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Systemic Interbank Network Risks in Russia

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Abstract: Modelling of contagion in interbank networks is discussed. A model taking into account bow-tie structure and disassortativity of interbank networks is developed. The model is shown to provide a good quantitative description of the Russian interbank market. Detailed arguments favouring the non-percolative nature of contagion-related risks in the Russian interbank market are given.

Keywords: Interbank market, systemic risk, contagion, credit risk

AMS Subject classification: 91B80, 91B74, 91G40

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1. Introduction

Quantitative analysis of systemic risks in financial networks presents one of the most important applications of network theory related ideas. These studies belong to a wide strand of literature devoted to analysis of cascading failures in complex networks, see e. g. [4, 19]. One of the important topics is here developing network-based mathematical models of contagion propagation in interbank markets [5, 10, 13, 14]. Constructing relevant mathematical models of default propagation is nontrivial due to the necessity of reproducing such observed features of these networks as their scale-free nature and significant disassortativity and clustering [1, 11, 12] and bow-tie structure [11–13]. In the literature one can find methods of taking into account disassortativity [3, 6, 17] and clustering [8, 15], but these and similar considerations have to be transplanted into developing mathematical models of interbank loan networks allowing to reproduce their main features.

2. Interbank network and contagion

In what follows we characterize interbank credit market in terms of a weighted oriented graph characterized by the weighted adjacency matrix $W = \{w_{ij} \geq 0\}$ where link variables $w_{ij} > 0$ correspond to netted obligations of the bank i towards the bank j on the daily basis. A directed link $i \rightarrow j$ corresponds to a credit to i provided by j. For a given node outgoing links correspond therefore to its obligations towards neighbouring nodes and incoming ones to claims of the node under consideration towards neighbouring nodes so that default contagion propagates through the outgoing links. Our considerations are based on the data on Russian interbank market (see Fig. 1) from January 11, 2011 till December 30, 2013. The data contain information on interbank loans to residents for 185 banks.

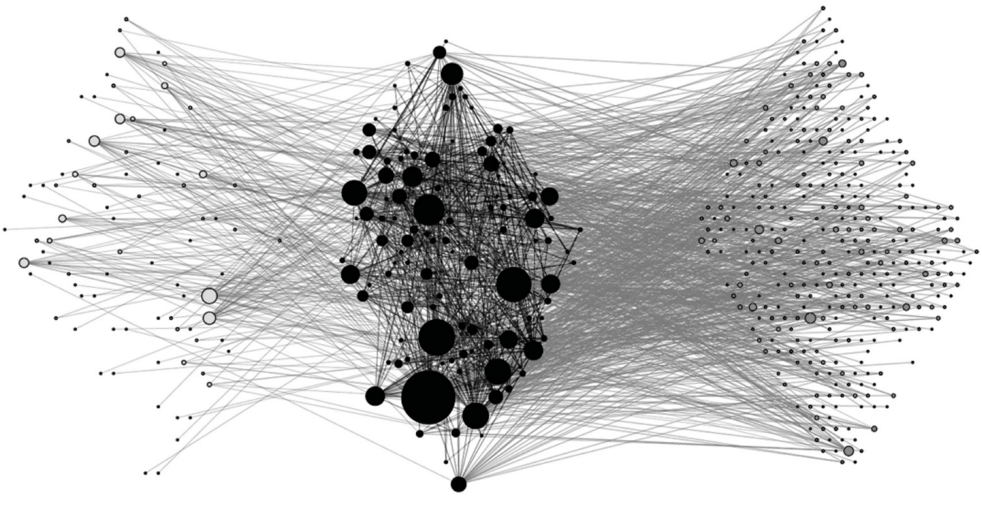

Fig. 1. Bow-tie structure of the Russian interbank network: Out component (left), In-Out component (center), In component (right). Diameter of the node in proportion to the sum of bank's claims and obligations

The structure of the interbank network shown in Fig. 1 reflects its generic so-called bow-tie decomposition into three components: nodes with outgoing links only, nodes with both incoming and outgoing links and nodes with incoming links only (Out, In-Out and In components correspondingly). A division of nodes among these components is, on average [13], about 20 % for Out and In-Out components and 60 % for the In one, so that the majority of the banks are pure lenders. As the origin of contagion can reside in the In and In-Out components only and, as we shall see below, the corresponding probabilities of spreading contagion to other nodes are very different, in the course of building a mathematical description of the problem it is essential to take into account the bow-tie structure of the interbank network. Let us also note that a disproportionately large fraction of about 50 % of the outstanding (amount of loans) resides in the In-Out component [13].

