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Linear independence and transcendence of values of hypergeometric functions

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Abstract: In this paper we obtain a criterion of linear independence of generalized hypergeometric functions over $\mathbb{C}(z)$ and a criterion of linear independence of their values over $\overline{\mathbb{Q}}$.

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We consider generalized hypergeometric functions

$$\psi_j(z) = 1 + \sum_{\nu=1}^{\infty} z^{\nu} \prod_{x=1}^{\nu} \frac{a_j(x)}{b_j(x)}, \quad j = \overline{1, t}, \tag{1}$$

where $a_j(x), b_j(x)$ are polynomials with complex coefficients and

$$m_j = \deg b_j(x) \geqslant \deg a_j(x), \quad m_j \geqslant 1, \quad b_j(x) \neq 0 \quad \text{for} \quad x = 1,\ 2, \dots .$$

The function $\psi_j(z)$ is a solution of a linear differential equation

$$b_j(\delta) y_j = a_j(\delta) z y_j + b_j(0), \quad \delta = z \frac{d}{d\,z}\,. \tag{2}$$

Certain sufficient conditions for the algebraic independence over $\mathbb{C}(z)$ for the collection (1) were obtained in [2]. In the present paper we consider the case when all the functions $\psi_j(z)$ are entire functions. In this case we prove that the conditions from [2] are necessary conditions. Thus we obtain a criterion of linear independence of functions (1) together with their derivatives.

F. Beukers [1] proved a theorem which gives a linear independence criterion for the values of the considered functions over the field of algebraic numbers, under certain additional conditions.

THEOREM 1. *Suppose that in* (1) *one has*

$$m_j = \deg b_j(x) > \deg a_j(x), \quad j = \overline{1, t}. \tag{3}$$

Then the functions in the collection

$$1,\ \psi_j^{(s)}(z), \quad j = \overline{1, t}, \quad s = \overline{0,\ m_j - 1}, \tag{4}$$

are linearly independent over $\mathbb{C}(z)$, *if and only if the following conditions hold*:

1) $a_j(x)b_j(x) \neq 0$ *for* $x = 1,\ 2, \dots;$
2) *For any two polynomials*

$$a_k(x)b_l(x) = c(x + \lambda_1)\dots(x + \lambda_N), \quad a_l(x)b_k(x), \quad 1 \leqslant k < l \leqslant t,$$

and any collection of rational integers $c_1, \dots, c_N$ *one has*

$$c(x + \lambda_1 + c_1)\dots(x + \lambda_N + c_N) \not\equiv a_l(x)b_k(x). \tag{5}$$

3) *For any* j, $1 \leqslant j \leqslant t$, *and any* $c \in \mathbb{Z}$, $c \geqslant 0$, *the two polynomials* $a_j(x)$, $b_j(x + c)$ *are relatively prime.*

Here we should notice that in the case when not all functions in the collection (1) are entire, that is when in (3) we should write $\geqslant$, the conditions 1)–3) are sufficient (see [2]) but not necessary.

Example. It is a simple corollary of Lemma 1 from § 2 of the book [3] that the functions

$$1, \quad \psi_1(z) = 1 + \sum_{\nu=1}^{\infty} z^\nu \prod_{x=1}^{\nu} \frac{x}{x+1}, \quad \psi_2(z) = 1 + \sum_{\nu=1}^{\infty} z^\nu \prod_{x=1}^{\nu} \frac{2x-1}{2x+1}$$

are linearly independent over $\mathbb{C}(z)$, though they do not satisfy the condition 2) of Theorem 1.

THEOREM 2. *Suppose that in* (1) *the first coefficients of all the polynomials* $a_j(x), b_j(x)$ *are equal to* 1. *Suppose that the roots of these polynomials are rational numbers. Let*

$$n = \deg b_j(x) - \deg a_j(x) \geqslant 1, \quad j = \overline{1, t}\,.$$

Suppose that $\omega_1, \dots, \omega_r$ *are different nonzero algebraic numbers.*
Then the elements of the collection

$$1, \quad \psi_j^{(s)}(\omega_k), \quad j = \overline{1, t}, \quad s = \overline{0,\ m_j - 1}, \quad k = \overline{1, r}, \tag{6}$$

are linearly independent over the field of algebraic numbers if and only if the polynomials $a_j(x),\ b_j(x),\ j = \overline{1, t},$ *satisfy the conditions of Theorem* 1.

COROLLARY. *Under the conditions of Theorems* 1 *and* 2 *each number*

$$\psi_j^{(s)}(\omega_k), \quad j = \overline{1, t}, \quad s = \overline{0,\ m_j - 1}, \quad k = \overline{1, r},$$

is transcendental.

PROOF OF THEOREM 1. Sufficiency was proved in [2].

Let us prove the necessity.

It is obvious that the first condition is necessary.

Let us prove that the condition 2) is necessary. First of all we should notice that the functions

$$f_{\lambda,\,\mu}(z) = \sum_{\nu=0}^{\infty} \frac{(\lambda+1)\dots(\lambda+\nu)}{(\mu+1)\dots(\mu+\nu)}\, c_\nu z^\nu, \quad c_\nu \in \mathbb{C},$$

satisfy the identity

$$(\delta+\lambda+1) f_{\lambda,\,\mu}(z) = (\lambda+1) f_{\lambda+1,\,\mu}(z), (\delta+\mu) f_{\lambda,\,\mu}(z) = \mu\, f_{\lambda,\,\mu-1}(z), \quad \delta = z\frac{d}{dz}\,.$$

Suppose that 2) is not true. Then for a certain choice of k and l, in (5) we have $\equiv$, instead of $\not\equiv$. It follows that one can find nonzero polynomials $P_k(x)$ and $P_l(x)$, such that

$$P_k(\delta)\psi_k(z) = P_l(\delta)\psi_l(z). \tag{7}$$

Applying (2) we can represent the left-hand side and the right-hand side of (7) as linear combinations of the functions

$$1,\ \psi_j^{(s)}(z), \quad s = \overline{0,\ m_j - 1}, \quad j = k,\ l, \tag{8}$$

respectively (with coefficients from $\mathbb{C}[z]$).

