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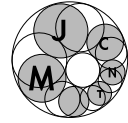
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Weighted fourth moments of Hecke zeta functions with groessencharacters

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Abstract: By using recently obtained bounds for certain sums of Kloosterman sums we prove new bounds for the sum $\sum_{-D \leq d \leq D} \int_{-D}^D |\zeta(1/2 + it, \lambda^d)|^4 \sum_{0 < |\mu|^2 \leq M} A(\mu) \lambda^d((\mu)) |\mu|^{-2it} dt$, where λ^d is the groessencharacter satisfying $\lambda^d((\alpha)) = \lambda^d(\alpha \mathbb{Z}[i]) = (\alpha/|\alpha|)^{4d}$, for $0 \neq \alpha \in \mathbb{Z}[i]$, and $\zeta(s, \lambda^d)$ is the Hecke zeta function that satisfies $\zeta(s, \lambda^d) = (1/4) \sum_{0 \neq \alpha \in \mathbb{Z}[i]} \lambda^d((\alpha)) |\alpha|^{-2s}$ for $\text{Re}(s) > 1$, while the numbers $D, M \in (0, \infty)$ and function $A : \mathbb{Z}[i] - \{0\} \rightarrow \mathbb{C}$ are arbitrary (though it is only in respect of cases in which M is relatively small, compared to D , that our results are new and interesting). One of our new bounds may have an application in enabling a certain improvement of a result of P. A. Lewis on the distribution of Gaussian primes.

Keywords: mean value, zeta function, Gaussian number field, groessencharakter, Hecke character, Gaussian primes, prime ideals, approximate functional equation, Kloosterman sum, sum formula, spectral theory, Hecke congruence group, Selberg eigenvalue conjecture, automorphic function, Fourier coefficient, large sieve

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1. Introduction

In [20, 21] Hecke discovered a class of zeta functions with applications in the study of the multidimensional distribution of prime ideals in algebraic number fields. Associated with the Gaussian number field $\mathbb{Q}(i)$, with ring of integers $\mathfrak{D} = \mathbb{Z}[i]$,

there is a family $(\zeta(s, \lambda^d))_{d \in \mathbb{Z}}$ of functions in Hecke's class that satisfy

$$\zeta(s, \lambda^d) = \sum_{\mathfrak{a} \in I} \lambda^d(\mathfrak{a}) (N(\mathfrak{a}))^{-s} = \frac{1}{4} \sum_{0 \neq \alpha \in \mathfrak{D}} \Lambda^d(\alpha) |\alpha|^{-2s} \quad (1.1)$$

($s \in \mathbb{C}$, $\operatorname{Re}(s) > 1$ and $d \in \mathbb{Z}$), where Λ^d is (for each $d \in \mathbb{Z}$) the endomorphism of the group $(\mathbb{Q}(i))^*$ given by

$$\Lambda^d(\alpha) = \left(\frac{\alpha}{|\alpha|} \right)^{4d} \quad (0 \neq \alpha \in \mathbb{Q}(i)), \quad (1.2)$$

while the elements of the set I are the non-zero ideals in \mathfrak{D} , the «norm» N satisfies $N((\alpha)) = |\mathfrak{D}/(\alpha)| = |\alpha|^2$ (when $(\alpha) = \alpha\mathfrak{D} = \{\alpha\beta : \beta \in \mathfrak{D}\}$), and each «groschencharacter» $\lambda^d : I \rightarrow \mathbb{C}^*$ is given by:

$$\lambda^d((\alpha)) = \Lambda^d(\alpha) \quad (0 \neq \alpha \in \mathfrak{D}).$$

For each $d \in \mathbb{Z}$ the function $\zeta(s, \lambda^d)$ has a meromorphic continuation to all points $s \in \mathbb{C}$, with the only pole being that of the Dedekind zeta function $\zeta(s, \lambda^0)$ at $s = 1$.

Our subject in this paper is the mean value

$$E(D; M, A) = \sum_{-D \leq d \leq D} \int_{-D}^D \left| \zeta(1/2 + it, \lambda^d) \right|^4 \left| P_M(A; it, \lambda^d) \right|^2 dt, \quad (1.3)$$

where it is assumed that A is a mapping from $\mathfrak{D} - \{0\}$ to \mathbb{C} , that $0 < D, M < \infty$, and that

$$P_M(A; s, \lambda^d) = \sum_{0 < |\mu|^2 \leq M} A(\mu) \Lambda^d(\mu) |\mu|^{-2s} \quad \text{for } d \in \mathbb{Z} \text{ and } s \in \mathbb{C} \quad (1.4)$$

(note that here, and in all that follows, variables of summation designated by Greek letters, such as the « μ » above, are \mathfrak{D} -valued variables, whereas those designated by Latin letters are \mathbb{Z} -valued variables). The mean value $E(D; M, A)$ is a «Gaussian integer analogue» of the mean value

$$S(T, N) = \int_0^T \left| \zeta(1/2 + it) \right|^4 \left| \sum_{0 < n \leq N} a_n n^{-it} \right|^2 dt$$

which has been studied by Iwaniec [27], Deshouillers and Iwaniec [9] and Watt [48]. In [8] and [9] a connection was established between $S(T, N)$ and the spectral theory of the non-Euclidean Laplace—Beltrami operator Δ_2 acting on certain spaces of functions defined on 2-dimensional hyperbolic space \mathbb{H}_2 and automorphic with respect to the action of some Hecke congruence group $\Gamma \leq SL(2, \mathbb{Z})$ on \mathbb{H}_2 ; in particular, it was observed by Deshouillers and Iwaniec in [9] that, if one could assume Selberg’s conjecture that, except for the eigenvalue 0 associated with the space of constant functions, the relevant eigenvalues of $-\Delta_2$ lie in the interval $[1/4, \infty)$, then their method would yield the estimate

$$S(T, N) \ll_{\varepsilon} \left(1 + T^{-1/2}N^2\right) T^{1+\varepsilon} \sum_{0 < n \leq N} |a_n|^2, \quad (1.5)$$

for $\varepsilon > 0$, $N \geq 1$, $T \geq 1$, and any complex sequence $(a_n)_{n \in \mathbb{N}}$. Selberg, in [45], had already shown that the eigenvalues in question do all lie in the interval $[3/16, \infty)$, and the lower bound $3/16$ in this result has since been improved to $975/4096 = (1/4) - (7/64)^2$ by Kim and Sarnak [32]. The application of this spectral theory to the mean value $S(T, N)$ has been significantly refined in the penetrating works [39, 40] of Motohashi: see the paragraph containing Equation (1.39) for some discussion of this matter.

By a non-trivial application of Selberg’s lower bound for the above mentioned eigenvalues, Deshouillers and Iwaniec obtained, in [9, Theorem 1], the unconditional estimate

$$S(T, N) \ll_{\varepsilon} \left(1 + T^{-1/2}N^2 + T^{-1/4}N^{5/4}\right) T^{1+\varepsilon} \sum_{0 < n \leq N} |a_n|^2, \quad (1.6)$$

for $\varepsilon > 0$, $N \geq 1$, $T \geq 1$, and any complex sequence $(a_n)_{n \in \mathbb{N}}$. If Kim and Sarnak’s improvement of Selberg’s lower bound for the eigenvalues had been available at the time, then Deshouillers and Iwaniec would have been able to replace the term $T^{-1/4}N^{5/4}$ (in brackets, in (1.6)) by $T^{\alpha-1/2}N^{2-3\alpha}$, where $\alpha = 7/64$.

By refining one aspect of the method used by Deshouillers and Iwaniec in their proof of (1.6), we obtained, in [48, Theorem 1], the further unconditional estimate

$$S(T, N) \ll_{\varepsilon} \left(1 + T^{-1/2}N^2\right) T^{1+\varepsilon} N \max_{0 < n \leq N} |a_n|^2, \quad (1.7)$$

for $\varepsilon > 0$, $N \geq 1$, $T \geq 1$, and any complex sequence $(a_n)_{n \in \mathbb{N}}$. For certain applications to the theory of the distribution of prime numbers (see, for example, [1], or [18, Chapters 7 and 9]) the estimate in (1.7) is more effective than that in (1.6), and is (moreover) just as effective as the conditional estimate in (1.5) would be (were an unconditional proof of (1.5) to be discovered). We shall mention below some refinements of (1.6) and (1.7) that have been obtained by other authors.

Our principal new result in this paper (Theorem 1, below) is an upper bound for $E(D; M, A)$ analogous to the bound (1.7) for $S(T, N)$. Our proof of this new result depends on estimates for a certain sum of Kloosterman sums; these estimates are supplied by Lemma 14 (below), which is a reformulation of a result from our paper [49]. Note that our work in [49] depends on the spectral large sieve inequality that we proved in [50]. In both [50] and [49], and in the present paper, we employ methods analogous to those pioneered by Iwaniec, in [27], and by Deshouillers and Iwaniec, in [8] and [9]. In particular, with the aid of a slight extension of Lokvenec—Guleska’s sum formula [38, Theorem 12.3.2] (which itself is a generalisation of the sum formula [6, Theorem 13.1] of Bruggeman and Motohashi), we establish, in [49], a connection between the sum of Kloosterman sums occurring in the equation (7.7) (below) and the spectral theory of the non-Euclidean Laplace—Beltrami operator

$$\Delta_3 = r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) - r \frac{\partial}{\partial r} \quad (1.8)$$

acting on certain spaces

$$\mathfrak{D} = \mathfrak{D}_\Gamma = \left\{ f \in L^2(\Gamma \backslash \mathbb{H}_3) \cap C^2(\mathbb{H}_3) : \Delta_3 f \in L^2(\Gamma \backslash \mathbb{H}_3) \right\},$$

the members of which are complex valued functions that are defined on $\mathbb{H}_3 = \{(x + iy, r) : x, y \in \mathbb{R}, r > 0\}$ (the upper half-space model for 3-dimensional hyperbolic space) and that are automorphic with respect to the action on \mathbb{H}_3 of some Hecke congruence group $\Gamma = \Gamma_0(\omega) \leq SL(2, \mathfrak{D})$, where ω is some non-zero Gaussian integer and

$$\Gamma_0(\omega) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathfrak{D}) : \gamma \in \omega \mathfrak{D} \right\} \quad (1.9)$$

(see [13, Chapters 1–4] for definitions of the space $L^2(\Gamma \backslash \mathbb{H}_3)$ and other terminology used here).

A significant feature of the spectral theory just mentioned is that, even if one allows for repetitions (consistent with the relevant multiplicities), the eigenvalues of the operator $-\Delta_3 : \mathfrak{D}_\Gamma \rightarrow L^2(\Gamma \backslash \mathbb{H}_3)$ are the terms of an unbounded monotonically increasing sequence $(\lambda_n(\Gamma))_{n \in \mathbb{N}}$ such that $\lambda_1(\Gamma) > \lambda_0(\Gamma) = 0$ (see, for example, [13, Theorem 4.1.8 and Chapter 8.9]), and so the operator $-\Delta_3$ with domain \mathfrak{D}_Γ has a smallest positive eigenvalue, which is $\lambda_1(\Gamma)$. Preparatory to stating our new results, we define

$$\Theta(\omega) = \sqrt{\max\{0, 1 - \lambda_1(\Gamma)\}} \in [0, 1) \quad (0 \neq \omega \in \mathfrak{D} \text{ and } \Gamma = \Gamma_0(\omega) \leq SL(2, \mathfrak{D})) \quad (1.10)$$

and

$$\vartheta = \sup_{0 \neq \omega \in \mathfrak{D}} \Theta(\omega). \quad (1.11)$$

Note that it is trivially the case that

$$\lambda_1(\Gamma_0(\omega)) \geq 1 - (\Theta(\omega))^2 \geq 1 - \vartheta^2 \quad (0 \neq \omega \in \mathfrak{D}). \quad (1.12)$$

Selberg's conjecture concerning the eigenvalues of $-\Delta_3$ for Hecke congruence subgroups of $SL(2, \mathfrak{D})$ is that for all non-zero $\omega \in \mathfrak{D}$ one has $\lambda_1(\Gamma_0(\omega)) \geq 1$; an equivalent conjecture is that the constant ϑ is zero. Work of Kim and Shahidi, [33] and [34], has shown that $\Theta(\omega) < 2/9$ for $0 \neq \omega \in \mathfrak{D}$, so that one has

$$0 \leq \vartheta \leq 2/9 \quad (1.13)$$

and, by (1.12), $\lambda_1(\Gamma_0(\omega)) > 77/81$ for $0 \neq \omega \in \mathfrak{D}$.

We now state our new bounds for the mean value $E(D; M, A)$ that is defined in (1.3).

THEOREM 1. *Let ϑ be the real constant defined in (1.10) and (1.11). Let $\varepsilon > 0$. Then, for $D \geq 1$, $M \geq 1$ and all functions $A : \mathfrak{D} - \{0\} \rightarrow \mathbb{C}$, one has both*

$$E(D; M, A) \ll_\varepsilon \left(D^{2+\varepsilon} + (1 + DM^{-3/2})^\vartheta D^{1+\varepsilon} M^2 \right) \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \quad (1.14)$$

and

$$E(D; M, A) \ll_{\varepsilon} D^{2+\varepsilon} \left(\sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \right) + \\ + \left(1 + DM^{-2} \right)^{\vartheta} D^{1+\varepsilon} M^3 \max \left\{ |A(\mu)|^2 : \mu \in \mathfrak{D} \text{ and } 0 < |\mu|^2 \leq M \right\}. \quad (1.15)$$

In our proof of Theorem 1 we follow quite standard practice in utilising an approximate functional equation for Hecke's zeta-functions $\zeta(s, \lambda^d)$ ($d \in \mathbb{Z}$). We remark that it might, perhaps, have been both interesting and profitable to have adopted a more novel approach, as Sarnak does in his proof (without the use of any approximate functional equation) of the sharp fourth power moment estimate

$$\sum_{-D \leq d \leq D} \int_{-D}^D \left| \zeta \left(1/2 + it, \lambda^d \right) \right|^4 dt \ll D^2 \log^4 D \quad (D \geq 2), \quad (1.16)$$

which is [44, Theorem 1], and as Motohashi does in [39] (where he obtains an explicit formula for fourth moments of the Riemann zeta function); we however decided (at an early stage of this work) that we should simply concentrate on the approach in which we had the greatest confidence. We are similarly unadventurous in our use of the Poisson summation formula in proving Theorem 2 (below): we might instead have attempted, there, to emulate the approach taken by Blomer, Harcos and Michel in [4, Section 4.1], in which Jutila's variant of the circle method, from [29], is used to detect a condition of summation of the form $m_1 n_1 - m_2 n_2 = h$.

A suitable approximate functional equation for $\zeta(s, \lambda^d)$ is proved in Section 3 of this paper: the proof we give is an implementation of a general method developed by Ivić, in [25]. Although this approximate functional equation (which is Lemma 6, below) is very nearly contained in a more general theorem of Harcos [15, Theorem 2.5], it does have the merit of implying a slightly sharper bound for the relevant error term.

By means of the approximate functional equation (Lemma 6), we show in Section 4 that Theorem 1 is a corollary of the asymptotic estimates contained in the results (1.30)–(1.32) of Theorem 2, below. Before coming to the statement of Theorem 2, we first define some of the notation used there (and subsequently).

For $p \in [1, \infty]$, and for any function $b : X \rightarrow \mathbb{C}$ with domain $X \neq \emptyset$, we define

$$\|b\|_p = \begin{cases} \left(\sum_{x \in X} |b(x)|^p \right)^{1/p} & \text{if } 1 \leq p < \infty; \\ \sup \{|b(x)| : x \in X\} & \text{if } p = \infty. \end{cases} \tag{1.17}$$

For $\alpha, \beta \in \mathfrak{D}$, we shall (unless $\alpha = \beta = 0$) take (α, β) to denote a highest common factor of α and β ; at the same time $[\alpha, \beta]$ will denote a least common multiple of α and β (so that if $\gamma = [\alpha, \beta]$ and $\delta = \alpha\beta/(\alpha, \beta)$ then γ and δ are associates in the ring \mathfrak{D}).

Throughout this paper the notation d_+z denotes the standard Lebesgue measure on the set \mathbb{C} (hence $d_+z = dx dy$, where x and y are the real and imaginary parts of the complex variable z).

THEOREM 2. *Let ϑ be the real constant defined in (1.10) and (1.11). Let $0 < \varepsilon \leq 1/6$ and $0 < \eta \leq (\log 2)/3$. Let $K_1, K_2, M_1 \geq K_0 = 1$, and let $T > 0$ satisfy*

$$T \gg \max \{K_1^2, K_2^2, M_1^2\}. \tag{1.18}$$

Let $w_0, w_1, w_2 : (0, \infty) \rightarrow \mathbb{C}$ be infinitely differentiable functions satisfying, for $i \in \{0, 1, 2\}, j = 0, 1, 2, \dots$ and $x > 0$,

$$w_i^{(j)}(x) = \begin{cases} O_j \left((\eta x)^{-j} \right) & \text{if } e^{-\eta} K_i \leq x \leq e^{\eta} K_i, \\ 0 & \text{otherwise,} \end{cases} \tag{1.19}$$

and let $a : \mathfrak{D} - \{0\} \rightarrow \mathbb{C}$ be a function such that

$$|a(\mu)| > 0 \quad \text{only if } e^{-\eta} M_1 \leq |\mu|^2 \leq e^{\eta} M_1. \tag{1.20}$$

Define C to be the mapping, from $\mathfrak{D} - \{0\}$ to \mathbb{C} , given by

$$C(\xi) = \sum_{\substack{\kappa_1 \\ \kappa_1 \kappa_2 \mu = \xi}} \sum_{\kappa_2} \sum_{\mu} w_1 \left(|\kappa_1|^2 \right) w_2 \left(|\kappa_2|^2 \right) a(\mu) \quad (0 \neq \xi \in \mathfrak{D}), \tag{1.21}$$

and put

$$\mathcal{D}_0 = \left(\pi \int_0^\infty w_0(x) dx \right) T \|C\|_2^2, \quad (1.22)$$

$$C^*(\beta; z) = \overline{w_1(|\beta z|^2)} \sum_{\substack{\mu \\ \mu\kappa=\beta}} \sum_{\substack{\kappa \\ \mu\kappa=\beta}} a(\mu) w_2(|\kappa|^2) \quad (0 \neq \beta \in \mathfrak{D}, \quad 0 \neq z \in \mathbb{C}), \quad (1.23)$$

$$N = T^{\varepsilon-1} K_1 K_2 M_1, \quad (1.24)$$

$$\mathcal{D}_1^* = \left(-4\pi \int_0^\infty w_0(x) dx \right) T \int_{\mathbb{C}} \sum_{\substack{\beta_1 \neq 0 \\ |\beta_1, \beta_2|^2 < e^{-4\eta} T^{-\varepsilon} N}} \sum_{\substack{\beta_2 \neq 0 \\ |\beta_1, \beta_2|^2 < e^{-4\eta} T^{-\varepsilon} N}} C^*(\beta_1; z) \overline{C^*(\beta_2; z)} |(\beta_1, \beta_2)|^2 d_+ z, \quad (1.25)$$

$$p(\alpha) = \sum_{\nu \neq 0} |\nu|^{-4} \exp(2\pi i \operatorname{Re}(\alpha\nu)) \quad (\alpha \in \mathbb{C}), \quad (1.26)$$

$$I_{w_0}(\gamma) = \frac{|\gamma|^4}{\pi^4} \int_{\mathbb{C}} \left(\left(x \frac{d^2}{dx^2} + \frac{d}{dx} \right)^2 w_0(x) \Big|_{x=|s|^2} \right) p\left(\frac{s}{\gamma}\right) d_+ s \quad (0 \neq \gamma \in \mathbb{C}), \quad (1.27)$$

and

$$\mathcal{D}_2^* = 4T \int_{\mathbb{C}} \sum_{\substack{\beta_1 \neq 0 \\ e^{-4\eta} T^{-\varepsilon} N \leq |(\beta_1, \beta_2)|^2 < e^{\eta} N}} \sum_{\substack{\beta_2 \neq 0 \\ e^{-4\eta} T^{-\varepsilon} N \leq |(\beta_1, \beta_2)|^2 < e^{\eta} N}} C^*(\beta_1; z) \overline{C^*(\beta_2; z)} |(\beta_1, \beta_2)|^2 I_{w_0}\left(\frac{[\beta_1, \beta_2]z}{T^{1/2}}\right) d_+ z. \quad (1.28)$$

For $d \in \mathbb{Z}$ and $t \in \mathbb{R}$, put

$$c(d, it) = \left(\sum_{\kappa_1 \neq 0} \frac{w_1(|\kappa_1|^2) \Lambda^d(\kappa_1)}{|\kappa_1|^{2it}} \right) \left(\sum_{\kappa_2 \neq 0} \frac{w_2(|\kappa_2|^2) \Lambda^d(\kappa_2)}{|\kappa_2|^{2it}} \right) \left(\sum_{\mu \neq 0} \frac{a(\mu) \Lambda^d(\mu)}{|\mu|^{2it}} \right). \quad (1.29)$$

Then one has

$$\sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} |c(d, it)|^2 w_0\left(\frac{|2d + it|^2}{\pi^2 T}\right) dt = 2\pi \mathcal{D}_0 + (\pi/2) (\mathcal{D}_1^* + \mathcal{D}_2^*) + \mathcal{E}, \quad (1.30)$$

where \mathcal{E} is some complex number satisfying both

$$\frac{\mathcal{E}}{K_1 K_2} \ll_{\eta, \varepsilon} \left(\frac{M_1^2}{T^{1/2}} + \left(\frac{M_1^{2-(3/2)\vartheta}}{T^{(1-\vartheta)/2}} \right) \left(\frac{K_2}{T^{1/2}} \right)^\vartheta \right) T^{1+9\varepsilon} \|a\|_2^2 \tag{1.31}$$

and

$$\begin{aligned} \frac{\mathcal{E}}{K_1 K_2} &\ll_{\eta, \varepsilon} \left(\frac{M_1^2}{T^{1/2}} + \left(\left(\frac{K_1}{T^{1/2}} \right)^{\vartheta/2} \left(\frac{K_2}{T^{1/2}} \right)^{1-(\vartheta/2)} + \left(\frac{K_2}{T^{1/2} M_1^{1/2}} \right)^\vartheta \right) \left(\frac{M_1^2}{T^{1/2}} \right)^{1-\vartheta} \right) \\ &\times T^{1+11\varepsilon} M_1 \|a\|_\infty^2. \end{aligned} \tag{1.32}$$

The implicit constants in (1.31) and (1.32) are determined by those in (1.19) and (1.18), and by ε and η .

