$ satisfying the following conditions: a) they do not intersect with the set $\{v_1, \ldots, v_l\}$; b) columns of the matrix A with numbers* *from* $\widetilde{M}_j^i \cup \{v_1, \dots, v_l\}$ *are independent over the ring* $\mathbb{Z}_{p_j^{\alpha_j}}$. *Note that* $l = 0$ *always implies* $\widetilde{M}_j^i = M_j^i$. *If* $p_j \geq 5$, *then the following inequality holds*:

$$\left|\bigcup_i \widetilde{M}_j^i\right| \geq \frac{1}{2} \cdot \frac{\ln p_j^{m-l}}{\ln \ln p_j^{m-l}}.$$

Remark. It is easy to prove that the set $\bigcup_i \widetilde{M}_j^i$ contains the numbers corresponding to the columns of the matrix A which are non-zero and cannot be expressed as linear combinations of the l selected columns. On one hand, column numbers in $\bigcup_i \widetilde{M}_j^i$ clearly must satisfy these conditions. On the other hand, any set of independent columns can be completed to a set of m_j independent columns. Therefore, Lemma 2 states that for any l independent columns there exist sufficiently many columns which are linearly independent from the selected l columns.

PROOF OF LEMMA 2. Consider a sublattice Λ' of the initial lattice

$$\Lambda = \langle \mathbb{Z}^n, \mathbf{a}_1, \dots, \mathbf{a}_m \rangle_{\mathbb{Z}},$$

where for each $\mathbf{x} \in \Lambda'$ the coordinates numbered as $v_1, \dots, v_l$ equal zero in the ring $\mathbb{Z}_{p_j^{\alpha_j}}$. If we consider a system of l independent linear equations defined by the columns of the matrix A over in $\mathbb{Z}_{p_j^{\alpha_j}}$, then the number of solutions to this system is a lower bound for the cardinality of the quotient group $\Lambda'/\mathbb{Z}^n$ (here we require that the coordinates numbered as $v_1, \dots, v_l$ of a point $\mathbf{x}$ in the lattice Λ' are equal to zero). Applying Lemma 1 leads directly to $|\Lambda'/\mathbb{Z}^n| \geq p_j^{m-l}$, which implies $\det \Lambda' \leq \frac{1}{p_j^{m-l}}$.

Let $\bigcup_i \widetilde{M}_j^i = \{u_1, \dots, u_k\}$. Consider an intersection of the unit octahedron and the lattice Λ' with a subspace spanned by coordinate axes numbered as $u_1, \dots, u_k$. The octahedron obtained by this intersection must also be admissible with respect to the intersection of the lattice with the same subspace, and the determinant of the lattice still cannot exceed $\frac{1}{p_j^{m-l}}$. Since the octahedron's volume is $\frac{2^k}{k!}$, by Minkowski's Theorem ([1], [2]) we have:

$$\frac{2^k}{k!} \leq \frac{2^k}{p_j^{m-l}} \Longrightarrow k! \geq p_j^{m-l} \Longrightarrow k \geq \frac{1}{2} \cdot \frac{\ln p_j^{m-l}}{\ln \ln p_j^{m-l}}.$$

The final inequality follows from the condition $p_j \geq 5$. The lemma is proved. □

LEMMA 3. *The following inequality holds*: $s \leq n$.

PROOF. The octahedron $O^n_{\mathcal{E}}$ is admissible with respect to the lattice Λ. At the same time, since the denominators of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are equal to q, then $\det \Lambda \leq \frac{1}{q}$. Thus, from Minkowski's Theorem, we have:

$$\frac{2^n}{n!} \leq \frac{2^n}{q} \Longrightarrow q \leq n!,$$

and $q = p_1^{\alpha_1} \ldots p_s^{\alpha_s} \geq s!$, which proves the lemma. □

4.3. A bound for $\theta(\mathfrak{M})$

Consider the system of families of sets $\mathfrak{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_s\}$. We can assume that n is sufficiently large (since n will only have impact on the constant C). We can also assume that $m \ll e^{(\ln n)^{1/3}}$ (see Section 2).

Let us start by defining L_j as the union of all sets from the family $\mathcal{M}_j$, $j = 1, \ldots, s$. Consider a family of sets $\mathcal{L} = \{L_1, \ldots, L_s\}$. Let us build a minimal SCR $\mathcal{L}$ (§ 3.3). Let us estimate the cardinality of this SCR or, in other words, obtain a bound for $\tau(\mathcal{L})$. If j is such that $p_j \leq \frac{n}{m}$, then let us take an arbitrary element from L_j and send it to the future SCR. From the prime number theorem [3], the number of such elements is $O\left(\frac{n}{m \ln(n/m)}\right)$. For the remaining values of j (the corresponding family of sets will be denoted as $\mathcal{L}'$), we can apply Lemma 2 with $l = 0$, which yields $|L_j| \geq \frac{1}{2} \cdot \frac{\ln p_j^m}{\ln \ln p_j^m}$. Here we choose n to be sufficiently large for the inequality $p_j > \frac{n}{m} > 5$ to be satisfied. For sufficiently large values of x, the function $\frac{\ln x}{\ln \ln x}$ is increasing, therefore we can write

$$|L_j| \geq \frac{1}{2} \cdot \frac{\ln \left(\frac{n}{m}\right)^m}{\ln \ln \left(\frac{n}{m}\right)^m}.$$