3. Gai—Kapadia contagion model

The first quantitative model of contagion at the interbank market taking into account network heterogeneity was developed in [10]. The mechanics of default propagation suggested in [10] is as follows. Let us consider a network node i having k incoming links, i. e. k banks having loans from i, with weights $\{w_{ji}\}$, $j = 1, \ldots, k$ corresponding to obligations of these k banks with respect to the bank i. Let us now assume [1)] that one of these k banks, the j^* one, defaults. The default propagates (contagion takes place) if this causes default of the node i. In the simplest setting this happens when the loss of $w_{j^* i}$ destroys the institutionally required balance between i'th assets A_i and liabilities L_i. The model of [10] assumes that for all nodes the sum of all node's interbank assets A_i^{IB}, $A_i^{\mathrm{IB}} \equiv \sum\limits_{j=1}^{k} w_{ji}$, is evenly distributed over k incoming links. A default of any of the neighbouring nodes causes contagion if it makes the capital buffer $K_i \equiv A_i - L_i$ negative [2)]:

$$K_i \equiv A_i - L_i < \frac{A_i^{\mathrm{IB}}}{k}. \tag{3.1}$$

A tractable analytical model of default propagation developed in [10] is based on a probabilistic description of contagion by introducing a probability v_k the probability

1) Here and in what follows we consider only the simplest case in which we have only one originally defaulting node.

2) Here it is assumed that the institutional requirement on the capital buffer is $K_i \geq 0$.

for any $i = 1, \ldots, N$ to be vulnerable conditioned on default of one of its borrowers

$$v_k = \text{Prob}\left[K_i < \frac{A_i^{\text{IB}}}{k}\right] \tag{3.2}$$

and describing an interbank network as a tree-like oriented random graph characterized by the degree distribution p_{jk}, the probability for a randomly chosen node to have j incoming and k outgoing links. A process of contagion is in these terms that of formation of a cluster of vulnerable nodes formed around the initially defaulting one, see Fig. 2.

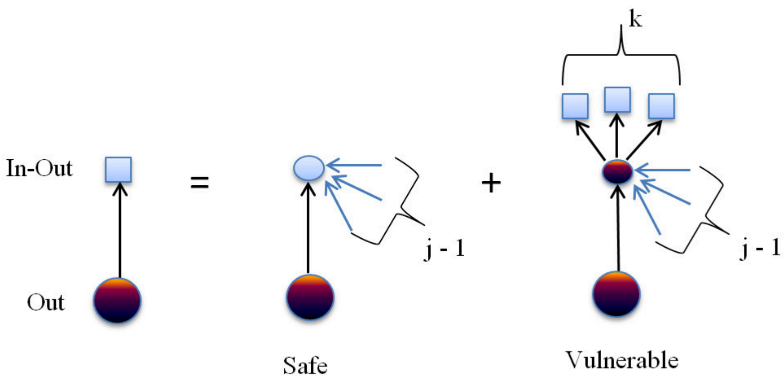

Fig. 2. Default propagation mechanism

A convenient analytical description of contagion can then be given in terms of the generating function $\mathcal{G}(x, y)$ describing the probability of having a vulnerable node with in-degree j and out-degree k:

$$\mathcal{G}(x, y) = \sum_{j,k} v_j p_{jk} x^j y^k. \tag{3.3}$$

The problem at hand is now a standard cite percolation problem, see e. g. [18]. Following the standard routine, let us introduce the generating functions $G_0(y)$ and $G_1(y)$ for the out-degrees of the vulnerable bank and its vulnerable neighbour respectively

$$G_0(y) = \sum_{j,k} v_j \cdot p_{jk} \cdot y^k, \qquad G_1(y) = \frac{\sum_{j,k} v_j \cdot j \cdot p_{jk} \cdot y^k}{\sum_{j,k} j \cdot p_{jk}}. \tag{3.4}$$

A standard set of equations for the size of a vulnerable cluster reached by following an arbitrary link leading from the initial vulnerable node $H_1(y)$ and the total size of the vulnerable cluster $H_0(y)$ then reads:

$$H_0(y) = 1 - G_0(1) + yG_0[H_1(y)],$$
$$H_1(y) = 1 - G_1(1) + yG_1[H_1(y)] \tag{3.5}$$

and, consequently, the following equation for the average size S of a default cluster formed by following the outgoing links joining vulnerable nodes:

$$S = G_0(1) + \frac{G_0'(1)G_1(1)}{1 - G_1'(1)}. \tag{3.6}$$

From (3.6) we see that the point of phase transition due to formation of a giant vulnerable cluster corresponds to $G_1'(1) = 1$ or, more explicitly,

$$\sum_{j,k} j \cdot k \cdot v_k \cdot p_{jk} = \langle k \rangle, \tag{3.7}$$

where $\langle k \rangle$ is an average vertex degree in the network under consideration. The main focus of [10] was precisely on systemic risk related to an appearance of the giant vulnerable cluster at the point $G_1'(1) = 1$. Let us note that numerical simulations in [10] were performed using the Poissonian distribution for the number of outgoing links. The analysis of [10] was further developed in [5] where a dependence of phase transition threshold on replacing Poisson degree distribution by a scale-free one and on degree-degree correlations was studied.