Remark 1.1. For future reference, we observe here that the hypotheses (1.19), (1.20) and the definitions (1.22), (1.21) and (1.17) imply the upper bound

$$\mathcal{D}_0 \ll T \sum_{|\kappa_1|^2 \asymp K_1} \sum_{|\kappa_2|^2 \asymp K_2} \sum_{|\mu_1|^2 \asymp M_1} \sum_{|\kappa_3|^2 \asymp K_1} \sum_{|\kappa_4|^2 \asymp K_2} \sum_{\substack{|\mu_2|^2 \asymp M_1 \\ \kappa_3 \kappa_4 \mu_2 = \kappa_1 \kappa_2 \mu_1}} |a(\mu_1) a(\mu_2)|.$$

From this, the inequality $2|a(\mu_1)a(\mu_2)| \leq |a(\mu_1)|^2 + |a(\mu_2)|^2$, and the bound (2.13) noted below, it follows that, when $K_1, K_2, M_1 \geq 1$ and $T > 0$ are such that (1.18) holds, one has:

$$\begin{aligned} |\mathcal{D}_0| &\leq T \sum_{|\kappa|^2 \asymp K_1} \sum_{|\lambda|^2 \asymp K_2} \sum_{|\mu|^2 \asymp M_1} |a(\mu)|^2 \sum_{\substack{\kappa' \\ \kappa' \lambda' \mu' = \kappa \lambda \mu}} \sum_{\lambda'} \sum_{\mu'} O(1) = \\ &= T \sum_{|\kappa|^2 \asymp K_1} \sum_{|\lambda|^2 \asymp K_2} \sum_{|\mu|^2 \asymp M_1} |a(\mu)|^2 O_\varepsilon(|\kappa \lambda \mu|^\varepsilon) \ll_\varepsilon T^{1+\varepsilon} K_1 K_2 \|a\|_2^2 \quad (\varepsilon > 0). \end{aligned} \tag{1.33}$$

Similar upper bounds may be obtained for the terms \mathcal{D}_1^* and \mathcal{D}_2^* occurring, alongside \mathcal{D}_0 , in the result (1.30). Indeed, by (1.23), (1.19) and the Cauchy–Schwarz inequality, it follows that

$$|C^*(\beta; z)|^2 \ll \tau_2(\beta) \sum_{\substack{|\mu|^\beta \\ |\mu|^2 \asymp M_1}} |a(\mu)|^2 \quad \text{for } 0 \neq \beta \in \mathfrak{D}, z \in \mathbb{C}$$

(with $\tau_2(\beta)$ denoting, here and henceforth, the number of Gaussian integer divisors of β) and that it is, moreover, the case that $C^*(\beta; z) = 0$ unless one has both $|\beta|^2 \asymp K_2 M_1$ and $|\beta z|^2 \asymp K_1$. Hence, and by the arithmetic-geometric mean inequality, one has, for some $z \in \mathbb{C}$,

$$\begin{aligned}
 \mathcal{D}_1^* &\ll T \sum_{\beta \neq 0} |C^*(\beta; z)|^2 \sum_{|\beta|^2 \asymp K_2 M_1} |(\beta, \beta')|^2 |\beta'|^{-2} K_1 \ll \\
 &\ll TK_1 \sum_{\beta \neq 0} |C^*(\beta; z)|^2 \tau_2(\beta) \ll \\
 &\ll TK_1 \sum_{|\beta|^2 \asymp K_2 M_1} \tau_2^2(\beta) \sum_{\substack{\mu|\beta \\ |\mu|^2 \asymp M_1}} |a(\mu)|^2 \leq \\
 &\leq \left(TK_1 \sum_{|\kappa|^2 \asymp K_2} \frac{\tau_2^2(\kappa)}{16} \right) \left(\sum_{|\mu|^2 \asymp M_1} \tau_2^2(\mu) |a(\mu)|^2 \right). \tag{1.34}
 \end{aligned}$$

Moreover, given the hypotheses (1.19), (1.20) and the definitions (1.23) and (1.24), it is implicit in (1.28) that

$$\left| \frac{[\beta_1, \beta_2] z}{\sqrt{T}} \right|^2 = \frac{|\beta_1 \beta_2 z|^2}{|(\beta_1, \beta_2)|^2 T} \ll \frac{K_2 M_1 K_1}{e^{-4\eta} N T^{-\varepsilon+1}} = e^{4\eta} \ll 1,$$

and so, as (1.26) and (1.27) trivially imply $I_{w_0}(\gamma) \ll |\gamma|^4$, one may deduce from (1.23) and (1.28) essentially the same upper bound for $|\mathcal{D}_2^*|$ as that found for $|\mathcal{D}_1^*|$ in (1.34). Therefore it follows by the bound (2.13) (below) that, when the hypotheses of Theorem 2 are satisfied, the terms \mathcal{D}_1^* and \mathcal{D}_2^* in (1.30) satisfy

$$\mathcal{D}_j^* \ll_{\varepsilon} T^{1+\varepsilon} K_1 K_2 \|a\|_2^2 \quad \text{for } j = 1, 2 \text{ and all } \varepsilon > 0. \tag{1.35}$$

The details of our proof of Theorem 2 appear in Sections 5, 6 and 7 of this paper.

The contents of Section 5 are several basic lemmas that are needed in Section 6: note that Lemma 9 is effectively the first step in our analysis of the sum occurring on the left hand side of Equation (1.30).

By means of the three (quite specialised) lemmas of Section 6, we transform the task of bounding the term \mathcal{E} in Equation (1.30) into a search for suitable bounds for the sum of Kloosterman sums occurring in the equation (6.47) of Lemma 13;

aside from the use made of Lemma 9 in the proof Lemma 11, it is fair to regard the proofs of Lemma 11, Lemma 12 and Lemma 13 as being exercises in the application of the Poisson Summation formulae that are contained in Lemma 7.

In Section 7 we reformulate (as Lemma 14) the results of [49, Theorem 11], and we use most of the remainder of the section in applying those results so as to obtain (in Lemma 18) the sought for bounds on the sum of Kloosterman sums in (6.47); the supplementary bound (7.12) of Lemma 15 is needed in order to handle certain extreme cases (in which the condition (7.2) of Lemma 14 becomes an obstacle). At the end of Section 7 we complete our proof of Theorem 2, using only Lemma 11, Lemma 12, Lemma 13 and Lemma 18; our proof of Theorem 1 is, thereby, also completed (for we show in Section 4 that Theorem 2 implies Theorem 1).

We conclude this introduction with a brief discussion of one likely application of Theorem 1, followed by a remark in respect of one immediate implication of Theorem 1, and some speculative remarks regarding what might be done to either refine, or generalise, the results of Theorem 1.

Our bound (1.7) for $S(T, N)$ played an essential part in work of Baker, Harman and Pintz on the distribution of rational primes; with its help, they showed, in [1], that there exists an $x_0 \in (0, \infty)$ such that

$$\left| \left\{ p : p \text{ is prime in } \mathbb{Z} \text{ and } x < p \leq x + x^{0.525} \right\} \right| \geq \frac{9x^{0.525}}{100 \log x} \text{ for all } x \geq x_0. \quad (1.36)$$

Similar progress on the distribution of Gaussian primes has been somewhat impeded by the the lack (until now) of a Gaussian integer analogue of (1.7). Nevertheless, in [19], Harman, Kumchev and Lewis have shown that there exist positive real numbers c_1 and r_0 such that, if one has $1 \geq \alpha \geq 0.53$, then

$$|\{ \pi_1 : \pi_1 \text{ is prime in } \mathbb{Z}[i] \text{ and } |\pi_1 - z| \leq |z|^\alpha \}| \geq \frac{c_1 |z|^{2\alpha}}{\log |z|} \text{ for all } z \in \mathbb{C} \text{ such that } |z| \geq r_0. \quad (1.37)$$

Moreover, Lewis has improved on this by showing, in his thesis [37], that (1.37) holds (for some $c_1, r_0 \in (0, \infty)$) whenever $1 \geq \alpha > 0.528$. Since our new estimate (1.15) implies the required Gaussian integer analogue of the bound (1.7), we expect that, by methods entirely analogous to those employed in the proof, in [1], of the result (1.36), it could now be proved that there exist positive real numbers c_1 and r_0 such that (1.37) holds whenever $1 \geq \alpha \geq 0.525$.

Our estimate (1.15) in Theorem 1 certainly does imply that, if $M \leq D^{1/2}$, then one has

$$\sum_{-D \leq d \leq D} \int_{-D}^D \left| \zeta \left(1/2 + it, \lambda^d \right) \right|^4 \left| \sum_{M/2 < |\mu|^2 \leq M} \frac{A(\mu) \Lambda^d(\mu)}{|\mu|^{1+2it}} \right|^2 dt \ll_{\varepsilon} D^{2+\varepsilon} \|A\|_{\infty}^2 \quad (\varepsilon > 0). \tag{1.38}$$

If it could be shown that (1.38) holds whenever $0 \leq M \leq D$, then, by virtue of Corollary 3 (below), it would follow that

$$\sum_{-D \leq d \leq D} \int_{-D}^D \left| \zeta \left(1/2 + it, \lambda^d \right) \right|^6 dt \ll_{\varepsilon} D^{2+\varepsilon} \quad \text{for all } \varepsilon > 0 \text{ and all } D \geq 1.$$

In [22] Hughes and Young have applied the «delta method» of [12] in order to study the «twisted» fourth moment of zeta,

$$I(h, k; w) = \int_{-\infty}^{\infty} \left(\frac{h}{k} \right)^{-it} \zeta(1/2 + \alpha + it) \zeta(1/2 + \beta + it) \zeta(1/2 + \gamma - it) \zeta(1/2 + \delta - it) w(t) dt,$$

in which h and k denote arbitrary positive integers, while α , β , γ and δ denote complex numbers (assumed to be sufficiently small) and w may be any non-negative valued infinitely differentiable real function that has its support contained in some interval $[T/2, 4T]$ and that, for some $T_0 \gg T^{1/2+\varepsilon}$, satisfies $w^{(j)}(t) \ll_j T_0^{-j}$ whenever $t \in \mathbb{R}$ and $j \in \mathbb{N} \cup \{0\}$. Theorem 1.1 of [22] implies, for any fixed choice of ϕ satisfying $0 < \phi < 1/11$, the asymptotic expansion of $S(T, N)$ when T is sufficiently large (in terms of ϕ) and N satisfies $1 \leq N \leq T^{\phi}$. This improves on a result of J. A. Gaggero Jara [14] that was conditional upon having $0 < \phi < 5/589$. In joint work (only very recently completed) Sandro Bettin, Hung Bui, Xiannan Li and Maksym Radziwiłł have made further progress by obtaining a refinement of the upper bound estimate (1.7). They therefore obtain, for any fixed choice of ϕ satisfying $0 < \phi < 1/4$, the asymptotic expansion of $S(T, N)$ when T is sufficiently large (in terms of ϕ) and N and the sequence $(a_n)_{n \in \mathbb{N}}$ are such that one has both $1 \leq N \leq T^{\phi}$ and, when ε is positive, $\sum_{0 < n \leq N} |a_n|^2 \gg_{\varepsilon} N^{1-\varepsilon} \max \{ |a_n|^2 : 0 < n \leq N \}$ (this last condition can certainly be weakened, but, since we currently lack some of the relevant information, we are unable to determine exactly how much weaker the

condition on $(a_n)_{n \in \mathbb{N}}$ can be). An account of this work is being prepared; some closely related work of Bettin, Radziwiłł and Vorrapan Chandee is described in the preprint [3].

In view of the above mentioned progress regarding asymptotic expansions for $S(T, N)$, we conjecture that it should be possible to obtain an asymptotic expansion of the sum $E(D; M, A)$ that would constitute a refinement of the upper bounds, (1.14) and (1.15), of Theorem 1. Another way in which one might seek to improve upon Theorem 1 is suggested by Motohashi’s work [40], in which he obtains the explicit spectral decomposition of the error term in the asymptotic expansion of the mean value

$$M_2(g) = \int_{-\infty}^{\infty} |\zeta(1/2 + it)|^4 \left| \sum_{0 < n \leq N} a_n n^{-1/2-it} \right|^2 g(t) dt, \tag{1.39}$$

provided only that $a_n = 0$ for all non-squarefree $n \in \mathbb{N}$, and that there exists a sufficiently wide horizontal strip $\{z \in \mathbb{C} : |\text{Im}(z)| \leq A\}$ on which $g(z)$ is a holomorphic function that satisfies $g(z) = g(-z) = \overline{g(\bar{z})} \ll (|z| + 1)^{-A}$. It might be possible to obtain an analogous explicit spectral decomposition of the error term in the asymptotic expansion of the mean value

$$\sum_{d \in \mathbb{Z}} \int_{-\infty}^{\infty} \left| \zeta(1/2 + it, \lambda^d) \right|^4 \left| \sum_{M/2 < |\mu|^2 \leq M} \frac{A(\mu)\Lambda^d(\mu)}{|\mu|^{1+2it}} \right|^2 g(|d + it|^2) dt.$$

We remark that in [6, Theorem 14.1] Bruggeman and Motohashi have already obtained an explicit spectral decomposition of the mean value

$$\mathcal{Z}_2(g, \mathbb{Q}(i)) = \int_{-\infty}^{\infty} \left| \zeta(1/2 + it, \lambda^0) \right|^4 g(t) dt,$$

subject only to conditions on g that are weaker than the set of conditions below (1.39).

In [11, Theorem 2.2] it was shown by Duke that, if F is a number field of degree n , if \mathfrak{q} is an ideal in \mathfrak{O}_F (the ring of integers of F), and if χ and $\{\lambda_1, \dots, \lambda_{n-1}\}$ are (respectively) a narrow class character mod \mathfrak{q} and a basis for

the torsion free Hecke characters mod \mathfrak{q} , then, for some $B \in (0, \infty)$, one has:

$$\sum_{\mathfrak{d} \in (\mathbb{Z} \cap [-D, D])^{n-1}} \int_{-D}^D \left| \zeta_F \left(1/2 + it, \chi \lambda_{\mathfrak{d}}^{\mathfrak{d}} \right) \right|^4 dt \ll_{F, \mathfrak{q}} D^n \log^B D \quad (D \geq 2), \quad (1.40)$$

where $\chi \lambda_{\mathfrak{d}}^{\mathfrak{d}}$ is the groessencharacter $\chi \lambda_1^{d_1} \cdots \lambda_{n-1}^{d_{n-1}}$, while $\zeta_F(s, \chi \lambda_{\mathfrak{d}}^{\mathfrak{d}})$ is the Hecke zeta function such that, for all $s_0 \in \mathbb{C}$ such that $\text{Re}(s_0) > 1$, one has $\zeta_F(s_0, \chi \lambda_{\mathfrak{d}}^{\mathfrak{d}}) = \sum_{\mathfrak{a} \in I(F)} (\chi \lambda_{\mathfrak{d}}^{\mathfrak{d}})(\mathfrak{a}) (N(\mathfrak{a}))^{-s_0}$, with $I(F)$ being the set of non-zero ideals in \mathfrak{D}_F , and with $N(\mathfrak{a})$ being the norm of \mathfrak{a} . This result of Duke comes close to being a generalisation of Sarnak’s bound in (1.16), falling short of that only by some power of $\log D$.

In contrast with the considerable generality of the bounds in (1.40), our focus in the present paper is exclusively on Hecke zeta functions that are associated with $\mathbb{Q}(i)$. Similarly, in the papers [49] and [50] (upon which the present paper depends), we use only the case $F = \mathbb{Q}(i)$ of the sum formulae [38, Theorem 11.3.3 and Theorem 12.3.2]. In [38] itself it is permitted that F be any given imaginary quadratic field (while the relevant discrete group Γ may be any Hecke congruence subgroup of $SL(2, \mathfrak{D}_F)$). We expect that, by means of the sum formulae in [38] (or some slight extension thereof), it would be possible to generalise our work in [49, 50] and the present paper so as to obtain new and useful upper bounds for sums of the form

$$\sum_{-D \leq d \leq D} \int_{-D}^D \left| \zeta_F \left(1/2 + it, \chi \lambda_1^d \right) \right|^4 \left| \sum_{\substack{\mathfrak{a} \in I(F) \\ N(\mathfrak{a}) \leq M}} \frac{A(\mathfrak{a}) (\chi \lambda_1^d)(\mathfrak{a})}{(N(\mathfrak{a}))^{1/2+it}} \right|^2 dt,$$

where F might be any imaginary quadratic field and A might be any complex valued function defined on the set $I(F)$ of ideals in \mathfrak{D}_F , while $\chi, \lambda_1, \zeta_F(s, \chi \lambda_1^d)$ and $N(\mathfrak{a})$ would all have the same meaning as in the above paragraph on Duke’s result (1.40) (given that F is now assumed to be a number field of degree $n = 2$); we conjecture that, by taking such an approach towards generalising Theorem 1, one might obtain mean value estimates for Hecke zeta functions capable of being used to obtain worthwhile improvements of certain results concerning the distribution of prime ideals in imaginary quadratic fields ([19, Theorem 2], in particular).

Notation and terminology that is fairly standard

\mathbb{N}	— the set $\{n \in \mathbb{Z} : n \geq 1\}$;
$ \mathcal{A} $	— the cardinality of the set \mathcal{A} (so that $ \{x \in \mathbb{R} : x^2 = 1\} = 2$, for example);
$\max \mathcal{X}$	— the greatest element of the set $\mathcal{X} \subset \mathbb{R}$ (where this exists);
$\min \mathcal{X}$	— the least element of $\mathcal{X} \subset \mathbb{R}$ (where this exists);
$\max_{A(x)} f(x)$	— equal to $\max\{f(x) : A(x) \text{ is true}\}$, when $A(x)$ is some statement about x ;
$\min_{A(x)} f(x)$	— equal to $\min\{f(x) : A(x) \text{ is true}\}$, when $A(x)$ is some statement about x ;
$\operatorname{Re}(z)$ and $\operatorname{Im}(z)$	— the real and imaginary parts of the complex number z ;
$ z $ and \bar{z}	— the absolute value and complex conjugate of the complex number z ;
$\operatorname{Arg}(z) \in (-\pi, \pi]$	— the principal value of the argument of the non-zero complex number z ;
$g \circ f$	— the function obtained by composing f with g (so that $(g \circ f)(x) = g(f(x))$);
$f^{(j)}(s_0)$	— the j -th derivative of the function f at the point $s = s_0$ (where this exists);
$f^{(0)}(s)$	— equal to $f(s)$;
π	— the ratio of the circumference of a circle to its diameter;
e	— the base of the natural logarithm function, $\ln(x)$;
$\log(z)$	— the principal value of the logarithm of z , equal to $\ln(z) + i\operatorname{Arg}(z)$;
$\exp(z)$	— equal to e^z , when $z \in \mathbb{C}$;
i	— most often denotes a square root of -1 , but sometimes is an integer variable;
$\Gamma(z)$	— Euler's Gamma function;

$\frac{\Gamma^{(1)}}{\Gamma}(z)$	— the logarithmic derivative of $\Gamma(z)$, equal to both $\Gamma^{(1)}(z)/\Gamma(z)$ and $\frac{d}{dz} \log \Gamma(z)$;
γ	— may denote either a variable or Euler's constant, $-\Gamma^{(1)}(1) = 0.5772157 \dots$;
$n!$	— is « n -factorial» (equal to $\Gamma(n+1)$), when $n \in \mathbb{N} \cup \{0\}$;
$\binom{n}{r}$	— the «binomial coefficient» $(n!/(n-r)!/r!)$, when $n, r \in \mathbb{Z}$ satisfy $0 \leq r \leq n$;
$[x]$	— equal to $\max\{m \in \mathbb{Z} : m \leq x\}$, when x is a real number;
$\mathbf{u} \cdot \mathbf{v}$	— equal to $\sum_{j=1}^n u_j v_j$, when $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$.