It follows from (3) that both sides of (7) are entire functions which are not polynomials. So, both sides of (7) are not rational functions. Hence (8) consists of functions linearly dependent over $\mathbb{C}(z)$. The necessity of the condition 2) is proved.

Now let us prove that the condition 3) is necessary.

Suppose that this condition is not valid. Then for some j, $1 \leqslant j \leqslant t$, one has

$$a_j(x) = (x + \lambda + c)\alpha_j(x), \quad b_j(x) = (x + \lambda)\beta_j(x),$$
$$\alpha_j(x), \beta_j(x) \in \mathbb{C}[x], \quad c \in \mathbb{Z}, \quad c \geqslant 0.$$

Let $\varphi_j(z)$ be a function of the kind defined by (1), with polynomials $\alpha_j(x)$ and $\beta_j(x)$ instead of $a_j(x)$ and $b_j(x)$. This function satisfies a differential equation of the order $m_j - 1$. Moreover the functions $\varphi_j(z)$, $\psi_j(z)$ satisfy the following relation:

$$\psi_j(z) = \frac{(\delta + \lambda + 1) \ldots (\delta + \lambda + c)\varphi_j(z)}{(\lambda + 1) \ldots (\lambda + c)} \,.$$

By (1) and (2) we see that every function in the collection of $m_j + 1$ functions

$$1,\ \psi_j^{(s)}(z), \quad s = \overline{0,\ m_j - 1}, \tag{9}$$

can be expressed as a linear combination of m_j functions

$$1,\ \varphi_j^{(s)}(z), \quad s = \overline{0,\ m_j - 2}$$

with coefficients from $\mathbb{C}(z)$. Thus, the functions (9) are linearly dependent over $\mathbb{C}(z)$, this implies the necessity of the condition 3) and proves Theorem 1. □

Proof of Theorem 2. First of all let us notice that it follows from the conditions of Theorem 2 that the assumptions 1)–3) for $a_j(x)$, $b_j(x)$, $j = \overline{1, t}$, imply that the same assumptions are valid for the polynomials

$$\omega_k a_j(x),\ b_j(x), \quad j=\overline{1,t}, \quad k=\overline{1,r}.$$

So, by Theorem 1, the conditions 1)–3) for $a_j(x)$, $b_j(x)$ are equivalent to linear independence over $\mathbb{C}(z)$ of the functions

$$1, \quad \psi_j^{(s)}(\omega_k z), \quad j=\overline{1,t}, \quad s=\overline{0,\ m_j-1}, \quad k=\overline{1,r}\,. \tag{10}$$

Let us prove the necessity of the conditions 1)–3) in Theorem 2.

Suppose that not all of them are valid. Then, as it was shown above, the functions (10) are linearly dependent over $\mathbb{C}(z)$. By Lemma 2 from § 2, Ch. 3 of the book [3], linear dependence over $\mathbb{C}(z)$ of the functions (10) leads to linear dependence over the algebraic numbers of the collection (6). Which implies the necessity of the conditions 1)–3).

Now let us show that these conditions are sufficient for the numbers (6) to be linearly independent. By Lemma 3 from § 2, Ch. 5 of the book [3] and by the conditions of Theorem 2, all the functions

$$1, \quad z^s\psi_j^{(s)}(\omega_k z^n), \quad j=\overline{1,t}, \quad s=\overline{0,\ m_j-1}, \quad k=\overline{1,r}, \tag{11}$$

are E-functions. The following Lemma enables us to show that these functions are linearly independent over $\mathbb{C}(z)$.

Lemma. *Let n be a positive integer. Suppose that the functions*

$$f_j(z)=\sum_{\nu=0}^{\infty} a_{j,\nu} z^{n\nu}, \quad j=\overline{1,m}, \tag{12}$$

are linearly independent over $\mathbb{C}(z^n)$. Then they are linearly independent over $\mathbb{C}(z)$.

Proof. Let us assume the contrary. Then the following identity holds:

$$\sum_{j=1}^{m} P_j(z) f_j(z) \equiv 0, \quad P_j(z)\in\mathbb{C}[z], \tag{13}$$

where $P_j(0) \neq 0$ for some j. It follows from (12) and (13) that

$$\sum_{j=1}^{m} P_j(ze^{2\pi ik/n}) f_j(z) \equiv 0, \quad P_j(z) \in \mathbb{C}[z], \quad k = \overline{0,\, n-1}\,. \tag{14}$$

Summing up all the equalities (14) we see that the functions (12) are linearly dependent over $\mathbb{C}(z^n)$, thus coming to a contradiction. □

Let us return to the proof of Theorem 2 and suppose that the conditions 1)–3) are valid. Then by Theorem 1 and by the Lemma the collection (11) of E-functions is linearly independent over $\mathbb{C}(z)$. By Beukers' theorem [1] the values of these functions at $z = 1$ are linearly independent over the algebraic numbers. This concludes the proof or Theorem 2. □

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