4. Contagion model with bow-tie structure and disassortativity

It is clearly of interest to develop a contagion model describing contagion risks on a real interbank market. The model should take into account both realistic topology of the corresponding interbank network and the structure of the banks' balance sheets. Such a model was developed for the Russian interbank market in [11–13]. In principle, with this data available, contagion risks can be studied through numerical simulations for any topology of the underlying network and default clusters. Possibility of analytical description depends however on how complex is the topology of contagion propagation and, therefore, that of the resulting default cluster. Extensive numerical simulations for the Russian interbank market have revealed several characteristic features of contagion propagation, namely:

1. Conditional contagion probability depends on positions of both spreader and recipient vertices within the bow-tie structure.
2. Although the underlying interbank network is characterized by significant local clustering, the default clusters are predominantly tree-like [3)]
3. The spreading process is sensitive to the probabilistic interrelations between the degrees of adjacent vertices of the underlying network. In fact, the interbank networks are, in all cases known, disassortative, see e. g. [17], i. e. there is a tendency for a link with a large degree to have first neighbours with small degrees. This is illustrated in Fig. (3), where we plot the following relations: in Fig. (3) (upper) we plot the mean number of outgoing links of the first neighbors of a node having a fixed number of incoming links; in Fig. (3) (lower) we plot the mean number of outgoing links of the first neighbours of a node having a fixed number of outgoing links. Both figures clearly show the above-described property of disassortativity.

The analysis of [11–13] uses the daily data on the interbank loans in the Russian interbank market and monthly data on banks' balance sheets. The model accounts for essential probabilistic patterns existing between adjacent nodes characterized by replacing the conditional probabilities of default propagation v_k and the bivariate degree distribution p_{jk} in the original model of [10] $v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)$ and degree distribution $P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)$, where conditional probabilities depend on the position of the corresponding nodes in the bow-tie structure so that the formalism includes $\mathrm{IO}\to\mathrm{IO}$ if both nodes belong to the In-Out component, u and t are the numbers of outgoing links from the borrower to In-Out and In components correspondingly and r the number of incoming links from In-Out component while k and l are the numbers of outgoing links for the lender. Interrelation between nodes from In-Out and In components is described by the corresponding conditional probabilities $v^{\mathrm{IO}\to\mathrm{In}}(r|k,l)$ and $P^{\mathrm{IO}\to\mathrm{In}}(r|k,l)$.

4.1. Existence of a giant cluster

We have already mentioned a theoretically appealing definition of systemic risk as of the appearance of the giant cluster [10]. The practical relevance of this

[3)] This observation is similar to the one made in [7]. We are grateful to C. Borgs for this reference.