$SL(2, R)$ denotes the group (under multiplication) of the 2×2 matrices M that have their elements in the integral domain R , and their determinants equal to 1.

When D is an open subset of \mathbb{C} , a function $f : D \rightarrow \mathbb{C}$ may be termed «smooth» if and only if it is the case that, for all $j, k \in \mathbb{N} \cup \{0\}$, the partial derivative $(\partial^{j+k}/\partial x^j \partial y^k)f(x+iy)$ is defined and continuous at all points $(x, y) \in \mathbb{R}^2$ such that $x+iy \in D$.

The «Schwartz space» contains $F : \mathbb{R}^2 \rightarrow \mathbb{C}$ if and only if, for all real $B \geq 0$ and all $j, k \in \mathbb{N} \cup \{0\}$, the partial derivative $(\partial^{j+k}/\partial x^j \partial y^k)F(x, y)$ is defined and continuous at all points $(x, y) \in \mathbb{R}^2$, and the mapping $(x, y) \mapsto (x^2+y^2)^B (\partial^{j+k}/\partial x^j \partial y^k)F(x, y)$ is a bounded function on \mathbb{R}^2 .

Algebraic and number-theoretic notation

R^*	— the group of units in R , when R is a ring with an identity;
\mathfrak{D}_F	— the ring of integers of F , when F is a number field;
\mathfrak{D}	— the integral domain $\mathfrak{D}_{\mathbb{Q}(i)} = \mathbb{Z}[i] = \{m+n\sqrt{-1} : m, n \in \mathbb{Z}\}$;
Gaussian integer	— a number in the ring \mathfrak{D} ;
Gaussian prime	— a prime in the ring \mathfrak{D} ;
(α) or $\alpha\mathfrak{D}$	— the ideal $\{\alpha\beta : \beta \in \mathfrak{D}\}$ of \mathfrak{D} , when α is a Gaussian integer;
$(\alpha) \mid (\delta)$	— signifies that the ideal (α) divides the ideal (δ) (i. e. that $\mathfrak{D} \supseteq \alpha\mathfrak{D} \supseteq \delta\mathfrak{D}$);

$\beta \equiv \alpha \pmod{\gamma\mathfrak{D}}$	— signifies that β is «congruent» to $\alpha \pmod{\gamma\mathfrak{D}}$ (i. e. that $(\gamma) \mid (\beta - \alpha)$);
$\mathfrak{D}/(\gamma\mathfrak{D})$	— the ring of residue classes $\pmod{\gamma\mathfrak{D}}$ (these being the cosets of $\gamma\mathfrak{D}$ in \mathfrak{D});
$\Gamma_0(\omega)$ or Γ	— a Hecke congruence subgroup of $SL(2, \mathfrak{D})$ (see Equation (1.9) for the definition);
$\delta \mid \gamma$	— signifies that δ is a divisor of γ (i. e. that one has both $\delta \in \mathfrak{D}$ and $\gamma \in \delta\mathfrak{D}$);
associate	— $\gamma \in \mathfrak{D}$ and $\delta \in \mathfrak{D}$ are associates if and only if one has both $\gamma \mid \delta$ and $\delta \mid \gamma$;
$\gamma \sim \delta$	— signifies that $\gamma, \delta \in \mathfrak{D}$ are associates, so that one has $\gamma/\delta \in \mathfrak{D}^* = \{i, -1, -i, 1\}$;
$\tau_n(\alpha)$	— the number of elements in the set $\{(\delta_1, \dots, \delta_n) \in \mathfrak{D}^n : \delta_1\delta_2 \cdots \delta_n = \alpha\}$.

(α, β) denotes a highest common factor of the Gaussian integers α and β ; this notation is somewhat ambiguous, for if δ is a highest common factor of α and β , then so too are the three other associates of δ (i. e. $i\delta$, $-\delta$ and $-i\delta$); it does not, however, lead to any serious difficulties, since relations of the form $(\alpha, \beta) \sim \delta$, or $|(\alpha, \beta)|^2 = n$, remain valid if the number (α, β) is replaced by any one of its associates. In the case of \mathbb{Z} -valued variables or constants, m and n (say), we unambiguously put $(m, n) = \max\{d \in \mathbb{N} : d \mid m \text{ and } d \mid n\}$.

$[\alpha, \beta]$ denotes a least common multiple of the Gaussian integers α and β ; like the notation for highest common factors, this notation for least common multiples is harmlessly ambiguous.

Specialised or customised notation

$\zeta(s)$	— the Riemann zeta-function;
$L(s, \chi)$	— the Dirichlet L -function associated with the Dirichlet character χ ;
Λ^d	— $(\Lambda^d)_{d \in \mathbb{N}}$ is the family of endomorphisms of $(\mathbb{Q}(i))^*$ that are given by (1.2);
λ^d	— the groessencharacter (or Hecke character) defined, for $d \in \mathbb{Z}$, just below (1.2);

- $\zeta(s, \lambda^d)$ — a Hecke zeta function (meromorphic on \mathbb{C} and, for $\text{Re}(s) > 1$, as stated in (1.1));
- $P_M(A; s, \lambda^d)$ — given by (1.4), if $M \in (0, \infty)$, A is a mapping from $\mathfrak{D} - \{0\}$ to \mathbb{C} , $s \in \mathbb{C}$, $d \in \mathbb{Z}$;
- $E(D; M, A)$ — the mean value defined in (1.1)–(1.4);
- \mathbb{H}_3 — a model for 3-dimensional hyperbolic space (see below (1.8), and [13, Chapter 1]);
- \mathfrak{D}_Γ — a certain subspace of $L^2(\Gamma \backslash \mathbb{H}_3)$ (see below (1.8), and refer to [13, Chapters 1–4]);
- Δ_3 — the Laplacian operator, from \mathfrak{D}_Γ into $L^2(\Gamma \backslash \mathbb{H}_3)$, that is given by (1.8);
- $\lambda_1(\Gamma)$ — the smallest positive eigenvalue of the operator $-\Delta_3$;
- $\Theta(\omega)$ — the function, from $\mathfrak{D} - \{0\}$ into $[0, 2/9)$, that is defined in (1.10);
- ϑ — the absolute constant defined in (1.11) (see also (1.12) and (1.13));
- $\|b\|_p$ — defined in (1.17), it is, when $p \in [1, \infty]$, a norm of the complex valued function b ;
- d_+z — a Lebesgue measure on \mathbb{C} ($d_+z = dx dy$, where $x = \text{Re}(z)$, $y = \text{Im}(z)$);
- $\int_{\mathbb{C}} f(z) d_+z$ — the integral of f , with respect to d_+z , over \mathbb{C} (equal to $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x + iy) dx dy$);
- $X_d(s)$ — when $d \in \mathbb{Z}$, the mapping $s \mapsto X_d(s)$ is the meromorphic function given by (2.6);
- $T(d, t)$ — when $d \in \mathbb{Z}$, the mapping $t \mapsto T(d, t)$ is the real function given by (2.6) and (2.8);
- $\delta(\Lambda^d, n)$ — when $d \in \mathbb{Z}$, the real sequence $(\delta(\Lambda^d, n))_{n \in \mathbb{N}}$ is given by (2.10) and (1.2);
- $z_{m,t}$ and $Z_{m,t}$ — the complex number in (3.2) and (in the proof of Lemma 3) its absolute value;
- $\delta_{a,b}$ — is equal to 1 if $a = b$, and is otherwise equal to zero (see (3.40));

- $e(x)$ — denotes $\exp(2\pi ix)$, when $x \in \mathbb{R}$;
 $\widehat{F} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ — defined as in Lemma 7, it is a Fourier transform of the function $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$;
 $\widehat{f} : \mathbb{C} \rightarrow \mathbb{C}$ — defined as in Lemma 7, it is a Fourier transform of the function $f : \mathbb{C} \rightarrow \mathbb{C}$;
 $\Delta_{\mathbb{R} \times \mathbb{R}}$ and $\Delta_{\mathbb{C}}$ — defined in Lemma 8, these are two Laplacian operators (of the Euclidean type);
 S_{α} — defined in Remark 5.1, it is (when $\alpha \in \mathbb{C}^*$) a certain «rotation-dilatation operator»;
 \mathfrak{N} — the function $\mathfrak{N} : \mathbb{C} \rightarrow [0, \infty)$ such that one has $\mathfrak{N}(z) = |z|^2$ for all $z \in \mathbb{C}$;
 $S(\alpha, \beta; \gamma)$ — defined in (5.5), a Kloosterman sum associated with the number field $\mathbb{Q}(i)$.

Conventions concerning certain notation associated with sums and products. With regard to products in which the factors are indexed by a variable non-zero ideal (α) (and in which the ideal itself determines the value of the corresponding factor), it should be understood that there is a one-to-one correspondence between the relevant set of ideals and the factors in the product: this effectively means that one may view the factors in the product as being indexed by the Gaussian integer α that is implicit in the notation « (α) », provided only that one constrains this α to satisfy the conditions $\text{Im}(\alpha) \geq 0$ and $\text{Re}(\alpha) > 0$.

When $\gamma \in \mathfrak{D}$ and $\gamma \neq 0$, the elements of the group of units $(\mathfrak{D}/(\gamma\mathfrak{D}))^*$ are the «reduced residue classes mod $\gamma\mathfrak{D}$ » (and each of these is, of course, a coset of $\gamma\mathfrak{D}$ in \mathfrak{D}). However, where there occurs a condition of summation of the form $\delta \in (\mathfrak{D}/(\gamma\mathfrak{D}))^*$, the variable of summation δ should, by an abuse of notation, be understood to have a range that is a fixed set of representatives $\{\delta_1, \delta_2, \dots, \delta_{|(\mathfrak{D}/(\gamma\mathfrak{D}))^*|}\} \subset \mathfrak{D}$ of the reduced residue classes mod $\gamma\mathfrak{D}$. Similarly, any condition of summation of the form $\kappa \in \mathfrak{D}/\gamma\mathfrak{D}$ signifies that κ has a range that is a fixed set of representatives $\{\kappa_1, \kappa_2, \dots, \kappa_{|\mathfrak{D}/(\gamma\mathfrak{D})|}\} \subset \mathfrak{D}$ of the residue classes mod $\gamma\mathfrak{D}$.

Variables of summation designated by Latin letters are \mathbb{Z} -valued variables, whereas those designated by Greek letters are instead \mathfrak{D} -valued variables (and may take any value in \mathfrak{D} permitted by the conditions attached to the summation).

Where there occur nested summations, such as sums of the form $\sum_{\alpha} f(\alpha) \sum_{\beta} g(\alpha, \beta)$ (for example), it should be understood that the variable α of the outer summation is constrained to satisfy the condition $f(\alpha) \neq 0$.

Superscripts and subscripts. The superscripts \star , \flat and \sharp , the dash, $'$, and double-dash, $''$, and the tilde and breve accents, \sim and \breve , have no intrinsic meaning (we simply found it convenient to use them in devising names for certain variables); when used as subscripts the symbols $*$ and \dagger are similarly devoid of meaning.

In relations such as $\alpha\delta^* \equiv \kappa \pmod{\gamma\mathfrak{D}}$, and in expressions such as $e(\operatorname{Re}(\alpha\delta^*/\gamma))$, or the highest common factor $(\alpha\delta^*, \gamma)$, it is to be understood that δ^* denotes a variable dependent upon a variable δ , and that δ and δ^* take values in \mathfrak{D} such that $\delta^*\delta \equiv 1 \pmod{\gamma\mathfrak{D}}$; where such notation is used, the relevant \mathfrak{D} -valued variables γ , δ and δ^* are implicitly constrained (by virtue of the congruence condition just mentioned) to take only combinations of values satisfying both $(\delta, \gamma) \sim 1$ and $(\delta^*, \gamma) \sim 1$. One notable use of this «*-notation» is in our definition, in (5.5), of the Kloosterman sum $S(\alpha, \beta; \gamma)$: note, in particular, that the summand in (5.5) may be expressed as the product $e(\operatorname{Re}(\alpha\delta^*/\gamma)) e(\operatorname{Re}(\beta\delta/\gamma))$, and note also the indication given above as to how the relation $\delta \in (\mathfrak{D}/\gamma\mathfrak{D})^*$ is to be interpreted when it occurs as a condition of summation.

Notation for bounds and asymptotic estimates. The notation $O_{\alpha_1, \dots, \alpha_n}(B)$, when it is the k -th instance of O -notation used in this paper, should be understood to denote a complex-valued variable ι_k that satisfies a condition of the form $|\iota_k| \leq C(\alpha_1, \dots, \alpha_n)B$, in which the «implicit constant» $C(\alpha_1, \dots, \alpha_n)$ is positive and depends only on $\alpha_1, \dots, \alpha_n$ and declared constants. Where this « O -notation» is used, the relevant variable or constant B must necessarily satisfy $B \geq 0$.

We frequently employ Vinogradov's notation, as an alternative to the O -notation. Thus we may use either the notation $\xi \ll_{\alpha_1, \dots, \alpha_n} B$, or the equivalent notation $B \gg_{\alpha_1, \dots, \alpha_n} \xi$, to signify that one has $\xi = O_{\alpha_1, \dots, \alpha_n}(B)$. Where $A \geq 0$ and $B \geq 0$, the notation $A \asymp_{\alpha_1, \dots, \alpha_n} B$ signifies that $A \ll_{\alpha_1, \dots, \alpha_n} B \ll_{\alpha_1, \dots, \alpha_n} A$ (i. e. that one has both $A \ll_{\alpha_1, \dots, \alpha_n} B$ and $B \ll_{\alpha_1, \dots, \alpha_n} A$). There are a few places where, instead of attaching subscripts (to the O , \ll , \gg or \asymp sign), we explicitly state the parameters upon which the relevant implicit constant, or constants, may depend.

Epsilon and Eta. In stating the results of this paper we treat ε and η as positive valued variables. However, in any meaningful application of our results in which ε

had a part to play, it would be necessary either to assign to ε a specific, fairly small, numerical value (such as 10^{-10} , say), or else to have ε function as an «arbitrarily small positive constant»; the same applies in respect of the variable η .

2. Essential properties of the zeta functions

By work of Hecke [21, Pages 34–35] it is known that if $d \in \mathbb{Z} - \{0\}$, then the function

$$\xi(s, \lambda^d) = \Gamma(s + 2|d|)\pi^{-s}\zeta(s, \lambda^d) \quad (2.1)$$

has an entire analytic continuation satisfying, for all $s \in \mathbb{C}$,

$$\xi(s, \lambda^d) = \xi(1 - s, \lambda^{-d}). \quad (2.2)$$

By substituting $\bar{\alpha}$ for α in (1.1) one finds that

$$\zeta(s, \lambda^d) = \zeta(s, \lambda^{-d}) \quad \text{and} \quad \xi(s, \lambda^d) = \xi(s, \lambda^{-d}). \quad (2.3)$$

The case $d = 0$ of (2.1) defines a function $\xi(s, \lambda^0)$ with a single valued analytic continuation to $\mathbb{C} - \{0, 1\}$ satisfying the case $d = 0$ of (2.2). This function has simple poles at $s = 0$ and $s = 1$, where the residues are $-1/4$ and $1/4$, respectively. These facts, combined with (2.1) and (2.2), show that $\zeta(0, \lambda^0) = -1/4$, so the pole of $\xi(s, \lambda^0)$ at $s = 0$ is purely an effect of the «Gamma factor» in (2.1). If $\text{Re}(s) > 1$, then it follows by (1.1), (1.2) and (2.3) that $\overline{\zeta(s, \lambda^d)} = \zeta(\bar{s}, \lambda^{-d}) = \zeta(\bar{s}, \lambda^d)$. Therefore, by Schwarz's reflection principle,

$$\zeta(\bar{s}, \lambda^d) = \overline{\zeta(s, \lambda^d)} \quad \text{for } s \in \mathbb{C} - \{1\}. \quad (2.4)$$

To summarise some of the above, one may note firstly that if $(0, 1) \neq (d, s) \in \mathbb{Z} \times \mathbb{C}$, then

$$\zeta(s, \lambda^d) = X_d(s)\zeta(1 - s, \lambda^{-d}), \quad (2.5)$$

where

$$X_d(s) = \frac{\pi^{2s-1}\Gamma(2|d| + 1 - s)}{\Gamma(2|d| + s)}, \quad (2.6)$$

and secondly that

$$\pi^{-1}\zeta\left(s, \lambda^0\right) = \frac{1/4}{s-1} + h_0 + O_\rho(|s-1|) \quad \text{for } \rho \geq |s-1| > 0, \quad (2.7)$$

where h_0 is some real absolute constant. Introducing further new notation, define $T(d, t)$ by

$$\log T(d, t) = -\frac{X_d^{(1)}\left(\frac{1}{2} + it\right)}{X_d\left(\frac{1}{2} + it\right)} \quad \text{for } d \in \mathbb{Z} \text{ and } t \in \mathbb{R}. \quad (2.8)$$

Due to the central part it plays in Lemma 6 below (an approximate functional equation for $\zeta(s, \lambda^d)$) the number $T(d, \text{Im}(s))$ figures significantly in much of what follows.

By the absolute convergence of the sums in (1.1),

$$\zeta(s, \lambda^d) = \sum_{n=1}^{\infty} \frac{\delta(\Lambda^d, n)}{n^s} \quad \text{for } d \in \mathbb{Z} \text{ and } \text{Re}(s) > 1, \quad (2.9)$$

where

$$\delta(\Lambda^d, n) = \frac{1}{4} \sum_{|\alpha|^2=n} \Lambda^d(\alpha) \quad (2.10)$$

(so that, by virtue of the definition (1.2), one has $\delta(\Lambda^d, n) \in \mathbb{R}$). By (2.10), (1.2) and a theorem of Jacobi [17, Theorem 278] one has $\delta(\Lambda^0, n) = \sum_{k|n} \chi_4(k)$ for $n \in \mathbb{N}$, where $\chi_4(n)$ is the non-principal Dirichlet character mod 4. Therefore

$$\zeta\left(s, \lambda^0\right) = \zeta(s)L\left(s, \chi_4\right) \quad (2.11)$$

and, for $\varepsilon > 0$, $n \in \mathbb{N}$ and $d \in \mathbb{Z}$,

$$\left|\delta(\Lambda^d, n)\right| \leq \frac{1}{4} \sum_{|\alpha|^2=n} \left|\Lambda^d(\alpha)\right| = \delta(\Lambda^0, n) = \sum_{k|n} \chi_4(k) \leq \sum_{k|n} 1 = O_\varepsilon(n^\varepsilon). \quad (2.12)$$

By using (2.11) and information about $\zeta(s)$ and $L(s, \chi_4)$ (such as [47, Equation (2.1.16)] and [7, Equation (15) of Chapter 6]) one can determine that (2.7)

holds with $h_0 = (1/4)(\gamma + \frac{L^{(1)}}{L}(1, \chi_4))$, where γ is Euler's constant. Note also that, by its definition in the section on notation, the «divisor function» $\tau_j(\alpha)$ satisfies, for $j \in \mathbb{N}$ and $\alpha \in \mathfrak{D} - \{0\}$, both $4 \leq \tau_j(\alpha) \leq (\tau_2(\alpha))^{j-1}$ and $\tau_2(\alpha) \leq \sum_{n|\alpha|^2} 4\delta(\Lambda^0, n)$,

so that by (2.12) one has

$$\tau_j(\alpha) \ll_{\varepsilon,j} |\alpha|^{2(j-1)\varepsilon} \quad \text{for } j \in \mathbb{N}, \varepsilon > 0 \text{ and } \alpha \in \mathfrak{D} - \{0\}. \quad (2.13)$$

As noted in [21, Equation (30)], the multiplicativity of the groessencharacters and absolute convergence of the series in (1.1) together imply the Euler product formula:

$$\zeta(s, \lambda^d) = \prod_{\substack{\mathfrak{p} \in I \\ \mathfrak{p} \text{ is prime}}} \frac{1}{1 - \lambda^d(\mathfrak{p})(N\mathfrak{p})^{-s}} \quad \text{for } \text{Re}(s) > 1, \quad (2.14)$$

where the product is taken over all prime ideals \mathfrak{p} in \mathfrak{D} , while the other notation used is as defined in, and just below, Equation (1.2). This Euler product representation of $\zeta(s, \lambda^d)$ is of fundamental significance in respect of its applications concerning the distribution of Gaussian primes. Here however it will serve merely to justify the assertion that there is no integer d for which the function $s \mapsto \zeta(s, \lambda^d)$ is identically equal to zero (the convergence of the Euler product guaranteeing that $\zeta(s, \lambda^d) \neq 0$ in the half plane where $\text{Re}(s) > 0$).