Let

$$k = \frac{1}{2} \cdot \frac{\ln \left(\frac{n}{m}\right)^m}{\ln \ln \left(\frac{n}{m}\right)^m}.$$

From Lemma 3 we have $s \leq n$, and thus Theorem 4 yields

$$\tau(\mathcal{L}') = O\left(\frac{n}{k} \ln k\right) = O\left(\frac{n}{\ln \left(\frac{n}{m}\right)^m} \cdot \left(\ln \ln \left(\frac{n}{m}\right)^m\right)^2\right).$$

Clearly, the same inequality holds for $\tau(\mathcal{L})$.

Let $\tau_1 = \tau(\mathcal{L})$, and let us denote the elements of the corresponding SCR as $v_1^1, \ldots, v_{\tau_1}^1$.

For each $j \in \{1, \ldots, s\}$ consider an element $v_{\nu(j)}^1$ that lies in the set L_j. Clearly, this element lies in a number of sets M_j^i in the family $\mathcal{M}_j$. For each identified set M_j^i, let $\widetilde{M}_j^i = M_j^i \setminus \left\{v_{\nu(j)}^1\right\}$. From Lemma 2 with $l = 1$, we have

$$\left|\bigcup_i \widetilde{M}_j^i\right| \geq \frac{1}{2} \cdot \frac{\ln p_j^{m-1}}{\ln \ln p_j^{m-1}} \geq \frac{1}{2} \cdot \frac{\ln \left(\frac{n}{m}\right)^{m-1}}{\ln \ln \left(\frac{n}{m}\right)^{m-1}},$$

for an arbitrary j satisfying the condition $p_j > \frac{n}{m-1}$. Repeating the above argument, we can construct an SCR $v_1^2, \ldots, v_{\tau_2}^2$ for the family of sets $\bigcup_i \widetilde{M}_j^i$:

$$\tau_2 = O\left(\frac{n}{\ln \left(\frac{n}{m}\right)^{m-1}} \cdot \left(\ln \ln \left(\frac{n}{m}\right)^{m-1}\right)^2\right) = \\ = O\left(\frac{n}{\ln \left(\frac{n}{m}\right)^{m-1}} \cdot \left(\ln \ln \left(\frac{n}{m}\right)^{m}\right)^2\right).$$

Now let us consider a pair of elements $v_{\nu(j)}^1$, $v_{\mu(j)}^2$ for each $j \in \{1, \ldots, s\}$. Both of them lie in a certain number of sets M_j^i from the family $\mathcal{M}_j$. Let sets $\widetilde{M}_j^i$ be defined as $\widetilde{M}_j^i = M_j^i \setminus \left\{v_{\nu(j)}^1, v_{\mu(j)}^2\right\}$. Considering the two cases $p_j \leq \frac{n}{m-2}$ and $p_j > \frac{n}{m-2}$ separately, it is possible to apply Lemma 2 with $l = 2$ to find an SCR $v_1^3, \ldots, v_{\tau_3}^3$:

$$\tau_3 = O\left(\frac{n}{\ln \left(\frac{n}{m}\right)^{m-2}} \cdot \left(\ln \ln \left(\frac{n}{m}\right)^{m-2}\right)^2\right) = \\ = O\left(\frac{n}{\ln \left(\frac{n}{m}\right)^{m-2}} \cdot \left(\ln \ln \left(\frac{n}{m}\right)^{m}\right)^2\right).$$

Repeating this procedure $f \leq m$ times yields the following set:

$$M = \left\{v_1^1, \ldots, v_{\tau_1}^1\right\} \sqcup \left\{v_1^2, \ldots, v_{\tau_2}^2\right\} \sqcup \ldots \sqcup \left\{v_1^f, \ldots, v_{\tau_f}^f\right\}.$$

It is clear that $\theta(\mathfrak{M}) \le |M|$, i. e., we can write

$$\theta(\mathfrak{M}) \le O\left(\frac{n}{\ln\left(\frac{n}{m}\right)^m}\cdot\left(\ln\ln\left(\frac{n}{m}\right)^m\right)^2\right) + \\ + O\left(\frac{n}{\ln\left(\frac{n}{m}\right)^{m-1}}\cdot\left(\ln\ln\left(\frac{n}{m}\right)^m\right)^2\right) + \ldots + \\ + O\left(\frac{n}{\ln\left(\frac{n}{m}\right)}\cdot\left(\ln\ln\left(\frac{n}{m}\right)^m\right)^2\right).$$

To simplify the right-hand side of this asymptotic equality, it is sufficient to compute the sum of the following expressions:

$$\frac{1}{\ln\left(\frac{n}{m}\right)^r} = \frac{1}{r}\cdot\frac{1}{\ln\left(\frac{n}{m}\right)}, \quad r = 1, \ldots, m.$$

Writing this sum as $O\left(\frac{\ln(m+1)}{\ln\left(\frac{n}{m}\right)}\right)$ proves the theorem.

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