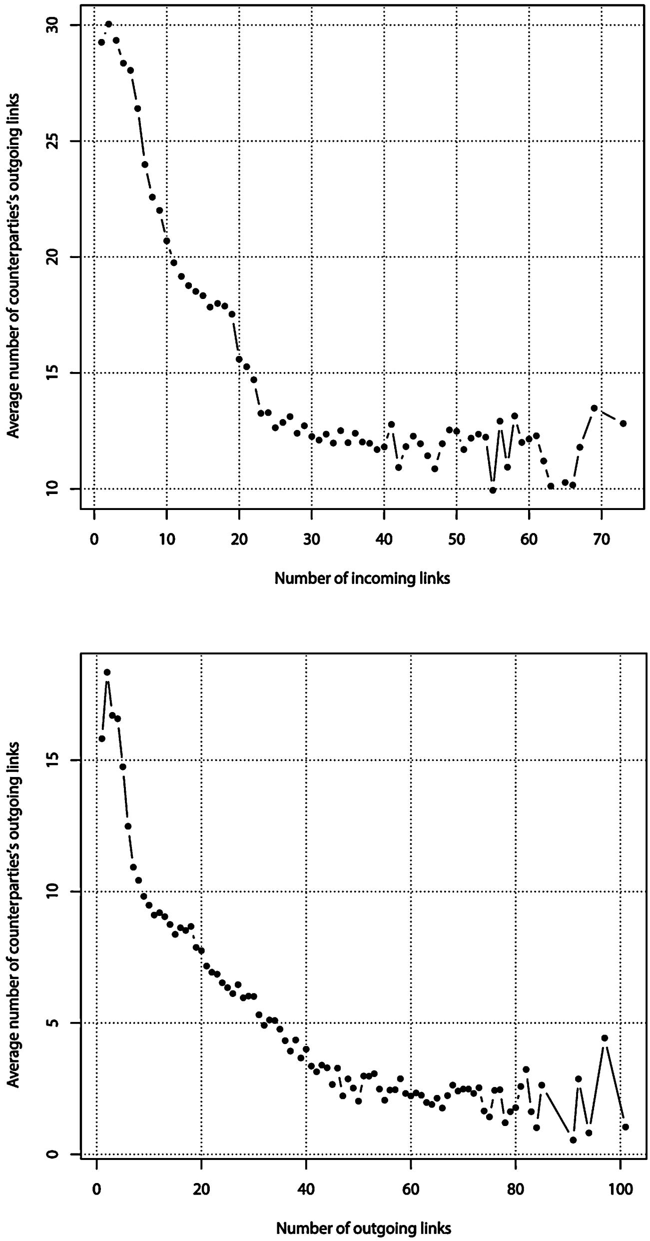

Fig. 3. Upper: mean number of outgoing links of the first neighbours of a node having a fixed number of incoming links k_{in}; Lower: mean number of outgoing links of the first neighbours of a node having a fixed number of outgoing links k_{out}.

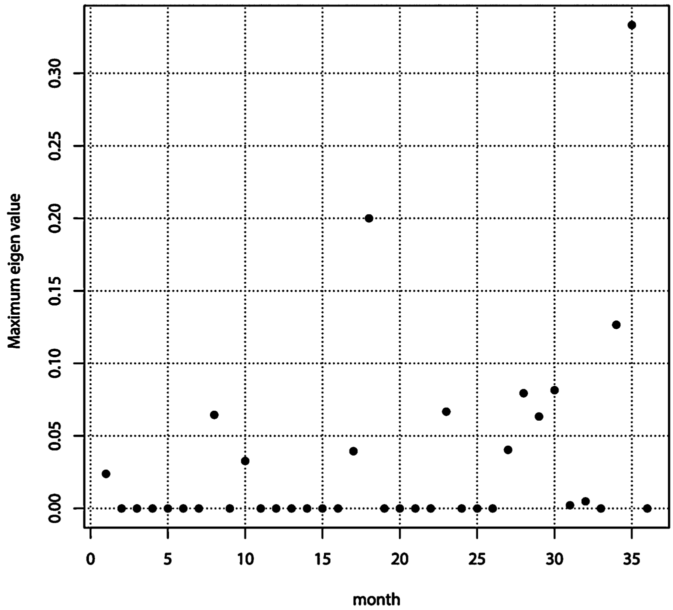

Fig. 4. Dynamics of the maximal eigenvalue of A. The data was sampled with a frequency of one month.

definition does however depend on whether formation of such a cluster is possible in real interbank networks. To answer this question one has to convert information on interbank loans and bank balance sheets into conditional probability distributions $v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)$ and $P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)$ thus specifying the structure of clusters of vulnerable nodes. For tree-like oriented graphs the condition of existence of a giant out-cluster can be formulated in terms of a condition $\lambda_{\max} > 1$ on the maximal eigenvalue of the matrix

$$A_{(k,l)(u,t)} = \sum_{r}^{\infty} uP^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l), \tag{4.1}$$

see e. g. [2, 3, 9] [4]. The details can be found in the Appendix. The dynamics of $\lambda_{\max}$ for the Russian interbank network is shown in Fig. 4. We see that the maximal

[4] In actual calculations it is convenient to change notations so that with each pair (k,l) and (u,t) one associates a natural number. In general matrix A may have infinite number of elements. Although in practice due to finite number of nodes in network the matrix size is bounded. A discussion of the origin of the criterion $\lambda_{\max} > 1$ can be found in [6].