3. An approximate functional equation

The principal result in this section, Lemma 6, is essentially a special case of the very general class of approximate functional equations given by Ivić's theorem in [25], though additional work has been done to make the result more uniform in the groessencharacter aspect. This uniformity issue has previously been addressed by Harcos in [15, Theorem 2.5], which is an application of the method of [25] in the context of L functions corresponding to irreducible cuspidal automorphic representations of general linear groups over number fields. For errata to [15] see [16].

Lemma 6 is applicable at any point in the strip $-1/3 \leq \text{Re}(s) \leq 4/3$, whereas [15, Theorem 2.5] applies only at points on the critical line $\text{Re}(s) = 1/2$. However, the case $\sigma = 1/2$ is the only case of Lemma 6 that is used in this paper. In light of [15, Remark 2.7] it is clear that the case $\sigma = 1/2$ of Lemma 6 is very nearly a direct corollary of [15, Theorem 2.5] (it surpasses what [15, Theorem 2.5] would

imply only to the extent of providing, in (3.50)–(3.52), bounds for the relevant error term that are better than the bound stated in [15, Equation (2.12)]). The above mentioned results of Harcos would meet our needs in this paper, but we think that Lemma 6 has enough merits of its own to justify the inclusion of its proof in this section (which also serves to make this paper more self-contained).

One can find, in the literature, other approximate functional equations for Hecke zeta functions with groessencharacters: one is Huxley’s result [23, Theorem 2], which is applicable to Hecke zeta functions associated with number fields of arbitrary degree, but appears formidably complicated; another is Lavrik’s theorem in [36], which is reasonably simple and involves only very small error terms, but is less convenient (for our purposes) than Lemma 6. See also [28, Theorem 5.3] for a very general approximate functional equation.

LEMMA 1. *Let $0 < \delta < \pi$. Then, for $|\text{Arg}(z)| \leq \pi - \delta$,*

$$\frac{\Gamma^{(1)}}{\Gamma}(z) = \log z - \frac{1}{2z} + O_\delta\left(\frac{1}{|z|^2}\right). \tag{3.1}$$

PROOF. See [24, Page 57], where it is noted that methods of complex analysis enable one to deduce the result (3.1) from the asymptotic estimate

$$\begin{aligned} \log \Gamma(z) &= \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + O_\delta\left(\frac{1}{|z|}\right) \\ &\quad (z \in \mathbb{C} - \{0\}, \quad |\text{Arg}(z)| \leq \pi - \delta < \pi). \end{aligned}$$

A proof of the latter estimate is given in [51, Section 13.6] □

LEMMA 2. *The equations (2.8) and (2.6) together define a function $T : \mathbb{Z} \times \mathbb{R} \rightarrow (0, \infty)$ such that, for $t, t_1 \in \mathbb{R}$, $d \in \mathbb{Z}$, $C_0 = 4\pi e^\gamma$ and*

$$z_{d,t} = 2|d| + 1/2 + it, \tag{3.2}$$

one has:

$$\log T(d, t) = \log T(|d|, |t|) = 2 \operatorname{Re} \left(\frac{\Gamma^{(1)}}{\Gamma}(z_{d,t}) - \log \pi \right); \tag{3.3}$$

$$T(d, t_1) > T(d, t) \quad \text{if } |t_1| > |t|; \tag{3.4}$$

$$\log T(d, 0) = -2 \log C_0 + 4 \sum_{k=1}^{|d|} \frac{1}{2k-1}; \tag{3.5}$$

$$\log T(0, 1/2) < -\log(2C_0); \tag{3.6}$$

$$T(d, t) = \frac{1}{\pi^2} \left(|z_{d,t}|^2 - \operatorname{Re}(z_{d,t}) \right) + O(1) = \frac{|2d + it|^2}{\pi^2} + O(1). \tag{3.7}$$

PROOF. Given the definition (2.8), the identity (3.3) follows by logarithmic differentiation of (2.6) and an appeal to the reflection principle. By (3.3), $T(d, t)$ is positive valued. One has, moreover,

$$\frac{d^2}{dz^2} \log \Gamma(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} \quad \text{for } |\operatorname{Arg}(z)| < \pi$$

(see, for example, [51, Section 12.16]), and so it follows from (3.3) and (3.2) that, for $d \in \mathbb{Z}$, one has

$$\frac{\partial}{\partial t} \log T(d, t) = 4t \sum_{n=0}^{\infty} \frac{\operatorname{Re}(n + z_{d,t})}{|n + z_{d,t}|^4}$$

at all points $t \in \mathbb{R}$. By this equation one has $\frac{\partial}{\partial t} \log T(d, -t) = \frac{\partial}{\partial t} \log T(d, t) > 0$ for $t > 0$, $d \in \mathbb{Z}$. Therefore, when the integer d is given, $\log T(d, t)$ is a strictly increasing function of $|t|$. The result (3.4) follows, given the strict monotonicity of the function $\exp(x)$ on \mathbb{R} .

Since $\frac{\Gamma^{(1)}}{\Gamma}(1/2) = \frac{\Gamma^{(1)}}{\Gamma}(1) - 2 \log 2 = -\gamma - \log 4$ (see [51, Sections 12.1 and 12.15]), one may deduce (3.5) from (3.3) and (3.2) by logarithmic differentiation of the identity $\Gamma(z) = (z-1)(z-2) \dots (z-2|d|)\Gamma(z-2|d|)$. Since

$$\frac{d}{dz} \log \Gamma(z) = -\gamma - \frac{1}{z} + z \sum_{n=1}^{\infty} \frac{1}{n(n+z)} \quad \text{for } |\operatorname{Arg}(z)| < \pi$$

(see [51, Section 12.6]), one has

$$\operatorname{Re} \left(\frac{\Gamma^{(1)} \left(\frac{1+i}{2} \right)}{\Gamma \left(\frac{1+i}{2} \right)} \right) = -\gamma - 1 + 2 \sum_{n=1}^{\infty} \frac{n+1}{n(4n(n+1)+2)} < -\gamma - 1 + 2 \left(\frac{1}{5} + \frac{\zeta(2)-1}{4} \right).$$

By this, combined with the equality $\zeta(2) = \pi^2/6$, the inequality $\pi^2/6 < \gamma + 11/5 - \log(8/\pi)$ and the case $d = 0, t = 1/2$ of (3.2) and (3.3), one obtains the inequality (3.6).

By (3.3) and (3.2), the case $\delta = \pi/2$ of (3.1) of Lemma 1 shows that, for $d \in \mathbb{Z}$ and $t \in \mathbb{R}$,

$$\log T(d, t) = 2 \log |z_{d,t}| - \frac{\operatorname{Re}(z_{d,t})}{|z_{d,t}|^2} - 2 \log \pi + O\left(\frac{1}{|z_{d,t}|^2}\right).$$

Here we have, by (3.2), $|z_{d,t}| \geq \operatorname{Re}(z_{d,t}) \geq 1/2$, so the estimate (3.7) follows. \square

COROLLARY 1. *With C_0 as in the above lemma, it follows by (3.2), (3.4), (3.5) and (3.7) that*

$$C_0^{-2} \leq T(d, t) \asymp |2|d| + 1/2 + it|^2 \quad \text{for } d \in \mathbb{Z}, t \in \mathbb{R}. \tag{3.8}$$

LEMMA 3. *Let $0 < \eta \leq 1/4$. Then, for $d \in \mathbb{Z}$ and $s = \sigma + it$ with*

$$-\frac{5}{2} \leq \sigma \leq \frac{5}{2} \quad \text{and } t \in \mathbb{R}, \tag{3.9}$$

one has

$$X_d(s) \ll_{\eta} T^{1/2-\sigma} \quad \text{if either } T = T(d, t) \geq T(0, 1/2) \text{ or } \sigma \notin \bigcup_{n=1}^2 (n - \eta, n + \eta), \tag{3.10}$$

where $X_d(s)$ and $T(d, t)$ are as given by (2.6) and (2.8).

PROOF. By (2.6) and [41, Lemma 3] one has

$$|X_d(s)| = \pi^{2\sigma-1} \left| \frac{\Gamma(2|d| + 1 - s)}{\Gamma(2|d| + s)} \right| \leq \left| \frac{2|d| + 1 + s}{\pi} \right|^{1-2\sigma} \quad \text{if } -1/2 \leq \sigma \leq 1/2.$$

Therefore, if one puts $Z_{d,t} = |2|d| + 1/2 + it|$, then

$$|X_d(s_1)| \ll Z_{d,t}^{1-2\operatorname{Re}(s_1)} \quad \text{for } s_1 \in \mathbb{C} \text{ such that } \operatorname{Im}(s_1) = t \text{ and } |\operatorname{Re}(s_1)| \leq 1/2. \tag{3.11}$$

For cases where $|\sigma| > 1/2$ an estimate similar to (3.11) is needed.

If $n \in \mathbb{N}$, then unless s_1 is an integer not satisfying $-2|d| - n < s_1 < 2|d| + 1$, it follows by (2.6) and the functional equation of Γ that

$$\frac{X_d(s_1)}{X_d(s_1 + n)} = \pi^{-2n} \prod_{k=0}^{n-1} (2|d| - s_1 - k)(2|d| + s_1 + k). \tag{3.12}$$

Therefore (3.12) holds for $|\operatorname{Re}(s_1)| < 3$ (and for all $n \in \mathbb{N}$) when the integer d is non-zero. If however $d = 0$, then one may observe that when $T(0, \operatorname{Im}(s_1)) \geq T(0, 1/2)$ it must follow by (3.3) and (3.4) of Lemma 2 that $|\operatorname{Im}(s_1)| \geq 1/2 > 0$. The case $d = 0$ of (3.12) therefore holds for $n \in \mathbb{N}$ and $s_1 \in \mathbb{C}$ such that $T(0, \operatorname{Im}(s_1)) \geq T(0, 1/2)$ (for $|\operatorname{Im}(s_1)| \geq 1/2$ certainly implies $s_1 \notin \mathbb{Z}$).

Suppose now that $d \in \mathbb{Z}$ and $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ such that (3.9) holds and $T(d, t) \geq T(0, 1/2)$. If $-5/2 \leq \sigma < -1/2$, then, by the discussion of the previous paragraph, (3.12) may be applied with $s_1 = s$ and $n = -[\sigma + 1/2] \in \{1, 2\}$. Consequently one obtains the bound

$$X_d(s) \ll Z_{d,t}^{2n} |X_d(s + n)|,$$

and it follows by (3.11) that

$$X_d(s) \ll Z_{d,t}^{2n+1-2(\sigma+n)} = Z_{d,t}^{1-2\sigma} \quad \text{if } -5/2 \leq \sigma < 1/2. \tag{3.13}$$

If instead $1/2 < \sigma \leq 5/2$, then (3.12) may be applied with $s_1 = s - n$ and $n = -[1/2 - \sigma] \in \{1, 2\}$. One has either $d = 0$ and $|t| \geq 1/2$, or $2|d| \geq 2 \geq 1/2 + \max\{|\sigma - 2|, |\sigma - 1|\}$, so that (3.12) yields the lower bound

$$\left| \frac{X_d(s - n)}{X_d(s)} \right| \geq \left(\frac{1}{\pi} \max \left\{ \frac{|d|}{2}, \frac{1}{2}, |t| \right\} \right)^{2n} \gg Z_{d,t}^{2n}.$$

One has also $|\operatorname{Re}(s - n)| = |\sigma - n| \leq 1/2$ and $\operatorname{Im}(s - n) = t$, so it follows by the above and (3.11) that

$$X_d(s) \ll Z_{d,t}^{-2n} \left| \frac{X_d(s - n)}{X_d(s)} \right| \ll Z_{d,t}^{-2n+1-2(\sigma-n)} = Z_{d,t}^{1-2\sigma} \quad \text{if } 1/2 < \sigma \leq 5/2. \tag{3.14}$$

By the result (3.8) of Corollary 1, one has $Z_{d,t} \asymp (T(d, t))^{1/2}$ in (3.11), (3.13) and (3.14), which therefore yield all those cases of the bound (3.10) in which $T(d, t) \geq T(0, 1/2)$.

The only cases of (3.10) requiring further consideration are therefore those in which $T(d, t) < T(0, 1/2)$ and $\min\{|\sigma - 1|, |\sigma - 2|\} \geq \eta$. In such cases it follows by (3.3)–(3.6) of Lemma 2 (and the inequality $\exp(4 - \gamma) > 2\pi$) that $d = 0$ and $|t| < 1/2$. Therefore it follows by (2.6) that in all these cases one has $X_d(s) \ll 1$, for the set $\{s \in \mathbb{C} : \operatorname{Re}(s) \in [-5/2, 1 - \eta] \cup [1 + \eta, 2 - \eta] \cup [2 + \eta, 5/2]\}$ and $|\operatorname{Im}(s)| \leq 1/2$ is a bounded closed region containing no pole of $X_0(s)$. By (3.8) one has $1 \ll T^{1/2 - \sigma}$ in (3.10), so the proof is now complete. \square

LEMMA 4. *Let $d \in \mathbb{Z}$. Suppose moreover that the functions $X_d(s)$ and $T(d, t)$ are as in (2.8) and (2.6), and that $s = \sigma + it$, where*

$$-\frac{1}{2} \leq \sigma \leq \frac{3}{2} \quad (3.15)$$

and $t \in \mathbb{R}$ satisfies

$$T(0, 1/2) \leq T(d, t) = T \quad (\text{say}). \quad (3.16)$$

Then one may define a meromorphic complex function $\tau \mapsto G_d(s, \tau)$ by:

$$G_d(s, \tau) = \frac{X_d(s - \tau)}{X_d(s)} T^{-\tau} - 1 \quad \text{for } \tau \in \mathbb{C} \text{ with } s - 2|d| - \tau \notin \mathbb{N}. \quad (3.17)$$

This function is analytic on the open disc $\{\tau \in \mathbb{C} : |\tau| < R_{d,s}\}$, where

$$R_{d,s} = |2|d| + 1 - s| \asymp T^{1/2}, \quad (3.18)$$

and one has $G_d(s, 0) = 0$. There is a unique complex sequence $(a_k(d, s))_{k \in \mathbb{N}}$ with the property that

$$\sum_{k=1}^{\infty} a_k(d, s) \tau^k = G_d(s, \tau) \quad \text{for all } \tau \in \mathbb{C} \text{ such that } |\tau| < R_{d,s}. \quad (3.19)$$

For some absolute constant $B \geq 1$, the above sequence $(a_k(d, s))_{k \in \mathbb{N}}$ satisfies:

$$a_k(d, s) \ll (B|t|/T)^{k/2} + (B/T)^{k/3} \ll B^k T^{-k/4} \quad \text{for } k \in \mathbb{N}, \quad (3.20)$$

$$a_1(d, s) = -\frac{2(\sigma - 1/2)ti}{|z_{d,t}|^2} + O(T^{-1}), \quad a_2(d, s) = \frac{ti}{|z_{d,t}|^2} + O(T^{-1}), \quad (3.21)$$

$$a_3(d, s) \ll T^{-1} \quad \text{and} \quad a_4(d, s) = -\frac{(1/2)t^2}{|z_{d,t}|^4} + O\left(T^{-9/8}\right) \ll T^{-1}, \quad (3.22)$$

where (as in Lemma 2) $z_{d,t} = 2|d| + 1/2 + it$.

PROOF. By (3.17) and (2.6),

$$G_d(s, \tau) = g_{d,s}(\tau) - 1, \quad (3.23)$$

where

$$\begin{aligned} g_{d,s}(\tau) &= (X_d(s))^{-1} X_d(s - \tau) T^{-\tau} = \\ &= \left(\frac{\Gamma(2|d| + 1 - s)}{\Gamma(2|d| + s)} \right)^{-1} \left(\frac{\Gamma(2|d| + 1 - s + \tau)}{\Gamma(2|d| + s - \tau)} \right) (\pi^2 T)^{-\tau}. \end{aligned} \quad (3.24)$$

Assuming that s is neither a pole nor zero of $X_d(s)$, the function $g_{d,s}(\tau)$ is analytic for

$$|\tau| < |2|d| + 1 - s| = R_{d,s},$$

and is non-zero for

$$|\tau| < |2|d| + s| = R_{d,s}^* \quad (\text{say}).$$

Therefore, and since $g_{d,s}(0) = 1$, it follows by (3.23) and the theory of Taylor and Laurent series (for which see [47, Sections 2.43 and 2.9]) that (3.19) holds if and only if $a_k(d, s) = g_{d,s}^{(k)}(0)/(k!)$ for all $k \in \mathbb{N}$.

Completion of the proof requires verification of (3.18) and (3.20)–(3.22). For the latter part it suffices to estimate of $g_{d,s}(\tau)$ in a subset of the region indicated in (3.19). Cauchy's inequality (stated in [47, Section 2.5]) is useful in deducing (3.20).

If $d \neq 0$, then by (3.15)

$$\min \{R_{d,s}, R_{d,s}^*\} \geq |2|d| - 1/2 + it| \geq (3/5) |z_{d,t}|,$$

where $z_{d,t} = 2|d| + 1/2 + it$. If $d = 0$, then by (3.3) and (3.4) of Lemma 2 the lower bound on $T(d, t)$ in (3.16) implies that $|t| > 1/2$, and so

$$\min \{R_{d,s}, R_{d,s}^*\} \geq |t| \geq 2^{-1/2} |1/2 + it| = 2^{-1/2} |z_{d,t}|$$

in this case. Since $2^{-1/2} > 3/5$, one may deduce that

$$\min \{|(2|d| + 1) - s|, |(-2|d|) - s|\} \geq 3/10 > 0,$$

so that, by virtue of (3.15), one is sure to have $X_d(s) \in \mathbb{C} - \{0\}$. Moreover, given (3.15), the facts already gathered are sufficient to justify the conclusions that

$$3 |z_{d,t}| \geq |z| + |\sigma - 1/2| \geq \min \{|2|d| + 1 - s|, |2|d| + s|\} \geq (3/5) |z_{d,t}| \quad (3.25)$$

and

$$\max \{|\text{Arg}(2|d| + 1 - s)|, |\text{Arg}(2|d| + s)|\} \leq (3/4)\pi,$$

in all cases covered by the lemma. Consequently, for $\tau \in \mathbb{C}$ such that

$$|\tau| \leq (3/5) |z_{d,t}| \sin(\pi/6) = (3/10) |z_{d,t}|, \quad (3.26)$$

one has

$$\max \{|\text{Arg}(2|d| + 1 - s + \tau)|, |\text{Arg}(2|d| + s - \tau)|\} \leq (3\pi/4) + (\pi/6) = (11/12)\pi. \quad (3.27)$$

Note that, by (3.25) and the result (3.8) of Corollary 1, one has $R_{d,s} \asymp T^{1/2}$, as claimed in (3.18).

