ISSN 2220-5438

Reprint from

Moscow Journal of Combinatorics and Number Theory

Moscow Journal of Combinatorics and Number Theory Volume 5 • Issue 1–2 2015

МФТИ

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Volume 5 • Issue 1–2 2015

Moscow Journal
of Combinatorics
and Number Theory
2015, vol. 5, iss. 1–2,
pp. 21–25

Three-term identity for products of Jacobi theta functions of one or two variables

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Abstract: The article offers the arithmetic method of proof of new three-term identity for products of Jacobi theta functions of one or two variables.

Keywords: theta function, three-term identity, arithmetic Liouville's methods

AMS Subject classification: 11A27

Received: 10.03.2014; **revised:** 25.09.2014

1. Introduction

In his book [1] D. Mumford wrote the following about identities for theta functions: "We have listed these at such length to illustrate a key point in the theory of theta functions: the symmetry of situation generates rapidly an overwhelming number of formulae, which do not however make a completely elementary pattern. To obtain a clear picture of algebraic implications of these formulae altogether is then not usually easy". Identities for theta functions can be proved in different ways. One of the most common methods is based on Liouville's theorem from the theory of functions of a complex variable. This theorem says that if a holomorphic function is bounded on the whole complex plane then it is constant. Liouville also developed elementary arithmetic methods for proving identities for theta functions. Following this tradition the authors of papers [2], [3], [4], [5] proposed methods of proving classical identities.

Let

$$\vartheta_3(z;q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz}$$

and

$$H(z,w;q) = z + w + 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{2mn}\sin(2mz + 2nw)$$

be Jacobi theta functions depending on one variable and two variables respectively.

In this paper a new result is proved with the help of arithmetic methods.

Theorem 1. *Given arbitrary complex v, u, w and $|q| < 1$, we have*

$$\vartheta_3(v;q^2)H(-u,-w;q) + \vartheta_3(v-2u;q^2)H(v+w-u,u;q) + \\ + \vartheta_3(v+2w;q^2)H(w,u-w-v;q) = 0. \quad (1)$$

The author is grateful to V. Bykovskii for setting the problem and discussing the results.

2. Proof of Theorem 1

The proof is based on the identity from [4].

Let A be an additive abelian group, and let $G : \mathbb{Z}^{\nvDash} \to A$ be an arbitrary function satisfying the condition

$$G(-m_1,-m_2,-m_3) = -G(m_1,m_2,m_3)$$

and

$$\Phi(m_1,m_2,m_3) = G(m_1,m_2,m_3) + G(m_1+2m_2-m_3,-m_1-m_2,-m_1) + \\ + G(-m_3,-m_2+m_3,-m_1-2m_2+m_3).$$

Then for any positive integer d

$$\sum_{\substack{a,b,c\in\mathbb{Z},\\ a,c>0,\\ b^2+ac=d}} \Phi(a,b,c) = \sum_{\substack{d=n^2\\ 0<k<2n}} \Phi(k, n-k, 2n-k). \qquad (2)$$

Let us apply

$$G(m_1, m_2, m_3) = x^{m_1} y^{m_2} z^{m_3} - x^{-m_1} y^{-m_2} z^{-m_3},$$
$$\text{where } x = e^{2iu}, y = e^{2iv}, z = e^{2iw},\ \ u, v, w \in \mathbb{R} \qquad (3)$$

for (2). The left-hand side of (2) is equal to

$$S_1(d) = \sum_{\substack{a,b,c\in\mathbb{Z},\\ a,c>0,\\ b^2+ac=d}} \Phi(a,b,c) =$$

$$= \sum_{m_2^2+m_1m_3=d} \left(x^{m_1}y^{m_2}z^{m_3} - x^{-m_1-2m_2+m_3}y^{m_1+m_2}z^{m_1} - x^{m_3}y^{m_2-m_3}z^{m_1+2m_2-m_3}\right) +$$

$$+ \sum_{m_2^2+m_1m_3=d} \Big(-x^{-m_1}y^{-m_2}z^{-m_3} + x^{m_1+2m_2-m_3}y^{-m_1-m_2}z^{-m_1}$$
$$+ x^{-m_3}y^{-m_2+m_3}z^{-m_1-2m_2+m_3}\Big).$$