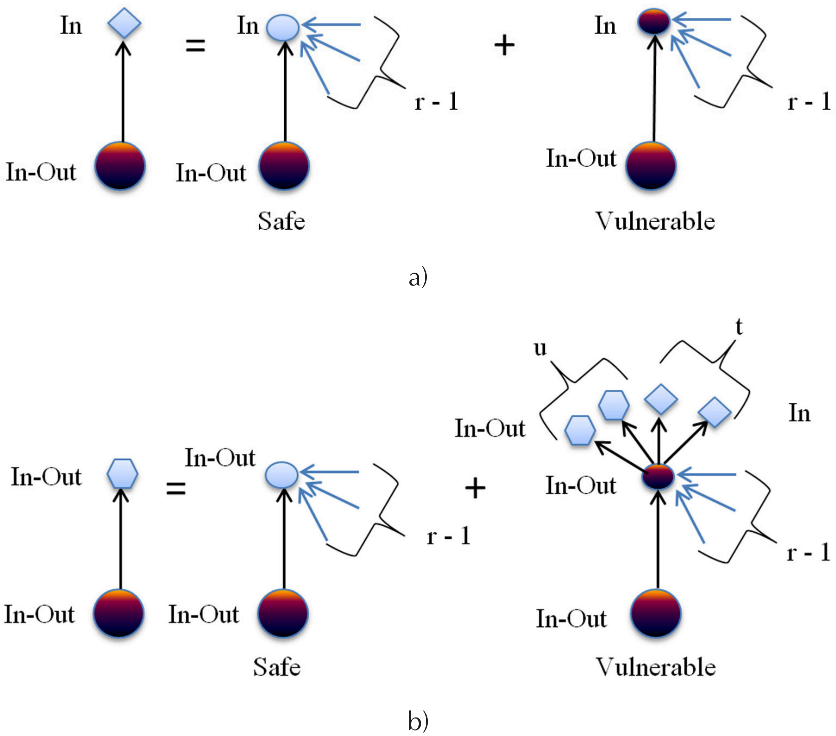

Fig. 5. Default spreading mechanism

observed value of $\lambda_{\max}$ is 0.35, so according to this criterion in the Russian interbank market the systemic risk as defined in [10] is absent.

4.2. Mean default cluster size

The mathematical model we use to describe systemic risks on the Russian interbank market developed in [13] generalizes the approach of [10] by explicitly taking into account the bow-tie structure of the network under consideration and its disassortativity. This means, in particular, that one has to consider two separate mechanisms of contagion propagation, those from the In-Out component to the In one, see Fig. 5 a, and within the In-Out component, see Fig. 5 b.

The equations for the corresponding generating functions read:

$$N_{k,l}(y) = \sum_{r}^{\infty} P^{\mathrm{IO}\to\mathrm{In}}(r|k,l)\left(1 - v^{\mathrm{IO}\to\mathrm{In}}(r|k,l) + y\, v^{\mathrm{IO}\to\mathrm{In}}(r|k,l)\right) \tag{4.2}$$

$$M_{k,l}(x,y) = \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)(1 - v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)) + \\ +x \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)[M_{u,t}(x,y)]^{u}[N_{u,t}(y)]^{t}. \tag{4.3}$$

Let us consider a bank from the In-Out component with $k + l$ outgoing links, where k of them lead to the In-Out component and l to In component respectively[5)] and take a randomly chosen edge linking the chosen node to a node in the In component which, in addition, has $r - 1$ incoming links, see Fig. 5 a. This is a simplest case where contagion goes from the In-Out component to the In one and stops there.

Let us introduce a generating function $F(x,y) = \sum_{k,l}^{\infty} P^{\mathrm{IO}}(k,l)x^k y^l$ for the probability for a bank from the In-Out component to have k and l first neighbours from the In-Out and In components respectively. Then $F(M,N)$ is the generating function for the number of vulnerable banks in the network. The mean size of vulnerable cluster $\langle s \rangle$ is then given by its derivative at $y = x = 1$:

$$\langle s \rangle = F'_x(y = x = 1). \tag{4.4}$$

We have

$$F'(M,N) = \sum_{k,l}^{\infty} P^{\mathrm{IO}}(k,l)(kM'_{k,l} + lN'_{k,l}). \tag{4.5}$$

Straightforward calculations [13] lead to the following expressions for N' and M':

$$N'_{k,l|x=1} = \sum_{r}^{\infty} P^{\mathrm{IO}\to\mathrm{In}}(r|k,l) v^{\mathrm{IO}\to\mathrm{In}}(r|k,l), \tag{4.6}$$

$$M'_{k,l} = \sum_{u,t}^{\infty} \beta_{k,l,u,t}\gamma_{u,t}. \tag{4.7}$$

5) As discussed in [13], nodes from the Out component generate very small systemic risks so that the corresponding effects will be neglected.