Since one may define a single valued and analytic branch of $\log \Gamma(z)$ on $\mathbb{C} - (-\infty, 0]$, it follows by (3.24), (3.27) and the result (3.3) of Lemma 2 that, for $\tau \in \mathbb{C}$ satisfying (3.26), one has:

$$\begin{aligned} \frac{d}{d\tau} \log g_{d,s}(\tau) &= \frac{\Gamma^{(1)}}{\Gamma}(2|d| + 1 - s + \tau) + \frac{\Gamma^{(1)}}{\Gamma}(2|d| + s - \tau) - (\log T + 2 \log \pi) = \\ &= \left(\frac{\Gamma^{(1)}}{\Gamma}(2|d| + 1 - s + \tau) - \frac{\Gamma^{(1)}}{\Gamma}(z_{d,-t}) \right) + \left(\frac{\Gamma^{(1)}}{\Gamma}(2|d| + s - \tau) - \frac{\Gamma^{(1)}}{\Gamma}(z_{d,t}) \right). \end{aligned} \quad (3.28)$$

If $|z_{d,t}| \leq (10/3)^3$, then for τ as in (3.26) it follows by (3.27), (3.28) and the case $\delta = \pi/12$ of the result (3.1) of Lemma 1 that

$$\begin{aligned} \left| \frac{d}{d\tau} \log g_{d,s}(\tau) \right| &\leq |\log(2|d| + 1 - s + \tau) - \log(2|d| + 1/2 - it)| + \\ &\quad + |\log(2|d| + s - \tau) - \log(2|d| + 1/2 + it)| + |O(|z_{d,t}|^{-1})| \leq \\ &\leq 2(\log(10/3) + 17\pi/12) + O(1) \ll 1. \end{aligned}$$

From this it follows, since $\log g_{d,s}(0) = \log 1 = 0$, that if $|z_{d,t}| \leq (10/3)^3$ then, for τ satisfying (3.26), one has $\log g_{d,s}(\tau) \ll |\tau| \leq (10/3)^2$, and so $g_{d,s}(\tau) \ll 1$. Consequently (given (3.18), (3.19) and (3.23)) it follows by Cauchy's inequality that, in cases where $|z_{d,t}| \leq (10/3)^3$, one has

$$a_k(d, s) \ll ((3/10) |z_{d,t}|)^{-k} \quad \text{for } k \in \mathbb{N}.$$

This confirms (3.20) in those cases, since (3.8) shows $|z_{d,t}| \gg T^{1/2} \gg T^{1/3}$. One may also verify that in those same cases, in which $T \asymp 1$, the estimates in (3.21) and (3.22) are no stronger than those provided (for $k = 1, 2, 3, 4$) by the confirmed bound (3.20).

The above completes the proof of the lemma for cases with $|z_{d,t}| \leq (10/3)^3$, so henceforth it is to be supposed that $|z_{d,t}| > (10/3)^3$. This supposition is more than sufficient to ensure that, for τ satisfying (3.26) (and with σ as in (3.15)), one has

$$(1/2) |z_{d,t}| \geq \left((3/10) + (3/10)^3 \right) |z_{d,t}| \geq |\tau| + 1 \geq |\tau_\sigma|, \quad (3.29)$$

where

$$\tau_\sigma = \tau - (\sigma - 1/2). \quad (3.30)$$

Assuming (3.26), it follows by (3.28), the estimate (3.1) of Lemma 1, and (3.29)–(3.30), that

$$\begin{aligned} \frac{d}{d\tau} \log g_{d,s}(\tau) &= (\log(z_{d,-t} + \tau_\sigma) - \log(z_{d,-t})) + (\log(z_{d,t} - \tau_\sigma) - \log(z_{d,t})) + \\ &\quad + \frac{1}{2} \left(\frac{1}{z_{d,-t}} - \frac{1}{z_{d,-t} + \tau_\sigma} \right) + \frac{1}{2} \left(\frac{1}{z_{d,t}} - \frac{1}{z_{d,t} - \tau_\sigma} \right) + O(|z_{d,t}|^{-2}) = \\ &= \frac{\tau_\sigma}{z_{d,-t}} + \frac{(-\tau_\sigma)}{z_{d,t}} + O\left(\frac{|\tau_\sigma|^2}{|z_{d,t}|^2}\right) + O\left(\frac{|\tau_\sigma| + 1}{|z_{d,t}|^2}\right) = \\ &= \frac{2it\tau_\sigma}{|z_{d,t}|^2} + O\left(\frac{1 + |\tau_\sigma|^2}{|z_{d,t}|^2}\right) = \frac{2it\tau}{|z_{d,t}|^2} - \frac{2it(\sigma - 1/2)}{|z_{d,t}|^2} + O\left(\frac{1 + |\tau|^2}{|z_{d,t}|^2}\right). \end{aligned}$$

Since $\log g_{d,s}(0) = 0$, it follows by the above that if τ satisfies (3.26) then

$$\log g_{d,s}(\tau) = \beta_1 \tau + \beta_2 \tau^2 + O\left(\frac{|\tau| + |\tau|^3}{|z_{d,t}|^2}\right), \quad (3.31)$$

where one has

$$\beta_1 = \beta_1(d, s) = -\frac{2i(\sigma - 1/2)t}{|z_{d,t}|^2} \quad \text{and} \quad \beta_2 = \beta_2(d, s) = \frac{it}{|z_{d,t}|^2}, \quad (3.32)$$

which, by (3.15), implies:

$$|\beta_j| \leq 2|t|/|z_{d,t}|^2 \leq 2/|z_{d,t}| \quad \text{for } j = 1, 2. \quad (3.33)$$

Now, if τ satisfies

$$|\tau| \leq \frac{|z_{d,t}|}{|t|^{1/2} + |z_{d,t}|^{1/3}}, \quad (3.34)$$

then, since we are to suppose that $|z_{d,t}|^{1/3} > 10/3$, it is certainly the case that τ also satisfies (3.26). Therefore it may be deduced from (3.31) and (3.33) that, for all $\tau \in \mathbb{C}$ satisfying the condition (3.34), one has

$$g_{d,s}(\tau) = \exp\left(O\left(|z_{d,t}|^{-1/3}\right) + O(1) + O\left(|z_{d,t}|^{-4/3} + 1\right)\right) = \exp(O(1)) \ll 1.$$

By this last bound, together with (3.18), (3.19), (3.23) and Cauchy's inequality, one finds that

$$a_k(d, s) \ll \left(|t|^{1/2} |z_{d,t}|^{-1} + |z_{d,t}|^{-2/3}\right)^k \quad \text{for } k \in \mathbb{N}. \quad (3.35)$$

By (3.35) and (3.8) one obtains the bounds stated in (3.20).

One may next observe that, since $|z_{d,t}| \geq (10/3)^3$, the condition (3.26) is certainly satisfied whenever $|\tau| \leq |z_{d,t}|^{1/3}$ (say), so that in such cases (3.31)–(3.33) apply. Moreover it follows by (3.31)–(3.33) that, for τ satisfying $1 \leq |\tau| \leq |z_{d,t}|^{1/3}$, one has:

$$\begin{aligned} g_{d,s}(\tau) &= \exp(\beta_1 \tau) \exp\left(\beta_2 \tau^2\right) \left(1 + O\left(|\tau|^3 |z_{d,t}|^{-2}\right)\right) = \\ &= (1 + \beta_1 \tau) \left(1 + \beta_2 \tau^2 + \beta_2^2 \tau^4 / 2\right) + O\left(\left(|\tau|^2 + |\tau|^3\right) |z_{d,t}|^{-2} + |\tau|^6 |z_{d,t}|^{-3}\right) = \\ &= 1 + \beta_1 \tau + \beta_2 \tau^2 + \beta_2^2 \tau^4 / 2 + O\left(|\tau|^3 |z_{d,t}|^{-2}\right). \end{aligned} \quad (3.36)$$

Recalling now that $|z_{d,t}| > (10/3)^3 > 2^4$, it is implied by (3.19), (3.23), (3.25) and (3.35) that

$$\begin{aligned}
 g_{d,s}(\tau) &= 1 + \sum_{k=1}^4 a_k(d, s)\tau^k + \sum_{k=5}^{\infty} O\left(\left(\frac{2|\tau|}{|z_{d,t}|^{1/2}}\right)^k\right) = \\
 &= 1 + \sum_{k=1}^4 a_k(d, s)\tau^k + O\left(|\tau|^3 |z_{d,t}|^{-2}\right) \quad \text{if } |\tau| \leq |z_{d,t}|^{1/4}. \quad (3.37)
 \end{aligned}$$

Given (3.32), and given that $|z_{d,t}|^2 \asymp T$ (by (3.8)), a comparison of (3.37) with (3.36) at the four points $\tau = \pm 1, \pm i$ (for example) is sufficient to establish the first three estimates of (3.21) and (3.22). The final estimate of (3.22) then follows by comparing (3.37) with (3.36) at the point $\tau = |z_{d,t}|^{1/4}$. \square

COROLLARY 2. *Subject to the hypotheses of the above lemma, and with the same absolute constant $B \geq 1$ as in (3.20), one has, in (3.18) and (3.19), $R_{d,s} = |2|d| + 1 - s| \geq \geq B^{-1}T^{1/4}$ and, for all integers $K \geq 0$,*

$$\sum_{k=K+1}^{\infty} a_k(d, s)\tau^k \ll_K \left((|t|/T)^{(K+1)/2} + T^{-(K+1)/3} \right) |\tau|^{K+1} \quad (3.38)$$

if $\tau \in \mathbb{C}$ and $|\tau| \leq (2B)^{-1}T^{1/4}$.

LEMMA 5 (CONVEXITY AND SUB-CONVEXITY ESTIMATES). *Let $0 < \varepsilon \leq 1/4$, $d \in \mathbb{Z}$ and $s = \sigma + it$, where $t \in \mathbb{R}$ and $\sigma \geq -5/2$. Put $T = T(d, t)$ (where $T(d, t)$ is given by (2.8) and (2.6)). Then*

$$\zeta(s, \lambda^d) - \frac{(\pi/4)\delta_{d,0}}{s-1} = O\left(\varepsilon^{-2}T^{\max\{0, (1-\sigma)/2 + \varepsilon, 1/2 - \sigma\}}\right), \quad (3.39)$$

where

$$\delta_{a,b} = \begin{cases} 1 & \text{if } a=b, \\ 0 & \text{otherwise.} \end{cases} \quad (3.40)$$

In the case $\sigma = 1/2$ this may be strengthened to:

$$\zeta\left(1/2 + it, \lambda^d\right) \ll_{\varepsilon} T^{1/6 + \varepsilon}. \quad (3.41)$$

PROOF. In cases where $\sigma > 3/2$, the bound (3.39) follows since $|1/(s - 1)| < 2$, and since, by (2.9) and the case $\varepsilon = 1/3$ of (2.12),

$$\left| \zeta(s, \lambda^d) \right| \leq \sum_{n=1}^{\infty} \frac{|\delta(\Lambda^d, n)|}{n^\sigma} \ll \zeta(\sigma - 1/3) < \zeta(7/6)$$

(this suffices, given that $\varepsilon^{-2} > 1$ and $\max\{0, (1 - \sigma)/2 + \varepsilon, 1/2 - \sigma\} = 0$ in (3.39)).

If instead $-5/2 \leq \sigma \leq 3/2$ and $T(d, t) < T(0, 1/2)$, then by (3.4)–(3.6) of Lemma 2 (and the inequality $2\pi < \exp(4 - \gamma)$) one must have $d = 0$ and $|t| < 1/2$, which implies that $|s - 1| = |\sigma - 1 + it| < |\sigma - 1| + 1/2 \leq 4$. Moreover, it follows by the case $\rho = 4$ of (2.7) that (3.39) holds whenever $|s - 4| \leq 4$ (one need only note that, for such s , $\max\{0, (1 - \sigma)/2 + \varepsilon, 1/2 - \sigma\} \in [0, 7/2]$, and that $T^{7/2} \gg 1$, by (3.8)).

Given the above, the only cases left to consider are those in which one has both $-5/2 \leq \sigma \leq 3/2$ and $|s - 1| > 4$, so that $T(d, t) \geq T(0, 1/2)$ (as follows on account of the inference established in the first sentence of the previous paragraph). If, in such a case, one has also $\sigma < -1/2$, so that $-5/2 \leq \sigma < -1/2$, $|s - 1| > 4$ and $T(d, t) \geq T(0, 1/2)$, then by Lemma 3 and the functional equation (2.5) one finds that

$$\zeta(s, \lambda^d) \ll T^{1/2-\sigma} \left| \zeta(1 - s, \lambda^{-d}) \right|.$$

where $\zeta(1 - s, \lambda^{-d}) \ll 1$, since $\text{Re}(1 - s) = 1 - \sigma > 3/2$. The bound (3.39) then follows, given that one has $|1/(s - 1)| < 1/4$ and $5/4 \geq -\sigma/2 > 1/4 \geq \varepsilon$ and, by (3.8), $T \gg 1$.

In those cases of the lemma still to be considered one may make use of Rademacher’s bounds in [41, Theorem 4 and Theorem 5], which imply that if $-1/2 \leq \sigma \leq 3/2$ and $|s - 1| > 4$, then

$$\zeta(s, \lambda^d) \ll \zeta^2(1 + \eta) |2|d| + 1 + s|^{\eta+1-\sigma},$$

for all $\eta \in (0, 1/2]$ such that $-\eta \leq \sigma \leq 1 + \eta$. Now $\zeta(1 + \eta) \ll \eta^{-1}$ for $\eta \in (0, 1/2]$, while for $-1/2 \leq \sigma \leq 3/2$ one has $|2|d| + 1 + s| \asymp |2|d| + 1/2 + it|$, and so, by taking (in the above)

$$\eta = \max\{2\varepsilon, |\sigma - 1/2| - 1/2\} = \max\{2\varepsilon + 1 - \sigma, |1/2 - \sigma| + 1/2 - \sigma\} - (1 - \sigma),$$

one obtains:

$$\zeta(s, \lambda^d) \ll \varepsilon^{-2} |2|d| + 1/2 + it|^{\max\{2\varepsilon+1-\sigma, 0, 1-2\sigma\}}. \tag{3.42}$$

By the result (3.8) of Corollary 1, it follows that (3.42) implies (3.39) whenever $-1/2 \leq \sigma \leq 3/2$ and $|s - 1| > 4$. This completes the proof of (3.39) in all the cases within the scope of the lemma.

The bound (3.41) follows from Kaufman’s theorem in [30], which in fact gives the sharper bound $\zeta(1/2 + it, \lambda^d) \ll T^{1/6} \log^C(T + 2)$ (where C is some positive absolute constant). See [31] and [46] for similar «sub-convexity» bounds on Hecke zeta functions for general number fields. \square

LEMMA 6 (AN APPROXIMATE FUNCTIONAL EQUATION). *Let $C_0 = 4\pi e^\gamma$. Let $\varepsilon \in (0, 1/4)$ and $b \in (1, \infty)$, and suppose moreover that the infinitely differentiable function $\rho : (0, \infty) \rightarrow \mathbb{R}$ satisfies*

$$\rho(u) + \rho(1/u) = 1 \quad \text{for } u > 0 \tag{3.43}$$

and

$$\rho(u) = 0 \quad \text{for } u \geq b. \tag{3.44}$$

Let $d \in \mathbb{Z}$ and $\sigma + it = s \notin \{-2|d|, 2|d| + 1\}$, where $\sigma, t \in \mathbb{R}$ and

$$-1/3 \leq \sigma \leq 4/3, \tag{3.45}$$

and take $X_d(s) \in \mathbb{C}$, the real sequence $(\delta(\Lambda^d, n))_{n \in \mathbb{N}}$ and $T(d, t) \in (0, \infty)$ to be as in (2.6), (2.10) and (2.8), respectively. Then, for all non-negative integers K , and all $x, y, T \in (0, \infty)$ satisfying both

$$xy = T = T(d, t) \tag{3.46}$$

and

$$1/(2C_0) \leq by \leq 2C_0T, \tag{3.47}$$

one has

$$\begin{aligned} \zeta(s, \lambda^d) &= \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) \delta(\Lambda^d, n) n^{-s} + X_d(s) \sum_{n=1}^{\infty} \tilde{\rho}_K\left(\frac{n}{y}\right) \delta(\Lambda^{-d}, n) n^{s-1} + \\ &+ \frac{(\pi/4)\delta_{d,0}}{s-1} e^{-|s-1|} + E_K(s, \lambda^d), \end{aligned} \tag{3.48}$$

where $\delta_{a,b}$ is as defined in (3.40), while

$$\tilde{\rho}_K(u) = \rho(u) + \sum_{k=1}^K (-1)^k a_k(d, s) \left(u \frac{d}{du}\right)^k \rho(u) \tag{3.49}$$

(with the same coefficients $a_k(d, s)$ as in (3.17)–(3.19) of Lemma 4) and

$$E_K(s, \lambda^d) \ll_{\varepsilon, K} x^{1/2-\sigma} \left((|t|/T)^{\alpha_K} + T^{-\beta_K} \right) \int_{-T^\varepsilon}^{T^\varepsilon} \left| \zeta\left(1/2+i(t+v), \lambda^d\right) \right| \frac{dv}{1+v^2} \tag{3.50}$$

$$\ll_{\varepsilon, K} T^{1/6+\varepsilon} x^{1/2-\sigma} \left((|t|/T)^{\alpha_K} + T^{-\beta_K} \right) \tag{3.51}$$

with exponents

$$(\alpha_K, \beta_K) = \begin{cases} (1, 1) & \text{if } K = 0, 1, \\ (2, 1) & \text{if } K = 2, \\ ((K+1)/2, (K+1)/3) & \text{if } K \geq 3, \end{cases} \tag{3.52}$$

and implicit constants that depend only upon ε , K , the function $\rho(u)$ and b .

PROOF. Note firstly that in this proof we generally omit to indicate where there is some dependence of an implicit constant upon either the function $\rho(u)$ or the related number b , so a statement such as $\ll U \ll_\varepsilon 1$ might mean only that $U \ll_{\varepsilon, \rho} 1$, or that $U \ll_{\varepsilon, b} 1$.

It helps to distinguish two cases: a case in which T , in (3.46), is «sufficiently large», and the complementary case, in which T is «of bounded magnitude». Taking the latter case first, one supposes that

$$T \leq (6B)^{4/(1-4\varepsilon)}, \tag{3.53}$$

where $B \geq 1$ is the absolute constant of Lemma 4 and Corollary 2. Then, by (3.8) and (3.53),

$$1 \ll T \ll_\varepsilon 1, \quad d \ll_\varepsilon 1 \quad \text{and} \quad t \ll_\varepsilon 1, \tag{3.54}$$

$$1 \ll x, y \ll_\varepsilon 1 \tag{3.55}$$

and

$$\int_{-T^\varepsilon}^{T^\varepsilon} \left| \zeta \left(1/2 + i(t+v), \lambda^d \right) \right| \frac{dv}{1+v^2} \gg_\varepsilon \int_{t-1/C_0}^{t+1/C_0} \left| \zeta \left(1/2 + iw, \lambda^d \right) \right| dw \gg_\varepsilon 1, \tag{3.56}$$

where the last lower bound follows since (3.54) restricts d to a finite subset of \mathbb{Z} determined by ε , and restricts t to a closed bounded interval $[-h, h]$ also determined by ε , while the relevant integrand, $|\zeta(1/2 + iw, \lambda^d)|$, is a continuous non-negative valued real function with only isolated zeros (each function $\zeta(z, \lambda^d)$ being analytic for $z \neq 1$, and, by the Euler product (2.14), non-zero for $\text{Re}(z) > 1$), so that the last integral in (3.56) is a continuous positive valued function of t with an infimum over $t \in [-h, h]$ that is also positive (this infimum necessarily being an attained minimum of the function).

Therefore, for proof of the case of the lemma that we are currently considering, it suffices to check that:

$$\zeta(s, \lambda^d) - \frac{(\pi/4)\delta_{d,0}}{s-1} e^{-|s-1|} \ll_\varepsilon 1, \tag{3.57}$$

$$\sum_{n=1}^\infty \rho\left(\frac{n}{x}\right) \delta(\Lambda^d, n) n^{-s} \ll_\varepsilon 1, \tag{3.58}$$

and

$$X_d(s) \sum_{n=1}^\infty \tilde{\rho}_K\left(\frac{n}{y}\right) \delta(\Lambda^{-d}, n) n^{s-1} \ll_{\varepsilon, K} 1, \tag{3.59}$$

where (both here and subsequently) the implicit constants may depend on the choice of function $\rho(u)$ and associated constant $b > 1$.

Since $(1 - \exp(-|s - 1|))/|s - 1| < 1$, the bound (3.57) is a consequence of (3.45), (3.54) and the bound (3.39) of Lemma 5. The bound (3.58) is straightforward to verify, given (3.44), (3.45), (3.55) and the upper bound in (2.12). If $T \geq T(0, 1/2)$, then it is similarly straightforward to verify (3.59), since then (3.45), (3.54) and the case $\eta = 1/4$ of the bound (3.10) in Lemma 3 show that $X_d(s) \ll \ll T^{5/6} \ll_\varepsilon 1$, while (3.20) of Lemma 4 implies that $a_k(d, s) \ll_k T^{-k/4} \ll_K 1$, for $k = 1, 2, \dots, K$.