Substituting m_2 with $-m_2$ we get

$$S_1(d) = \sum_{m_2^2+m_1m_3=d} \Bigg(- y^{m_2}\left(x^{-m_1}z^{-m_3} - x^{m_1}z^{m_3}\right) +$$
$$+ \left(\frac{y}{x^2}\right)^{m_2}\left(x^{-m_3}\left(\frac{yz}{x}\right)^{-m_1} - x^{m_3}\left(\frac{yz}{x}\right)^{m_1}\right) +$$
$$+ \left(yz^2\right)^{m_2}\left(\left(\frac{x}{yz}\right)^{-m_3} z^{-m_1} - \left(\frac{x}{yz}\right)^{m_3} z^{m_1}\right)\Bigg).$$

The right-hand side of (2), in view of the relation

$$\sum_{0<k<2n} x^k = \frac{x(x^{2n-1}-1)}{x-1},$$

can be written as

$$S_2(d) = \sum_{\substack{d=n^2\\0<k<2n}} \Phi(k, n-k, 2n-k) =$$

$$= \sum_{d=n^2} \left(\frac{x(1-yz)}{(x-yz)(1-x)} \left(x^{2n}y^{-n} + x^{-2n}y^n\right) + \right.$$

$$\left. + \frac{z(y-x)}{(yz-x)(1-z)} \left(y^n z^{2n} + y^{-n} z^{-2n}\right) + \frac{xz-1}{(x-1)(z-1)} \left(y^n + y^{-n}\right) \right).$$

Let us multiply both sides of (2) by q^d and sum up the result over d. Then, if we put

$$\widetilde{H}(x, y; q) = \sum_{d_1, d_2 \in \mathbb{N}} (x^{-d_1} y^{-d_2} - x^{d_1} y^{d_2}) q^{2d_1 d_2}, \quad \widetilde{\vartheta}_3(x; q) = \sum_{b=-\infty}^{\infty} x^b q^{b^2},$$

and use the relations

$$\widetilde{H}(x, y; q) = -\widetilde{H}(1/x, 1/y; q), \quad \widetilde{\vartheta}_3(x; q) = 1 + \sum_{b=1}^{\infty} (x^b + x^{-b}) q^{b^2},$$

we reduce the left-hand side to

$$\widetilde{\vartheta}_3\,(y; q)\, \widetilde{H}\left(\frac{1}{x}, \frac{1}{z}; \sqrt{q}\right) + \widetilde{\vartheta}_3\left(\frac{y}{x^2}; q\right) \widetilde{H}\left(\frac{yz}{x}, x; \sqrt{q}\right) +$$

$$+ \widetilde{\vartheta}_3\left(yz^2; q\right) \widetilde{H}\left(z, \frac{x}{yz}; \sqrt{q}\right).$$

Notice that

$$\frac{x(yz-1)}{(x-yz)(x-1)} + \frac{z(y-x)}{(x-yz)(z-1)} + \frac{xz-1}{(x-1)(z-1)} = 0.$$

Thus, the right-hand side turns into

$$\frac{x(yz-1)}{(x-yz)(x-1)} \widetilde{\vartheta}_3\left(\frac{y}{x^2}; q\right) + \frac{z(y-x)}{(x-yz)(z-1)} \widetilde{\vartheta}_3\left(yz^2; q\right) + \frac{xz-1}{(x-1)(z-1)} \widetilde{\vartheta}_3\,(y; q)\,.$$

Therefore,

$$\widetilde{\vartheta}_3(y;q)\widetilde{H}\left(\frac{1}{x},\frac{1}{z};\sqrt{q}\right)+\widetilde{\vartheta}_3\left(\frac{y}{x^2};q\right)\widetilde{H}\left(\frac{yz}{x},x;\sqrt{q}\right)+\widetilde{\vartheta}_3\left(yz^2;q\right)\widetilde{H}\left(z,\frac{x}{yz};\sqrt{q}\right)=$$

$$=\frac{x(yz-1)}{(x-yz)(x-1)}\widetilde{\vartheta}_3\left(\frac{y}{x^2};q\right)+\frac{z(y-x)}{(x-yz)(z-1)}\widetilde{\vartheta}_3\left(yz^2;q\right)+\frac{xz-1}{(x-1)(z-1)}\widetilde{\vartheta}_3(y;q).$$

Taking the relation

$$H(u,v;q)=2i\left(\widetilde{H}(x,y;q)+\frac{xy-1}{(x-1)(y-1)}\right)$$

into account, along with (3), we obtain (1).

Finally, we notice that by the principle of analytic continuation the obtained identity can be extended to the whole complex plane. For any complex numbers v, u, w the identity of the theorem applies by principle of analytic continuation. □

Acknowledgements

This research was supported by the Russian Foundation for Basic Research (project no. 14–11–00335).

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