where $\beta_{k,l,u,t}$ is an element $B_{(u,t),(k,l)}$ of the matrix $B = (I - A)^{-1}$ and A is a matrix with the elements $A_{(k,l)(u,t)} = \alpha_{u,t,k,l}$, where in turn

$$\alpha_{u,t,k,l} = \sum_{r}^{\infty} u P^{\mathrm{IO} \to \mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO} \to \mathrm{IO}}(u,t,r|k,l) \tag{4.8}$$

and

$$\begin{aligned} \gamma_{k,l} = & \sum_{u,t,r}^{\infty} P^{\mathrm{IO} \to \mathrm{IO}}(u,t,r|k,l) v^{IO \to IO}(u,t,r|k,l) + \\ & + \sum_{u,t,r}^{\infty} P^{\mathrm{IO} \to \mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO} \to \mathrm{IO}}(u,t,r|k,l) \times \\ & \times t \sum_{r1}^{\infty} P^{\mathrm{IO} \to \mathrm{In}}(r_1|u,t) v^{\mathrm{IO} \to \mathrm{In}}(r_1|u,t). \end{aligned} \tag{4.9}$$

Plugging in empirical conditional probability distributions $P^{\mathrm{IO} \to \mathrm{In}}(r|k,l)$, $v^{\mathrm{IO} \to \mathrm{In}}(r|k,l)$, $P^{\mathrm{IO} \to \mathrm{IO}}(u,t,r|k,l)$ and $v^{\mathrm{IO} \to \mathrm{IO}}(u,t,r|k,l)$ calculated on the monthly basis we compute the corresponding values of $\langle s \rangle$. A comparison of the model predictions and results of stress testing are shown in Fig. 6. We see a very good agreement between the model and experiment provided one takes into account correlations between the degrees of adjacent nodes captured by $P^{\mathrm{IO} \to \mathrm{In}}(r|k,l)$ and $P^{\mathrm{IO} \to \mathrm{IO}}(u,t,r|k,l)$ and a much poorer one when these correlations are neglected.

5. Conclusions

Let us formulate once again the main conclusions of the present paper:

1. Analysis of data on Russian interbank market shows that default contagion risks can be classified as those characteristic of non-percolative phase.
2. To build a successful mathematical model of contagion propagation the bow-tie structure of the corresponding network and its disassortativity have to be taken into account.

Appendix

Let us illustrate the formation of the giant cluster in In-Out component of the bow-tie graph structure, starting from the node belonging to In-Out component.

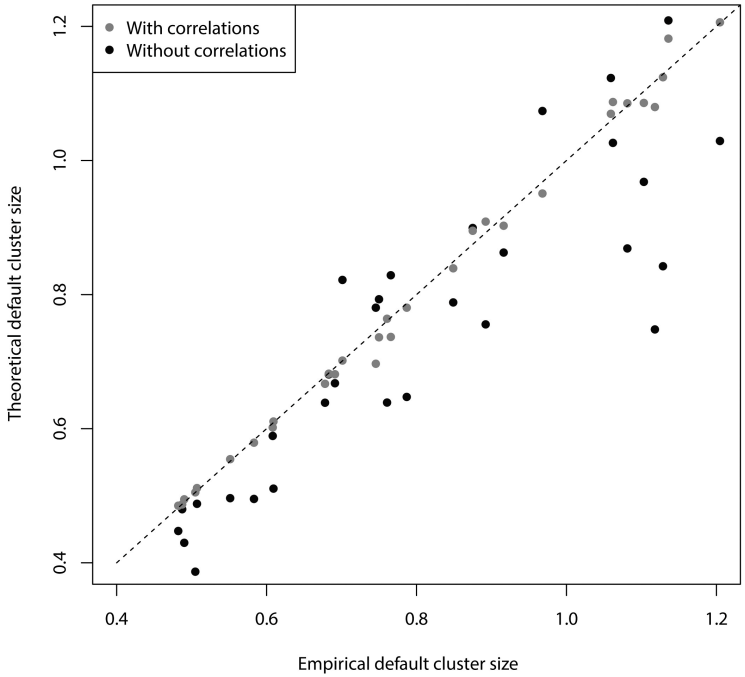

Fig. 6. Theoretical and empirical mean default cluster size

The giant cluster in an oriented graph is characterized by the degree distribution, i. e. the probability of having k outgoing links to In-Out component and l outgoing links to In component $P^{\mathrm{IO}}(k,l)$, and probability distribution $P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)$, the conditional probability of having u outgoing links to In-Out component, t outgoing links to In component and r incoming links from In-Out component for a first neighbour of the node having k and l outgoing links. The node we have arrived in may be vulnerable with conditional probability $v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)$ or safe with probability $1 - v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)$. We are interested in the mean number of the vulnerable nodes from In-Out component related with each other through the outgoing links.