If $T < T(0, 1/2)$, then by (3.3)–(3.6) of Lemma 2 one has $d = 0$ and $|t| < 1/2$. Though Lemma 4 itself does not cover this, one may in such a case nevertheless persist in defining the sequence $(a_k(d, s))_{k \in \mathbb{N}}$ through (3.17)–(3.19), given that (3.45) and the hypothesis that $s \notin \{-2|d|, 2|d| + 1\} = \{0, 1\}$ combine to imply that $G_0(s, \tau)$, in (3.17), is well defined and analytic on the non-empty open set $\{\tau \in \mathbb{C} : |\tau| < |1 - s|\}$. In particular, if $d = 0$ then $X_d(s) = X_0(s) = \pi^{2s-1} \times \Gamma(1 - s)/\Gamma(s) \in \mathbb{C} - \{0\}$ and, by (3.17), $G_d(s, 0) = T^0 - 1 = 0$. Therefore, when $T < T(0, 1/2)$ in (3.46), one may verify (3.59) just by noting, firstly, that (3.47) and the inequality (3.6) of Lemma 2 then imply the inequality $by < 1$, and, secondly, that by this inequality, (3.44) and (3.49), it follows that each summand in (3.59) equals zero.

The above completes the verification of (3.57)–(3.59), and so the case of the lemma in which (3.53) holds has been proved. The rest of this proof need only deal with the case complementary to (3.53), so it is henceforth to be supposed that

$$T > (6B)^{4/(1-4\varepsilon)}. \tag{3.60}$$

As in [25, Equation (21)], put

$$R(z) = \int_0^\infty \rho(u)u^{z-1} du \quad \text{for } \operatorname{Re}(z) > 0. \tag{3.61}$$

Then (again as in [25]) it follows by (3.43), (3.44) and integration by parts that, where defined,

$$R(z) = -\frac{1}{z} \int_{1/b}^b \rho^{(1)}(u)u^z du, \tag{3.62}$$

and that by means of this representation, (3.62), the function $R(z)$ has an analytic continuation to $\mathbb{C} - \{0\}$ with, at $z = 0$, a simple pole having residue equal to $\rho(1/b) - \rho(b) = 1 - 0 = 1$. Given (3.62), the condition (3.43) has the pleasing effect of ensuring that

$$R(-z) = -R(z) \quad \text{for all } z \in \mathbb{C} - \{0\}. \tag{3.63}$$

By (3.62) and the application of further integrations by parts, one obtains bounds for $R(z)$ which, in combination with (3.63), imply the bounds:

$$R(z) \ll_j \frac{b^{|\operatorname{Re}(z)|}}{|z|(|z| + 1)^{j-1}} \quad \text{for } j \in \mathbb{N} \text{ and } z \in \mathbb{C} - \{0\}, \quad (3.64)$$

with the relevant implicit constants being $b^{j-1} \int_{1/b}^b |\rho^{(j)}(u)| du$ ($j = 1, 2, \dots$).

Since $\rho(u)$ is infinitely differentiable, the hypotheses (3.43) and (3.44) ensure that, given (3.61) and the case $j = 2$ of (3.64), it follows from the Mellin inversion formula of [26, Equation (A.2)] that one has

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} R(z)u^{-z} dz = \rho(u) \quad \text{for } c, u > 0. \quad (3.65)$$

Differentiating under the integral sign here one obtains:

$$-\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} zR(z)u^{-z-1} dz = \rho^{(1)}(u)$$

(a result justified by virtue of the fact that the case $j = 3$ of (3.64) implies that the improper integral on the left hand side of the equation is uniformly convergent for all u in any given bounded closed interval $[u_0, u_1] \subset (0, \infty)$). Upon multiplying both sides of the above equation by u , one obtains the case $k = 1$ of the equation

$$\frac{(-1)^k}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^k R(z)u^{-z} dz = \left(u \frac{d}{du}\right)^k \rho(u) = \rho_k(u) \quad (\text{say}), \quad \text{for } c, u > 0. \quad (3.66)$$

The general case (all $k \in \mathbb{N}$) follows by induction (appealing, at the k -th step, to the case $j = k + 3$ of (3.64)).

Applying either (3.65) or (3.66), for $u = n/x$, and then multiplying the result by $\delta(\Lambda^d, n)n^{-s}$, before summing over $n \in \mathbb{N}$, one finds by (3.44), (2.9) and (2.12) (which guarantee the required uniform absolute convergence of $\sum_{n=1}^{\infty} \delta(\Lambda^d, n)n^{-s-z}$)

that if $x > 0$ and $c > \max\{0, 1 - \sigma\}$, then

$$\frac{(-1)^k}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^k R(z) x^z \zeta(s+z, \lambda^d) dz = \sum_{n=1}^{\infty} \rho_k\left(\frac{n}{x}\right) \frac{\delta(\Lambda^d, n)}{n^s} \quad \text{for } k \in \mathbb{N} \cup \{0\}, \tag{3.67}$$

where $\rho_k(u) = \rho(u)$ if $k = 0$, and otherwise is as in (3.66).

A deduction may now be made from (3.67) which, though it is a digression from the central theme of this proof, will later serve to keep the error term estimate (3.50) as simple as it is. For this purpose it is helpful to temporarily restrict the discussion to the special case of a function $\rho(u)$ such that (3.44) holds with $b = \sqrt{2}$ (there need be no question concerning the existence of such a function $\rho(u)$, since the hypotheses of the lemma imply that if the given $\rho(u)$ is not itself of the type sought, then for some choice of constant $q > 0$ the functions $\rho(u^q)$ will be). In this special case, the application of (3.67) with $k = 0$, $s = 1/4 + it$ and $x = b = \sqrt{2}$ shows that

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} R(z) b^z \zeta\left(1/4 + it + z, \lambda^d\right) dz = 1.$$

Indeed, by (3.43), (3.44) and (2.10), one has $\rho(n/x) = 0$ for $n \geq 2$, while $\rho(1/x) = \rho(1/b) = 1 - \rho(b) = 1$ and $\delta(\Lambda^d, 1) = 1$. By the estimate (3.39) of Lemma 5, together with (3.8) and the bounds of (3.64), it is permissible to shift the above line of integration to $\text{Re}(z) = 1/4$, though if $d = 0$ then the pole of $\zeta(1/4 + it + z, \lambda^d)$ at $z = 3/4 - it$ makes it necessary to adjust the shifted integral by addition of the appropriate residue. Then, by substituting $z = 1/4 + iv$, one obtains:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} R\left(\frac{1}{4} + iv\right) b^{1/4+iv} \zeta\left(\frac{1}{2} + i(t+v), \lambda^d\right) dv + (\pi/4) \delta_{d,0} b^{3/4-it} R\left(\frac{3}{4} - it\right) = 1, \tag{3.68}$$

where $\delta_{m,m} = 1$ and $\delta_{m,n} = 0$ if $m \neq n$. Using (3.8), the bound (3.39) of Lemma 5 and the case $j = [3/(2\epsilon)] + 3$ (say) of (3.64), one deduces from equation (3.68) that

$$\frac{1}{2\pi} \int_{-T^\epsilon}^{T^\epsilon} R\left(\frac{1}{4} + iv\right) b^{1/4+iv} \zeta\left(\frac{1}{2} + i(t+v), \lambda^d\right) dv + O_\epsilon\left(T^{-1}\right) = 1$$

(any dependence of the implicit constant here upon $\rho(u)$ may be effectively nullified through a «one time only» selection of $\rho(u)$ from amongst the set of all functions having the required properties). Since it is evidently not possible that both of the two terms on the left side of this last equation have absolute value less than $1/2$, one therefore must have either

$$T \ll_{\varepsilon} 1, \tag{3.69}$$

or else

$$\int_{-T^{\varepsilon}}^{T^{\varepsilon}} \left| R\left(\frac{1}{4} + iv\right) \zeta\left(\frac{1}{2} + i(t+v), \lambda^d\right) \right| dv \geq \pi b^{-1/4}. \tag{3.70}$$

As a corollary of the dichotomy just discerned one obtains a useful lower bound,

$$\int_{-T^{\varepsilon}}^{T^{\varepsilon}} \left| \zeta\left(\frac{1}{2} + i(t+v), \lambda^d\right) \right| \frac{dv}{1+v^2} \gg_{\varepsilon} 1, \tag{3.71}$$

for if T satisfies (3.69) then (3.71) follows by reasoning similar to that used to obtain (3.56) (subject to (3.53)), whereas if it is instead (3.70) which holds, then (3.71) follows from (3.70) by use of the case $j = 2$ of (3.64). It is certain that the lower bound (3.71) could be greatly improved, and refined upon, by very generally applicable methods of Balasubramanian and Ramachandra (for which see [42,2,43]).

With the bound (3.71) established, the digression on the special case $b = \sqrt{2}$ is now ended. Returning to consideration of the general case (though with (3.60) still in force) suppose now that x and y are as in (3.46) and (3.47). By applying (3.67) for $k = 0$ and $c = 3/2$ (so that $c + \sigma \geq 7/6 > 1$) and then shifting the contour of integration to the line $\text{Re}(z) = -c = -3/2$ (say), one arrives at the equation

$$\begin{aligned} \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} R(z)x^z \zeta(s+z, \lambda^d) dz + \zeta(s, \lambda^d) + \delta_{d,0}R(1-s)x^{1-s}\pi/4 = \\ = \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) \frac{\delta(\Lambda^d, n)}{n^s}, \end{aligned}$$

in which appear the residue of the integrand at $z = 0$ (the pole of $R(z)$) and, if $d = 0$, the residue at $z = 1 - s \neq 0$ (the pole of $\zeta(s+z, \lambda^0)$): the shifting of

the contour may be justified using the bounds (3.64), along with the result (3.8) of Corollary 1 and the estimate for $\zeta(s, \lambda^d)$ implicit in the result (3.39) of Lemma 5, which also contains the information needed for the residue calculation at $z = 1 - s$. Using the functional equation (2.5), and the substitution $z = -\tau$, one may rewrite the integral in the equation that was just arrived at, and so obtain a reformulation of that equation; by this reformulation, followed by two applications of (3.63), one finds that

$$\zeta(s, \lambda^d) = \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) \frac{\delta(\Lambda^d, n)}{n^s} + (\pi/4)\delta_{d,0} \frac{R(s-1)}{x^{s-1}} + I(d, s), \tag{3.72}$$

where

$$I(d, s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} X_d(s-\tau) \zeta(1-s+\tau, \lambda^{-d}) \frac{R(\tau)}{x^\tau} d\tau, \tag{3.73}$$

with (as before) $c = 3/2$.

In the integral of (3.73) one has $\text{Re}(1-s+\tau) = 1-\sigma+3/2 \geq 7/6 > 1$, by (3.45), and therefore $\zeta(1-s+\tau, \lambda^d) \ll 1$ (as follows, for example, by the case $\varepsilon = 1/12$ of the result (3.39) of Lemma 5). By (3.64) and (3.45), one has also

$$R(\tau) \ll_j b^{3/2} |\tau|^{-j} \ll |\tau|^{-j} \quad \text{if } \sigma - 1/2 \leq \text{Re}(\tau) \leq 3/2, \tau \neq 0 \quad \text{and } j \in \mathbb{N},$$

while, by (3.45) and (3.8), the bound (3.10) of Lemma 3 shows that

$$X_d(s-\tau) \ll |2|d|+1/2+i(t-\text{Im}(\tau))|^{2(2-\sigma)} \ll (T|\tau|^2)^{2-\sigma} \ll T^3 |\tau|^5 \text{ when } \text{Re}(\tau)=3/2.$$

Using the last two bounds to estimate parts of the integral in (3.73), one finds that

$$I(d, s) = \frac{1}{2\pi i} \int_{c-iT^\varepsilon}^{c+iT^\varepsilon} X_d(s-\tau) \zeta(1-s+\tau, \lambda^{-d}) \frac{R(\tau)}{x^\tau} d\tau + O_j\left(T^{3-(j-6)\varepsilon} x^{-3/2}\right), \tag{3.74}$$

for each integer j such that $j \geq 7$. By (3.45), $1/2 - \sigma > -3/2$, while by (3.46) and (3.47) one has $T \gg x \gg 1$ here, so that on restricting to the case $j = [(A+3)/\varepsilon] + 7$ one obtains, in (3.74),

$$O_j\left(T^{3-(j-6)\varepsilon} x^{-3/2}\right) \ll_{A,\varepsilon} x^{1/2-\sigma} T^{-A} \quad \text{for } 0 \leq A < \infty. \tag{3.75}$$

By (3.45), (3.46), (3.60) and the inequality (3.6) of Lemma 2, the conditions (3.15) and (3.16) of Lemma 4 are satisfied, so by Lemma 4 and Corollary 2 one has:

$$X_d(s - \tau) = X_d(s)T^\tau (1 + G_d(s, \tau)), \quad (3.76)$$

where $G_d(s, \tau)$ is analytic (as a function of τ) on the open disc $\{\tau \in \mathbb{C} : |\tau| < R_{d,s}\}$ with radius $R_{d,s} = |2|d| + 1 - s| \geq T^{1/4}/B$. Now, if τ satisfies

$$\sigma - 1/2 \leq \operatorname{Re}(\tau) \leq 3/2 \quad \text{and} \quad |\operatorname{Im}(\tau)| \leq T^\varepsilon, \quad (3.77)$$

then, by (3.45) and (3.60), one has also

$$|\tau| < 3/2 + T^\varepsilon < T^{1/4}/(4B) + T^{1/4}/(6B) < T^{1/4}/(2B), \quad (3.78)$$

so that the bound (3.38) of Corollary 2 applies. For later reference, observe moreover that the last two inequalities of (3.78) certainly demonstrate that if $d = 0$ then one has

$$|1 - \sigma| + |t| \geq |1 - s| = R_{0,s} \geq T^{1/4}/B > 3 + 2T^\varepsilon > |1 - \sigma| + 2T^\varepsilon$$

(given (3.45)). Consequently

$$|t| > 2T^\varepsilon \quad \text{if } d = 0. \quad (3.79)$$

Suppose now that K is an arbitrary non-negative integer. Since the conditions in (3.77) imply the inequalities in (3.78), and since $R_{d,s} \geq T^{1/4}/B$, it follows by (3.76), and the identity (3.19) of Lemma 4, that one may rewrite the integrand in (3.74) by using:

$$X_d(s - \tau) = X_d(s) (T^\tau P_{K,d,s}^+(\tau) + T^\tau P_{K,d,s}^-(\tau)), \quad (3.80)$$

where

$$P_{K,d,s}^+(\tau) = 1 + \sum_{k=1}^K a_k(d, s)\tau^k = \sum_{k=0}^K a_k(d, s)\tau^k \quad (\text{with } a_0(d, s) = 1) \quad (3.81)$$

and

$$P_{K,d,s}^-(\tau) = 1 + G_d(s, \tau) - P_{K,d,s}^+(\tau) = \sum_{k=K+1}^{\infty} a_k(d, s) \tau^k. \quad (3.82)$$

Using (3.80) and (3.75) in (3.74), one finds that, for arbitrary $A \in [0, \infty)$, one has

$$I(d, s) = X_d(s) (J_{\varepsilon,K}^+(d, s) + J_{\varepsilon,K}^-(d, s)) + O_{A,\varepsilon} \left(x^{1/2-\sigma} T^{-A} \right), \quad (3.83)$$

where (given (3.46) and (2.3))

$$J_{\varepsilon,K}^{\pm}(d, s) = \frac{1}{2\pi i} \int_{c-iT^{\varepsilon}}^{c+iT^{\varepsilon}} P_{K,d,s}^{\pm}(\tau) R(\tau) y^{\tau} \zeta \left(1 - s + \tau, \lambda^{\mp d} \right) d\tau. \quad (3.84)$$

By (3.81), the bounds (3.20) of Lemma 4, and the lower bound on $T(d, t)$ in (3.60), one has

$$P_{K,d,s}^+(\tau) \ll 1 + \sum_{k=1}^K B^k T^{-k/4} |\tau|^k \ll_K |\tau|^K \quad \text{for } |\tau| \geq T^{\varepsilon}.$$

By this last bound, together with (3.64) and the fact (noted while obtaining (3.74)) that $\zeta(z, \lambda^n) \ll 1$ when $\text{Re}(z) \geq 7/6$, it follows from (3.84) that

$$J_{\varepsilon,K}^+(d, s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} P_{K,d,s}^+(\tau) R(\tau) y^{\tau} \zeta \left(1 - s + \tau, \lambda^{-d} \right) d\tau + O_{K,j} \left(T^{-(j-K-1)\varepsilon} y^{3/2} \right), \quad (3.85)$$

for each integer j that satisfies $j \geq K + 2$. With regard to the O -term in (3.85), observe that if $A \in [0, \infty)$, and if one puts $j = [(A + 3)/\varepsilon] + K + 2$, then by (3.45), (3.47) and (3.60) it follows that

$$T^{-(j-K-1)\varepsilon} y^{3/2} \ll y^{\sigma-1/2} T^{3-(j-K-1)\varepsilon} \leq y^{\sigma-1/2} T^{-A}. \quad (3.86)$$

By (3.85), (3.86), (3.81), and (3.67), one now has (when $0 \leq A < \infty$)

$$\begin{aligned} J_{\varepsilon,K}^+(d,s) &= O_{A,K,\varepsilon} \left(y^{\sigma-1/2} T^{-A} \right) + \sum_{k=0}^K \frac{a_k(d,s)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tau^k R(\tau) y^\tau \zeta(1-s+\tau, \lambda^{-d}) d\tau = \\ &= O_{A,K,\varepsilon} \left(y^{\sigma-1/2} T^{-A} \right) + \sum_{k=0}^K (-1)^k a_k(d,s) \sum_{n=1}^{\infty} \rho_k \left(\frac{n}{y} \right) \frac{\delta(\Lambda^{-d}, n)}{n^{1-s}}, \end{aligned}$$

where $a_0(d,s) = 1$, while $\rho_k(u)$ is as in (3.66). Therefore, with $\tilde{\rho}_K(u)$ defined as in (3.49), one has

$$J_{\varepsilon,K}^+(d,s) = O_{A,K,\varepsilon} \left(y^{\sigma-1/2} T^{-A} \right) + \sum_{n=1}^{\infty} \tilde{\rho}_K \left(\frac{n}{y} \right) \delta(\Lambda^{-d}, n) n^{s-1} \quad \text{for all } A \in [0, \infty). \tag{3.87}$$

By (3.72), (3.83), and (3.87), the term $E_K(s, \lambda^d)$ in (3.48) satisfies

$$\begin{aligned} E_K(s, \lambda^d) &= (\pi/4) \delta_{d,0} \left(\frac{R(s-1)}{x^{s-1}} - \frac{e^{-|s-1|}}{s-1} \right) + \\ &+ X_d(s) \left(J_{\varepsilon,K}^-(d,s) + O_{A,K,\varepsilon} \left(y^{\sigma-1/2} T^{-A} \right) \right) + O_{A,\varepsilon} \left(x^{1/2-\sigma} T^{-A} \right), \end{aligned} \tag{3.88}$$

where the constant $A \in [0, \infty)$ is arbitrary. If $d \neq 0$, then $\delta_{d,0} = 0$, whereas if $d = 0$ then $\delta_{d,0} = 1$ and, by (3.64) and (3.79), and (3.45)–(3.47), one has

$$\left| \frac{R(s-1)}{x^{s-1}} \right| + \left| \frac{e^{-|s-1|}}{s-1} \right| \leq O_j \left(\frac{b^{4/3}}{T^{j\varepsilon}} x^{1-\sigma} \right) + \frac{(j-1)!}{T^{j\varepsilon}} \ll_{A,\varepsilon} x^{1/2-\sigma} T^{1-j\varepsilon} \leq x^{1/2-\sigma} T^{-A},$$

for each pair $(j, A) \in \mathbb{Z} \times [0, \infty)$ such that $j = [(A+1)/\varepsilon] + 1$. In either of the cases just mentioned it follows by (3.45)–(3.47), (3.60), and (3.6) of Lemma 2, that the bound (3.10) of Lemma 3 gives

$$X_d(s) \ll T^{1/2-\sigma} = (xy)^{1/2-\sigma}.$$

Therefore, and by (3.88), one has:

$$E_K(s, \lambda^d) \ll (xy)^{1/2-\sigma} J_{\varepsilon,K}^-(d,s) + O_{A,K,\varepsilon} \left(x^{1/2-\sigma} T^{-A} \right),$$

for an arbitrary constant $A \in [0, \infty)$. Since one may choose to put $A = \beta_K$ here (with β_k as in (3.50)–(3.52)), and since it was found that (3.71) holds, the bound (3.50) will now follow if it can be proved that

$$J_{\varepsilon, K}^-(d, s) \ll_{K, \varepsilon} y^{\sigma-1/2} \left((|t|/T)^{\alpha_K} + T^{-\beta_K} \right) \int_{-T^\varepsilon}^{T^\varepsilon} \left| \zeta \left(1/2 + i(t+v), \lambda^d \right) \right| \frac{dv}{1+v^2}, \quad (3.89)$$

with α_K and β_K as in (3.52).

To obtain (3.89) consider first the analytic function $P_{K, d, s}^-(\tau)$ occurring in (3.82) and (3.84). For τ satisfying the conditions (3.77), the inequalities in (3.78) hold, so it follows by (3.82) and Corollary 2 (the bound (3.38) in particular) that $P_{K, d, s}^-(\tau)$ is analytic at all points of the rectangular subset of \mathbb{C} given by (3.77), and on that subset satisfies:

$$P_{K, d, s}^-(\tau) \ll_K \left((|t|/T)^{(K+1)/2} + T^{-(K+1)/3} \right) |\tau|^{K+1} \quad (3.90)$$

(having, in particular, a zero of order $K + 1 \geq 1$ at $\tau = 0$). By this and (3.79), the integrand in the « $-$ » case of (3.84) has no poles either on, or inside of, the rectangle given by (3.77). Therefore

$$2\pi i J_{\varepsilon, K}^-(d, s) = \mathcal{J}(\overline{\kappa}, \overline{\mu}) + \mathcal{J}(\overline{\mu}, \mu) + \mathcal{J}(\mu, \kappa) \text{ for } \kappa = c + iT^\varepsilon, \mu = \sigma - 1/2 + iT^\varepsilon, \quad (3.91)$$

where $c = 3/2$ and, for the relevant pairs of points, $\nu, \omega \in \mathbb{C}$,

$$\mathcal{J}(\nu, \omega) = \int_{\nu}^{\omega} P_{K, d, s}^-(\tau) R(\tau) y^\tau \zeta \left(1 - s + \tau, \lambda^d \right) d\tau \quad (3.92)$$

with the contour of integration being the line segment between ν and ω .

By (3.8) and (3.45), the bound (3.90) (valid in the rectangle (3.77)) certainly implies that one has

$$P_{K, d, s}^-(\tau) \ll_K T^{-(K+1)/4} |\tau|^{K+1} \ll_K T^{-(1/4-\varepsilon)(K+1)} \ll_K 1$$

in the integrands of $\mathcal{J}(\overline{\kappa}, \overline{\mu})$ and $\mathcal{J}(\mu, \kappa)$ in (3.91)–(3.92). Moreover, in those integrands $\text{Re}(1 - s + \tau) \geq 1/2$ and, by (3.79) and (3.8), $\delta_{d,0} T^\varepsilon < |\text{Im}(1 - s + \tau)| \leq |t| + T^\varepsilon \ll |t| + |d|$ (where $\delta_{d,0}$ is as in (3.68)), with it therefore following by the

estimate (3.39) of Lemma 5 (and (3.8), again) that one has there:

$$\zeta \left(1 - s + \tau, \lambda^d \right) \ll_{\varepsilon} (1 + |t| + |d|)^{1/2+2\varepsilon} \asymp T^{1/4+\varepsilon}.$$

Given (3.45) and (3.47), the bound (3.64) (for $R(z)$) and points noted after (3.92) imply that

$$\max\{|\mathcal{J}(\overline{\kappa}, \overline{\mu})|, |\mathcal{J}(\mu, \kappa)|\} \ll_{K, \varepsilon, j} (2 - \sigma)(by)^{3/2} T^{-j\varepsilon+1/4+\varepsilon} \ll y^{3/2} T^{1/4-(j-1)\varepsilon} \text{ for } j \in \mathbb{N}. \tag{3.93}$$

A suitable bound for the integral $\mathcal{J}(\overline{\mu}, \mu)$ in (3.91)–(3.92) may be obtained by applying the bound (3.90) for the factor $P_{K,d,s}^-(\tau)$ in (3.92), while using the case $j = K + 3$ of (3.64) as a bound for $R(\tau)$ in (3.92); by following this with an application of the substitution $\tau = \sigma - 1/2 - iv$ and an application of the identity $|\zeta(z, \lambda^d)| = |\zeta(\overline{z}, \lambda^d)|$ (which is a corollary of (2.4)), one finds, in particular, that

$$\mathcal{J}(\overline{\mu}, \mu) \ll_K y^{\sigma-1/2} \left(\left(\frac{|t|}{T} \right)^{(K+1)/2} + T^{-(K+1)/3} \right) \int_{-T^\varepsilon}^{T^\varepsilon} \frac{|\zeta(1/2 + i(t+v), \lambda^d)| |b|^{\sigma-1/2}}{(1 + |\sigma - 1/2 + iv|)^2} dv. \tag{3.94}$$

If $K \geq 3$, then, given (3.52) and (3.71), the desired bound for $J_{\varepsilon,K}^-(d, s)$ (which is (3.89)) follows from (3.91), (3.93) and (3.94) on choosing j , in (3.93), to be large enough to ensure that

$$y^{3/2} T^{1/4-(j-1)\varepsilon} \ll y^{\sigma-1/2} T^{-(K+1)/3}$$

(by (3.45) and (3.47), the choice $j = [(K+9)/(3\varepsilon)] + 2$ achieves this). Consequently (given the discussion which preceded (3.89)) the bound (3.50) of the lemma has now been shown to hold when $K \geq 3$.

In those cases in which the estimate (3.50) is, as yet, unproved (that is, the cases in which one has $K \in \{0, 1, 2\}$), one can improve on the bound (3.94) for $\mathcal{J}(\overline{\mu}, \mu)$ by combining the estimate

$$P_{K,d,s}^-(\tau) = \sum_{k=K+1}^3 a_k(d, s) \tau^k + O \left(\left((|t|/T)^2 + T^{-4/3} \right) |\tau|^4 \right)$$

(from (3.82) and the case $K = 3$ of (3.90)) with the estimates for a_1, a_2, a_3 in (3.20)–(3.22) of Lemma 4. These combined estimates show that if $\tau = \sigma - 1/2 + iv$ with

$-T^\epsilon \leq v \leq T^\epsilon < T^{1/4}$, then, for $K = 0, 1, 2$, one has

$$P_{K,d,s}^-(\tau) = O\left(\frac{|\tau|^{K+1} + |\tau|^3}{T} + \left(\frac{|t|}{T}\right)^2 |\tau|^4\right) + \begin{cases} 0 & \text{if } K = 2, \\ it|z_{d,t}|^{-2}\tau^2 & \text{if } K = 1, \\ -it|z_{d,t}|^{-2}|\tau|^2 & \text{if } K = 0, \end{cases}$$

so that

$$P_{K,d,s}^-(\tau) \ll \left(|t|/T\right)^{\alpha_K} + T^{-\beta_K} \left|\tau\right|^{K+1} (1 + |\tau|)^2 \quad (K = 0, 1, 2), \quad (3.95)$$

with α_K, β_K as in (3.52) (note that $z_{d,t} = 2|d| + 1/2 + it$ here, so that by (3.8) one has $|z_{d,t}|^2 \asymp T$ and $|t|/|z_{d,t}|^2 \asymp |t|/T \ll T^{-1/2}$, while $|\tau|^2 = (\sigma - 1/2)^2 + v^2 < 1 + T^{2\epsilon} \ll \ll T^{1/2}$).

By using the bound (3.95) and the case $j = K + 5$ of (3.64) to estimate the integral $\mathcal{J}(\bar{\mu}, \mu)$ in (3.91)–(3.92), one improves on the bound (3.94) to the extent that the exponents $(K+1)/2$ and $-(K+1)/3$ (in the factor $(|t|/T)^{(K+1)/2} + T^{-(K+1)/3}$ on the right-hand side of (3.94)) are sharpened to α_K and $-\beta_K$, respectively. By reasoning similar to that employed in the paragraph below (3.94) it now follows (given the improvement of (3.94) just obtained) that (3.89) holds for each $K \in \{0, 1, 2\}$, and that the result (3.50) of the lemma has, as a consequence, now been proved for these cases, which complement the case $K \geq 3$ established earlier.

The proof may now be completed with the help of Corollary 1 and the sub-convexity bound (3.41) of Kaufman. Indeed, by virtue of those results, the factor $|\zeta(1/2 + i(t+v), \lambda^d)|$ occurring in the integrand in (3.50) is uniformly bounded above by $O_\epsilon(T^{1/6+\epsilon})$, and so, given that one has $\int_{-\infty}^\infty (1+v^2)^{-1} dv < \infty$, the estimate (3.51) follows from (3.50). □

COROLLARY 3. *Let $\rho : (0, \infty) \rightarrow \mathbb{R}$ and $b \in (1, \infty)$ be as in the above lemma. Suppose that $h \geq 1$, that $0 < \eta \leq 1 \leq T_*$ and that $\epsilon > 0$. Then, provided that T_* is sufficiently large (in terms of b and η), one has*

$$\zeta^h\left(1/2 + it, \lambda^d\right) \ll \frac{3^h}{\eta} \int_{-2\eta}^{2\eta} \left| \sum_{n=1}^\infty \rho\left(\frac{n}{e^{\theta T_*^{1/2}}}\right) \delta\left(\Lambda^d, n\right) n^{-1/2-it} \right|^h d\theta +$$

$$+ O_{\rho,h,\varepsilon} \left(T_*^{(\varepsilon-1/3)h} \right),$$

for all $d \in \mathbb{Z}$ and all $t \in \mathbb{R}$ such that

$$e^{-\eta T_*} \leq \pi^{-2} |2d + it|^2 \leq e^{\eta T_*}. \tag{3.96}$$

PROOF. By (2.6), $|X_d(s)| = 1$ if $\text{Re}(s) = 1/2$, so, for d, t as in (3.96), it follows from the case $\sigma = 1/2, K = 0$ of Lemma 6 that

$$\begin{aligned} \left| \zeta \left(1/2 + it, \lambda^d \right) \right| &\leq \left| \sum_{n=1}^{\infty} \rho \left(\frac{n}{x} \right) \delta \left(\Lambda^d, n \right) n^{-1/2-it} \right| + \\ &+ \left| \sum_{n=1}^{\infty} \rho \left(\frac{n}{y} \right) \delta \left(\Lambda^{-d}, n \right) n^{-1/2+it} \right| + \\ &+ O \left(T_*^{-1/2} \exp \left(-\pi \sqrt{T_*/e} \right) \right) + O_{\rho,\varepsilon} \left(T^{1/6+\varepsilon} \left((|t|/T) + T^{-1} \right) \right), \end{aligned} \tag{3.97}$$

were $T = T(d, t)$, and x, y is any pair of real numbers satisfying (3.46) and (3.47). By the condition (3.96), the result (3.7) of Lemma 2 and the hypothesis that T_* is sufficiently large (in terms of η), one may assume here that

$$e^{-2\eta T_*} \leq T \leq e^{2\eta T_*}. \tag{3.98}$$

By this and (3.96), the O -terms in (3.97) are not greater than $O_{\rho,\varepsilon} (T_*^{\varepsilon-1/3})$. Moreover, using (3.98) one can check that the condition (3.47) of Lemma 6 is satisfied by all y lying in the interval $[e^{-\eta\sqrt{T}}, e^{\eta\sqrt{T}}]$ (given the hypotheses that $\eta \leq 1$ and that T_* is sufficiently large in terms of b). Since (3.46) implies $y^{-1}dy = -x^{-1}dx$, it follows by (3.98) and the above discussion that, through an application of Hölder’s inequality, followed by integration with respect to y , one may deduce from (3.97) that

$$\begin{aligned} &2\eta \left| \zeta \left(1/2 + it, \lambda^d \right) \right|^h \leq \\ &\leq 3^{h-1} \sum_{w=\pm 1} \int_{e^{-\eta\sqrt{T}}}^{e^{\eta\sqrt{T}}} \left| \sum_{n=1}^{\infty} \rho \left(\frac{n}{x} \right) \delta \left(\Lambda^{wd}, n \right) n^{-1/2-wit} \right|^h x^{-1} dx + O_{\rho,h,\varepsilon} \left(\eta T_*^{(\varepsilon-1/3)h} \right). \end{aligned}$$

Since $\rho(u) = \overline{\rho(u)}$ and $\delta(\Lambda^{-d}, n) = \overline{\delta(\Lambda^d, n)}$, and since $|\bar{z}| = |z| \geq 0$ for $z \in \mathbb{C}$, the corollary now follows by appealing to (3.98) and then making the substitution $x = T_*^{1/2} e^\theta$. \square

Remark 3.1. Suppose that $b > 1$. Then one example of a function $\rho : (0, \infty) \rightarrow \mathbb{R}$ satisfying the hypotheses of Lemma 6 is that which is given by:

$$\rho(u) = \left(\int_{-\infty}^{\infty} \Phi(t) dt \right)^{-1} \int_{(\log u)/(\log b)}^{\infty} \Phi(t) dt \quad (u > 0), \quad (3.99)$$

with

$$\Phi(t) = \begin{cases} \exp\left(-\left(1-t^2\right)^{-1}\right) & \text{if } -1 < t < 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.100)$$

We leave it to the reader to verify that this ρ is indeed an infinitely differentiable function on $(0, \infty)$ satisfying both of the conditions (3.43) and (3.44). Since the function Φ defined in (3.100) takes only non-negative real values, it follows from (3.99) and (3.100) that the range of the function ρ is the interval $[0, 1]$. It can moreover be shown (by induction) that this function ρ is such that

$$\rho^{(j)}(u) = u^{-j} \sum_{k=0}^{j-1} r(j, k) (\log b)^{-(k+1)} \Phi^{(k)}\left(\frac{\log u}{\log b}\right) \quad (j \in \mathbb{N} \text{ and } u > 0),$$

where each coefficient $r(j, k)$ is an integer that is completely determined by the pair j, k . Given the definition (3.100), one can furthermore establish that $\sup\{|\Phi^{(k)}(t)| : t \in \mathbb{R}\} < \infty$ for all $k \in \mathbb{N} \cup \{0\}$. It therefore follows that if $1 < b \leq e$, and if $\rho : (0, \infty) \rightarrow \mathbb{R}$ is given by (3.99) and (3.100), then one has

$$\rho^{(j)}(u) \ll_j (u \log b)^{-j} \quad (j \in \mathbb{N} \cup \{0\} \text{ and } u > 0). \quad (3.101)$$

4. Proof that Theorem 2 implies Theorem 1

In this section we show that Theorem 1 is a corollary of Theorem 2: the proof of the latter theorem is divided between Sections 6 and 7.

Throughout this section we assume that the hypotheses of Theorem 1 concerning ϑ and ε are satisfied. We assume also that ε satisfies $\varepsilon \leq 1$: no loss of generality results from this, for the cases of Theorem 1 in which one has $\varepsilon > 1$ are a trivial corollary of the case in which one has $\varepsilon = 1$. We suppose, furthermore, that A is some complex valued function with domain $\mathfrak{D} - \{0\}$. For $M \geq 1$ and $D \geq M$, we put

$$E^*(D; M, A) = E(D; M, A) - E(M; M, A), \tag{4.1}$$

where $E(D; M, A)$ is as defined in Section 1.

Given (1.2) and (1.4), it follows by the Cauchy–Schwarz inequality and elementary estimates that

$$\left| P_M(A; it, \lambda^d) \right|^2 \leq 8M \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \leq 64M^2 \max_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \quad (t \in \mathbb{R}, d \in \mathbb{Z}, M \geq 1). \tag{4.2}$$

By the definition (1.3)–(1.4) and the first part of (4.2), we find that

$$E(D; M, A) \ll \left(M \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \right) E(D; 1, U) \quad (D, M \geq 1), \tag{4.3}$$

where U is the complex function defined on $\mathfrak{D} - \{0\}$ by:

$$U(\mu) = \begin{cases} 1 & \text{if } \mu = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

We are going to show that Theorem 2 implies that, for $M \geq 1$ and $D \geq M$, one has both

$$E^*(D; M, A) \ll_{\varepsilon} \left(D^{2+\varepsilon} + (1 + DM^{-3/2})^{\vartheta} D^{1+\varepsilon} M^2 \right) \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \tag{4.5}$$

and

$$E^*(D; M, A) \ll_{\varepsilon} D^{2+\varepsilon} \left(\sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \right) + (1 + DM^{-2})^{\vartheta} D^{1+\varepsilon} M^3 \max_{0 < |\mu|^2 \leq M} |A(\mu)|^2. \tag{4.6}$$

Before proceeding to the proof of (4.5) and (4.6), we consider the implications of these bounds.

If (4.5) can be shown to hold for all $M \geq 1$ and all $D \geq M$, then (since we may substitute U , as defined in (4.4), in place of the function A) it will follow by (4.1), (4.5), (4.4), (1.3), (1.4), (1.2) and the bound (1.13) of Kim and Shahidi that, for $D \geq 1$, one has

$$\begin{aligned} E(D; 1, U) &= E^*(D; 1, U) + E(1; 1, U) = \\ &= \left(O_\varepsilon \left(D^{2+\varepsilon} \right) + O(1) \right) |U(1)|^2 \ll_\varepsilon D^{2+\varepsilon} \end{aligned} \quad (4.7)$$

(note that the implied bound $E(D; 1, U) \ll_\varepsilon D^{2+\varepsilon}$ is not a new, for it may also be obtained directly from Sarnak's sharper estimate in (1.16)). Moreover, given (4.7), it follows by (4.1) and (4.3) that, for $M \geq 1$ and $D \geq M$, one has

$$\begin{aligned} E(D; M, A) &= E^*(D; M, A) + E(M; M, A) = \\ &= E^*(D; M, A) + O \left(\left(M \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \right) E(M; 1, U) \right) = \\ &= E^*(D; M, A) + O_\varepsilon \left(M^{3+\varepsilon} \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \right). \end{aligned}$$

Therefore, given that we also have the second inequality of (4.2), we may conclude that the bounds asserted in (4.5) and (4.6) would imply that the results (1.14) and (1.15) of Theorem 1 are valid whenever one has both $M \geq 1$ and $D \geq M$. We observe also that, by (4.3), the corollary of (4.5) noted in (4.7), and the second inequality of (4.2), it follows that

$$E(D; M, A) \ll_\varepsilon D^{2+\varepsilon} M \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \ll D^{2+\varepsilon} M^2 \max_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \quad (D, M \geq 1).$$

Consequently we may also conclude that the bound asserted in (4.5) would imply that the results (1.14) and (1.15) of Theorem 1 are valid whenever one has both $M \geq 1$ and $1 \leq D < M$. This last conclusion, when combined with our earlier conclusion (in respect of the cases where $M \geq 1$ and $D \geq M$), is enough to show that Theorem 1 will follow if it can be proved that the bounds (4.5) and (4.6) hold whenever one has $M \geq 1$ and $D \geq M$.

Our task now is to show, as was promised, that Theorem 2 implies the validity of (4.5) and (4.6) for all $M \geq 1$ and all $D \geq M$; it follows from our conclusions in the preceding paragraph that the succesful completion of this task will be enough to show that Theorem 1 is a corollary of Theorem 2.

We assume henceforth that M and D are given real numbers satisfying

$$1 \leq M \leq D. \tag{4.8}$$

Note that it is a consequence of the definitions given in (1.3), (1.4) and (4.1) that all terms occurring on either side of the bounds stated in (4.5) and (4.6) are independent of the mapping that is the restriction of A to the set $\{\mu \in \mathfrak{D} : |\mu|^2 > M\}$. Therefore we may furthermore assume that $A(\mu) = 0$ for all $\mu \in \mathfrak{D}$ such that $|\mu|^2 > M$. This justifies our subsequent use of the convenient notation $\ll \|A\|_2^2 \gg$ and $\ll \|A\|_\infty^2 \gg$ to signify, respectively, the sum over μ and the maximum over μ occurring in (4.5)–(4.6) (i. e. our use of this notation will be consistent with our definition of it in (1.17)).

By (4.8), (4.1) and (1.3), it follows that

$$E^*(D; M, A) \leq \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \zeta\left(1/2+it, \lambda^d\right) \right|^4 \left| P_M\left(A; it, \lambda^d\right) \right|^2 \chi_{[M, \sqrt{5}D]}(|2d+it|) dt, \tag{4.9}$$

where $\chi_{[a,b]}(x) = |\{x\} \cap [a, b]|$.