The mean number of first neighbours from In-Out component that can be reached by following the outgoing links is conveniently computed using the corresponding generating function $F(x)$:

$$\widetilde{z}_1 = \left.\frac{dF(x)}{dx}\right|_{x=1} = \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l), \quad F(x) = \sum_{k=0,l=0}^{\infty} P^{\mathrm{IO}}(k,l)x^k. \tag{A.1}$$

But not all of the first neighbours are vulnerable. To compute mean number of the vulnerable first neighbours from In-Out component we need to introduce simple generating function describing whether random outgoing link goes to a vulnerable node. Let $M_{(k,l)}(x)$ be that generating function:

$$M_{(k,l)}(x) = \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)[1 - v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)] + \\ + x\sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l).$$

Then the average number of the vulnerable first neighbours is computed as follows:

$$m_1 = \left.\frac{dF\big(M(x)\big)}{dx}\right|_{x=1} = \frac{\partial F}{\partial M_{(k,l)}} \frac{\partial M_{(k,l)}}{\partial x} = \\ = \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l).$$

To calculate the mean number of second vulnerable neighbours we need to rewrite generating function $M_{(k,l)}(x)$ in a fashion described, e. g., in [16]:

$$M_{(k,l)}(x, M) = \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)\big[1 - v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)\big] + \\ + x\sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l)[M_{(u,t)}(x)]^u.$$

Let us denote the number of the n-th level vulnerable neighbours by z_n consider the total number m_n of neighbours up to level n:

$$m_1 = z_1, \quad m_2 = z_1 + z_2, \ldots, \quad m_n = \sum_{i=1}^{n} z_i.$$

Then

$$m_2 = \left.\frac{dF(M(M,x))}{dx}\right|_{x=1} = \frac{\partial F}{\partial M_{(k,l)}} \left[\frac{\partial M_{(k,l)}}{\partial M_{(u,t)}} \frac{\partial M_{(u,t)}}{\partial x} + \frac{\partial M_{(k,l)}}{\partial x}\right] =$$

$$= \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) +$$

$$+ \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) \times$$

$$\times \sum_{u_1,t_1,r_1}^{\infty} u_1 P^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_,r_1|u,t) v^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t),$$

$$z_2 = m_2 - m_1 = \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) \times$$

$$\times \sum_{u_1,t_1,r_1}^{\infty} u_1 P^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t) v^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t),$$

and, generically,

$$z_n = \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) \times$$

$$\times \sum_{u_1,t_1,r_1}^{\infty} u_1 P^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t) v^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t) \times \ldots \times$$

$$\times \sum_{u_{n-1},t_{n-1},r_{n-1}}^{\infty} u_{n-1} P^{\mathrm{IO}\to\mathrm{IO}}(u_{n-1},t_{n-1},r_{n-1}|u_{n-2},t_{n-2}) \times$$

$$\times \quad v^{\mathrm{IO}\to\mathrm{IO}}(u_{n-1},t_{n-1},r_{n-1}|u_{n-2},t_{n-2}).$$

The mean number of all neighbours is thus given by

$$\sum_{i=1}^{\infty} z_i = \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) \times \qquad \text{(A.2)}$$

$$\times [1 + \sum_{u_1,t_1,r_1}^{\infty} u_1 P^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t) v^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t) +$$

$$+ \sum_{u_1,t_1,r_1}^{\infty} u_1 P^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t) v^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t) \times$$

$$\times \sum_{u_2,t_2,r_2}^{\infty} u_2 P^{\mathrm{IO}\to\mathrm{IO}}(u_2,t_2,r_2|u_1,t_1) v^{\mathrm{IO}\to\mathrm{IO}}(u_2,t_2,r_2|u_1,t_1) + \ldots].$$

The giant component exists if the sum (A.3) diverges.

Let us now discuss the conditions for the existence of a giant component. Defining $A_{(u,t)(u_1,t_1)} = \sum_{r_1}^{\infty} u_1 P^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t) v^{\mathrm{IO}\to\mathrm{IO}}(u_1,t_1,r_1|u,t)$, we can rewrite (A.3) as follows:

$$\begin{aligned}\sum_{i=1}^{\infty} z_i &= \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) \times \\ &\quad \times \sum_{u_1,t_1}^{\infty} [I_{(u,t)(u_1,t_1)} + A_{(u,t)(u_1,t_1)} + (A^2)_{(u,t)(u_1,t_1)} + \ldots] = \\ &= \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) \times \\ &\quad \times \sum_{u_1,t_1}^{\infty} [I + A + A^2 + \ldots]_{(u,t)(u_1,t_1)}. \end{aligned} \tag{A.3}$$

Let us assume that

$$\sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) < \infty.$$

For the sum (A.3) to converge the operator A has to be linear and bounded, $||A|| < 1$. Then there exist an operator $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$ and the sum (A.3) can be rewritten in the following form:

$$\begin{aligned}\sum_{i=1}^{\infty} z_i &= \sum_{k,l}^{\infty} kP^{\mathrm{IO}}(k,l) \sum_{u,t,r}^{\infty} P^{\mathrm{IO}\to\mathrm{IO}} \times \\ &\quad \times (u,t,r|k,l) v^{\mathrm{IO}\to\mathrm{IO}}(u,t,r|k,l) \sum_{u_1,t_1}^{\infty} [I-A]^{-1}_{(u,t)(u_1,t_1)}. \end{aligned} \tag{A.4}$$

The condition $||A|| < 1$ leads us to a simple criterion for the absence of a giant cluster: if the maximal eigenvalue of A satisfies $\lambda_{\max} < 1$, there is no giant In-Out component [6]. Let us also note that according to Perron-Frobenius theorem the maximal eigenvalue satisfies

$$\lambda_{\max} \le \max_{(u,t)} \sum_{u_1,t_1}^{\infty} A_{(u,t)(u_1,t_1)}.$$

Bibliography

1. ***E. Bastos e Santos, R. Cont,*** *The Brazilian interbank network structure and systemic risk*, Banco Central do Brasil Working Paper 219 (2010).
2. ***S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D. U. Hwang,*** *Complex networks: Structure and dynamics*, Phys. Repts **64** (2006), 175–308.
3. ***M. Boguñá, M. Serrano,*** *Generalized percolation in random directed networks*, Phys. Rev. **E72** (2005), 016106.
4. ***J. Borge-Holthoefer, R. Baños,*** *Cascading behavior in complex socio-technical networks*, Journ. Complex Networks **1** (2013), 2–24.
5. ***F. Caccioli, T. A. Catanach, Farmer J. Doyne,*** *Heterogeneity, correlations and financial contagion*, Advances in Complex Systems **15** (2012), 1250058.
6. ***J. Gleeson,*** *Cascades on correlated and modular networks*, Phys. Rev. **E77** (2008), 046117.
7. ***S. Goel, A. Anderson, J. Hofman, D. Watts,*** *The structural virality of online diffusion*, Working paper (2013).
8. ***A. Hackett, S. Melnik, J. Gleeson,*** *Cascades on a class of clustered networks*, Phys. Rev. **E83** (2011), 056107.
9. ***T. Hurd, J. Gleeson,*** *A framework for analyzing contagion in financial networks*, ArXiv:1110.4312.
10. ***S. Kapadia, P. Gai,*** *Contagion in financial networks*, Proceedings of Royal Society A466(2) (2010), 2401–2423.
11. ***A. V. Leonidov, E. L. Rumyantsev,*** *Russian interbank networks: main characteristics and stability with respect to contagion* , Proc. Instabilities and Control of Excitable Networks: from macro- to nano-systems, MIPT, 2012, arXiv:1210.3814.
12. ***A. V. Leonidov, E. L. Rumyantsev,*** *Estimate of systemic risks of Russian interbank market based on network topology*, Journal of NEA 3 (19) (2013) 65–80 (in Russian).
13. ***A. V. Leonidov, E. L. Rumyantsev,*** *Default contagion risks in Russian interbank market*, ArXiv:1409.1071.

[6] Here we have used the spectral theorem stating that the spectral radius of A is equal to its norm and the Perron-Frobenius theorem, according to which for non-negative matrix its spectral radius is equal to its maximal eigenvalue.

14. ***A. Lo,*** *Complexity*, *Concentration and Contagion*: *A Comment*, Journal of Monetary Economics **58** (5) (2011), 471–479.

15. ***S. Melnik, A. Hackett, M. Porter, P. Mucha, J. Gleeson,*** *The unreasonable effectiveness of tree-based theory for networks with clustering*, Phys. Rev. **E83** (2011), 036112.

16. ***M. Newman,*** *Random graphs as models of networks*, ArXiv: cond-mat/0202208.

17. ***M. E. J. Newman,*** *Assortative Mixing in Networks*, Phys. Rev. Lett. **89** (2002), 208707.

18. ***M. E. J. Newman,*** *Networks. An Introduction*, Oxford, 2010.

19. ***R. Pastor-Santorias, C. Castellano, P. Van Mieghem, A. Vespignani,*** *Epidemic processes in complex networks*, ArXiv:1408.2701.

20. ***R. Vandermaliere,*** *Network analysis of the Russian interbank system*, Master of science thesis at the Gent University (2012).

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