We now put

$$\eta = (\log 2)/3 \quad \text{and} \quad b = e^{\eta/3} = 2^{1/9}, \tag{4.10}$$

and we choose an infinitely differentiable function $W : [0, \infty) \rightarrow [0, 1]$ such that

$$W(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x \geq e^\eta, \end{cases} \tag{4.11}$$

and

$$W^{(j)}(x) \ll_j x^{-j} \quad \text{for } x > 0 \text{ and } j = 0, 1, 2, \dots \tag{4.12}$$

(an example of such a function is the mapping $x \mapsto \rho(x/b)$, in which we take ρ to be as defined in (3.99) and (3.100)). Then, on taking

$$L = 1 + \left\lceil 2\eta^{-1} \log \left(\frac{\sqrt{5}D}{M} \right) \right\rceil \quad (4.13)$$

(which, by virtue of (4.8) and (4.10), makes $L \geq 7$), one has

$$W \left(\frac{x^2}{5D^2} \right) - W \left(\frac{x^2}{5D^2 e^{-(L+1)\eta}} \right) \geq \chi_{[M, \sqrt{5}D]}(x) \quad \text{for } x \geq 0,$$

so it follows from (4.9) that

$$E^*(D; M, A) \leq \sum_{\ell=0}^L \tilde{E}_\ell(D; M, A), \quad (4.14)$$

where

$$\tilde{E}_\ell(D; M, A) = \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \zeta \left(1/2 + it, \lambda^d \right) \right|^4 \left| P_M \left(A; it, \lambda^d \right) \right|^2 w_0 \left(\frac{|2d + it|^2}{5D^2 e^{-\ell\eta}} \right) dt \quad (4.15)$$

with

$$w_0(x) = W(x) - W(e^\eta x) \quad \text{for } x \geq 0.$$

Given (4.10), the conditions on $W(x)$, including (4.11) and (4.12), are enough to ensure that $w_0(x)$ is an infinitely differentiable function from $[0, \infty)$ into $[0, 1]$, and satisfies, for $x > 0$ and $j = 0, 1, 2, \dots$,

$$w_0^{(j)}(x) = \begin{cases} O_j \left((\eta x)^{-j} \right) & \text{for } e^{-\eta} \leq x \leq e^\eta, \\ 0 & \text{otherwise.} \end{cases} \quad (4.16)$$

For the purpose of achieving our goals in this section, it will be enough that we obtain suitable bounds for each sum $\tilde{E}_\ell(D; M, A)$ occurring in (4.14). In accordance with this latter objective, we suppose now that ℓ is one of the integers in the set

$\{0, 1, 2, \dots, L\}$, where L is as defined in (4.13). We also put

$$T_\ell = 5\pi^{-2}D^2e^{-\ell\eta}, \tag{4.17}$$

and we define ρ to be the mapping from $(0, \infty)$ to \mathbb{R} that is given by (3.99) and (3.100) (with b as specified in (4.10)). As we observed in Remark 3.1, the function $\rho : (0, \infty) \rightarrow \mathbb{R}$ satisfies the conditions (3.43) and (3.44) of Lemma 6 and is infinitely differentiable on $(0, \infty)$. By the case $b = 2^{1/9}$ of (3.101), one has, moreover,

$$\rho^{(j)}(u) \ll_j u^{-j} \quad \text{for all } u > 0 \text{ and all } j \in \mathbb{N} \cup \{0\}. \tag{4.18}$$

Since $\ell \leq L$, it follows by (4.17), (4.13) and (4.10) that we have now

$$T_\ell \geq 2^{-1/3}\pi^{-2}M^2. \tag{4.19}$$

Given (4.10), (4.15)–(4.17) and our specific definition (in absolute terms) of the function ρ , it follows by the case $\varepsilon = 1/12$, $h = 4$, $T_* = T_\ell$ of Corollary 3 that, if T_ℓ is greater than a certain positive absolute constant, B_0 (say), then one has a bound of the form

$$\begin{aligned} \tilde{E}_\ell(D; M, A) &\ll \frac{1}{\eta} \int_{-2\eta}^{2\eta} \check{E}(T_\ell; M, A; \theta) d\theta + \\ &+ \frac{1}{T_\ell} \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| P_M(A; it, \lambda^d) \right|^2 w_0\left(\frac{|2d + it|^2}{\pi^2 T_\ell}\right) dt, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} \check{E}(T_\ell; M, A; \theta) &= \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{n=1}^{\infty} \frac{\rho(ne^{-\theta T_\ell^{-1/2}})\delta(\Lambda^d, n)}{n^{1/2+it}} \right|^4 \times \\ &\times \left| P_M(A; it, \lambda^d) \right|^2 w_0\left(\frac{|2d + it|^2}{\pi^2 T_\ell}\right) dt, \end{aligned} \tag{4.21}$$

while the implicit constant associated with the \ll notation in (4.20) is absolute. We observe (separately) that, since $w_0(x) \geq 0$ for all $x \in \mathbb{R}$, and since each mapping

$t \mapsto \zeta(1/2 + it, \lambda^d)$ is bounded on any bounded real interval, it follows from the definition (4.21), in combination with (4.15)–(4.17) and (4.10), that

$$\check{E}(T_\ell; M, A; \theta) \geq 0 \quad (-2\eta \leq \theta \leq 2\eta), \tag{4.22}$$

and that, for all $B \in (0, \infty)$, one has:

$$\frac{1}{T_\ell} \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| P_M(A; it, \lambda^d) \right|^2 w_0 \left(\frac{|2d + it|^2}{\pi^2 T_\ell} \right) dt \gg_B \tilde{E}_\ell(D; M, A) \text{ if } T_\ell \leq B. \tag{4.23}$$

We note, in particular, that if T_ℓ is not greater than B_0 (the positive absolute constant just referred to, above (4.20)), then one has $T_\ell \leq B_0$, and so, by virtue of (4.22) and the case $B = B_0$ of (4.23), the upper bound stated in (4.20)–(4.21) is valid. We may therefore conclude that, regardless of whether or not the condition $T_\ell > B_0$ is satisfied, one does have the upper bound stated in (4.20)–(4.21).

Given (4.16), one may deduce from (4.20), (4.2) and (4.19) that

$$\begin{aligned} \tilde{E}_\ell(D; M, A) &\ll M \|A\|_2^2 + \max_{-2\eta \leq \theta \leq 2\eta} \check{E}(T_\ell; M, A; \theta) = \\ &= O \left(T_\ell^{1/2} \|A\|_2^2 \right) + \max_{-2\eta \leq \theta \leq 2\eta} \check{E}(T_\ell; M, A; \theta). \end{aligned} \tag{4.24}$$

By virtue of the conclusions reached in (4.14), (4.19) and (4.24), it will now suffice (for the estimation of $E^*(D; M, A)$) that, in the cases where (4.19) holds and θ satisfies

$$-2\eta \leq \theta \leq 2\eta, \tag{4.25}$$

we obtain suitable bounds for $\check{E}(T_\ell; M, A; \theta)$. Given that the function ρ satisfies the condition (3.44) of Lemma 6, and bearing in mind the definition (4.21) of $\check{E}(T_\ell; M, A; \theta)$, we either have

$$e^\theta T_\ell^{1/2} > b^{-1}, \tag{4.26}$$

or else have $\check{E}(T_\ell; M, A; \theta) = 0$. Therefore, in our subsequent discussion of $\check{E}(T_\ell; M, A; \theta)$, we may assume that θ and T_ℓ satisfy the conditions (4.25) and (4.26). We remark that it is only when one has $M \ll 1$ that (4.26) is not implied by (4.10), (4.19) and (4.25).

In order that we may make use of Theorem 2 in bounding $\check{E}(T_\ell; M, A; \theta)$, both the sum over n occurring in (4.21) and the sum $P_M(A; it, \lambda^d)$ defined in (1.4) must first be split up into smaller subsums. In preparation for the first part of this splitting procedure, we put

$$r(u) = \rho(u) - \rho(bu) \quad \text{for } u > 0, \tag{4.27}$$

and we set

$$L_1 = \left[\left(\theta + \frac{1}{2} \log T_\ell \right) / \log b \right] + 1 \tag{4.28}$$

(so that, by (4.26) and (4.10), one has $L_1 \geq 0$). Then, by (3.44),

$$\rho(u) = \sum_{\ell_1=0}^{L_1} r\left(b^{\ell_1} u\right) \quad \text{if } u \geq e^{-\theta} T_\ell^{-1/2},$$

and so it follows by Hölder’s inequality that, for all $d \in \mathbb{Z}$ and all $t \in \mathbb{R}$, one has

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \rho\left(\frac{n}{e^{\theta} T_\ell^{1/2}}\right) \delta\left(\Lambda^d, n\right) n^{-1/2-it} \right|^4 &\leq \\ &\leq (1 + L_1)^3 \sum_{\ell_1=0}^{L_1} \left| \sum_{n=1}^{\infty} r\left(\frac{n}{e^{\theta} T_\ell^{1/2} b^{-\ell_1}}\right) \delta\left(\Lambda^d, n\right) n^{-1/2-it} \right|^4. \end{aligned} \tag{4.29}$$

Likewise, by (1.4), (4.8), (4.10) and the Cauchy–Schwarz inequality, one has

$$\begin{aligned} \left| P_M\left(A; it, \lambda^d\right) \right|^2 &\leq (1 + L_2) \sum_{\ell_2=0}^{L_2} \left| \sum_{Mb^{-\ell_2-1} < |\mu|^2 \leq Mb^{-\ell_2}} A(\mu) \Lambda^d(\mu) |\mu|^{-2it} \right|^2 \\ &\quad (d \in \mathbb{Z}, t \in \mathbb{R}), \end{aligned} \tag{4.30}$$

where

$$L_2 = [(\log M) / \log b] \geq 0. \tag{4.31}$$

Given (4.10), (4.19), (4.21), (4.25), (4.26) and the definition (2.10) of $\delta(\Lambda^d, n)$, it follows by the bounds (4.28)–(4.29) and (4.30)–(4.31) that for some pair $K, M' > 0$

satisfying

$$\frac{1}{b} < K \leq e^\theta T_\ell^{1/2} \ll T_\ell^{1/2} \quad \text{and} \quad 1 \leq M' \leq M \ll T_\ell^{1/2}, \quad (4.32)$$

one has

$$\begin{aligned} \check{E}(T_\ell; M, A; \theta) &\ll (1 + |\log T_\ell|)^6 K^{-2} \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \sum_{\kappa \neq 0} w(|\kappa|^2) \Lambda^d(\kappa) |\kappa|^{-2it} \right|^4 \times \\ &\times \left| \sum_{M'/b < |\mu|^2 \leq M'} A(\mu) \Lambda^d(\mu) |\mu|^{-2it} \right|^2 w_0\left(\frac{|2d + it|^2}{\pi^2 T_\ell}\right) dt, \quad (4.33) \end{aligned}$$

with

$$w(x) = (x/K)^{-1/2} r(x/K) \quad \text{for } x > 0. \quad (4.34)$$

Since the function ρ satisfies the conditions (3.43) and (3.44) of Lemma 6, it follows by (4.27) that the support of the function $r : (0, \infty) \rightarrow \mathbb{R}$ is contained in the interval $[b^{-2}, b]$; we therefore find, by (4.34) and (4.10), that the support of the function $w : (0, \infty) \rightarrow \mathbb{R}$ is contained in the interval $[e^{-\eta} bK, bK]$. Moreover, by (4.34), (4.27), (4.18) and (4.10), the function w is infinitely differentiable on $(0, \infty)$ and satisfies $w^{(j)}(x) \ll_j (K/x)^{1/2} x^{-j}$ for all $x > 0$ and all $j \in \mathbb{N} \cup \{0\}$. Consequently (and since η is, here, the positive absolute constant $(\log 2)/3$) it follows that, for all $x > 0$ and all $j \in \mathbb{N} \cup \{0\}$, one has:

$$w^{(j)}(x) = \begin{cases} O_j\left((\eta x)^{-j}\right) & \text{if } e^{-\eta} bK \leq x \leq e^\eta bK, \\ 0 & \text{otherwise.} \end{cases} \quad (4.35)$$

By (4.35), (4.32), (4.16) and (4.10), it follows that if we put $K_0 = 1$, $K_1 = K_2 = bK$, $w_1 = w_2 = w$, $M_1 = M'$, $T = T_\ell$ and, for $\mu \in \mathfrak{D} - \{0\}$,

$$a(\mu) = \begin{cases} A(\mu) & \text{if } M'/b < |\mu|^2 \leq M', \\ 0 & \text{otherwise,} \end{cases} \quad (4.36)$$

and if (at the same time) we substitute $\varepsilon/33 \in (0, 1/33]$ for ε , then the hypotheses of Theorem 2 (up to, and including, (1.20)) will be satisfied. Therefore, by Theorem 2 and the elementary estimates (1.33) and (1.35) (all applied with $\varepsilon/33$ in place of ε), we are able to deduce from (4.33)–(4.34) that

$$\check{E}(T_\ell; M, A; \theta) \ll (1 + |\log T_\ell|)^6 K^{-2} (2\pi\mathcal{D}_0 + (\pi/2)\mathcal{D}_1^* + (\pi/2)\mathcal{D}_2^* + \mathcal{E}), \quad (4.37)$$

where, with $a : \mathfrak{D} - \{0\} \rightarrow \mathbb{C}$ defined as in (4.36), the terms \mathcal{D}_0 , \mathcal{D}_1^* , \mathcal{D}_2^* and \mathcal{E} satisfy:

$$\max \{ |\mathcal{D}_0|, |\mathcal{D}_1^*|, |\mathcal{D}_2^*| \} \ll_\varepsilon T_\ell^{1+(\varepsilon/33)} K^2 \|a\|_2^2 \ll T_\ell^{1+(\varepsilon/3)} K^2 \|A\|_2^2, \quad (4.38)$$

$$\begin{aligned} K^{-2}\mathcal{E} &\ll_\varepsilon \left(\frac{(M')^2}{T_\ell^{1/2}} + \left(\frac{(M')^{2-(3/2)\vartheta}}{T_\ell^{(1-\vartheta)/2}} \right) \left(\frac{K}{T_\ell^{1/2}} \right)^\vartheta \right) T_\ell^{1+(3/11)\varepsilon} \|a\|_2^2 \ll \\ &\ll \left(1 + \frac{T_\ell}{M^3} \right)^{\vartheta/2} \left(\frac{M^4}{T_\ell} \right)^{1/2} T_\ell^{1+(\varepsilon/3)} \|A\|_2^2 \end{aligned} \quad (4.39)$$

and

$$\begin{aligned} K^{-2}\mathcal{E} &\ll_\varepsilon \left(\frac{(M')^2}{T_\ell^{1/2}} + \left(\frac{K}{T_\ell^{1/2}} + \left(\frac{K}{T_\ell^{1/2}(M')^{1/2}} \right)^\vartheta \right) \left(\frac{(M')^2}{T_\ell^{1/2}} \right)^{1-\vartheta} \right) T_\ell^{1+(\varepsilon/3)} M' \|a\|_\infty^2 \ll \\ &\ll \left(1 + \frac{T_\ell}{M^4} \right)^{\vartheta/2} \left(\frac{M^4}{T_\ell} \right)^{1/2} T_\ell^{1+(\varepsilon/3)} M \|A\|_\infty^2 \end{aligned} \quad (4.40)$$

(note that, in each of (4.38), (4.39) and (4.40), the final upper bound follows by virtue of (1.13), (4.36), the hypothesis that ε lies in $(0, 1]$ and the bounds on K , M and T_ℓ that are implied by (4.32)).

By (4.37)–(4.40) and (4.32), we obtain both

$$\check{E}(T_\ell; M, A; \theta) \ll_\varepsilon \left(1 + \left(1 + \frac{T_\ell}{M^3} \right)^{\vartheta/2} \left(\frac{M^4}{T_\ell} \right)^{1/2} \right) T_\ell^{1+(\varepsilon/2)} \|A\|_2^2$$

and

$$\check{E}(T_\ell; M, A; \theta) \ll_\varepsilon T_\ell^{1+(\varepsilon/2)} \|A\|_2^2 + \left(1 + \frac{T_\ell}{M^4} \right)^{\vartheta/2} \left(\frac{M^4}{T_\ell} \right)^{1/2} T_\ell^{1+(\varepsilon/2)} M \|A\|_\infty^2.$$

Moreover, for the reasons that are mentioned below (4.8), we may assume here that the mapping A is such that $\|A\|_2^2 = \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2$ and $\|A\|_\infty = \max\{|A(\mu)| : \mu \in \mathfrak{D} \text{ and } 0 < |\mu|^2 \leq M\}$. Therefore, by (4.24), (4.17), (4.8) and the bounds just obtained for $\check{E}(T_\ell; M, A; \theta)$, it follows that one has both

$$\tilde{E}_\ell(D; M, A) \ll_\varepsilon \left(D^{1+\varepsilon} + \left(1 + \frac{D^2}{M^3} \right)^{\vartheta/2} D^\varepsilon M^2 \right) T_\ell^{1/2} \sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \quad (4.41)$$

and

$$\begin{aligned} \tilde{E}_\ell(D; M, A) \ll_\varepsilon & \left(\sum_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \right) D^{1+\varepsilon} T_\ell^{1/2} + \\ & + \left(\max_{0 < |\mu|^2 \leq M} |A(\mu)|^2 \right) \left(1 + \frac{D^2}{M^4} \right)^{\vartheta/2} M^3 D^\varepsilon T_\ell^{1/2}. \end{aligned} \quad (4.42)$$

Since our conclusions in the above paragraph are valid for any $\ell \in \{0, 1, 2, \dots, L\}$, and since the definitions (4.17) and (4.10) imply that we have $\sum_{\ell=0}^\infty T_\ell^{1/2} = (\sqrt{5}/\pi) D \sum_{\ell=0}^\infty 2^{-\ell/6} = O(D)$, it follows that the upper bounds (4.14), (4.41) and (4.42) imply the bounds for $E^*(D; M, A)$ that are stated in (4.5) and (4.6). Moreover, our only assumption concerning M and D has been that the condition (4.8) is satisfied, and so we have completed the set task of showing that Theorem 2 implies the validity of (4.5) and (4.6) for all $M \geq 1$ and all $D \geq M$; in view of our conclusions in the paragraphs preceding (4.8), we have thereby shown that Theorem 2 implies Theorem 1. \square

5. Some lemmas that we need for the proof of Theorem 2

In this section we prepare for our proof of Theorem 2 (in in Sections 6 and 7) by stating some of the more basic lemmas that are used in that proof. Before proceeding to these lemmas (and their proofs), we define one further piece of notation by putting:

$$e(x) = \exp(2\pi ix) \quad \text{for all } x \in \mathbb{R}.$$

This convenient notation will be used freely in this section, and in those that follow.

LEMMA 7 (POISSON SUMMATION OVER \mathbb{Z}^2 AND OVER $\mathfrak{D} = \mathbb{Z}[i]$). Let $f : \mathbb{C} \rightarrow \mathbb{C}$. Suppose that the function $F : \mathbb{R}^2 \rightarrow \mathbb{C}$ given by $F(x, y) = f(x + iy)$ lies in the Schwartz space: so that, for all real $A \geq 0$ and all integers $j, k \geq 0$, the function $|x + iy|^A \frac{\partial^{j+k}}{\partial x^j \partial y^k} F(x, y)$ is continuous and bounded on \mathbb{R}^2 . Then the Fourier transforms

$$\widehat{F}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) e(-ux - vy) dx dy$$

and

$$\widehat{f}(w) = \int_{\mathbb{C}} f(z) e(-\operatorname{Re}(wz)) d_+ z = \widehat{F}(\operatorname{Re}(w), -\operatorname{Im}(w))$$

are complex-valued functions defined on \mathbb{R}^2 and \mathbb{C} , respectively. Moreover, one has

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} F(x, y) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} \widehat{F}(u, v), \quad (5.1)$$

and, for all $\tau \in \mathbb{C}$ and all $\alpha, \gamma \in \mathfrak{D}$ with $\gamma \neq 0$:

$$\sum_{\nu \in \mathfrak{D}} f(\nu) e(\operatorname{Re}(\tau\nu)) = \sum_{\xi \in \mathfrak{D}} \widehat{f}(\xi - \tau), \quad (5.2)$$

$$\sum_{\nu \equiv \alpha \pmod{\gamma\mathfrak{D}}} f(\nu) = \frac{1}{|\gamma|^2} \sum_{\xi \in \mathfrak{D}} \widehat{f}\left(\frac{\xi}{\gamma}\right) e\left(\operatorname{Re}\left(\frac{\alpha\xi}{\gamma}\right)\right), \quad (5.3)$$

and

$$\sum_{\nu \in \mathfrak{D}} f(\nu) e\left(\operatorname{Re}\left(\alpha \frac{\nu^*}{\gamma}\right)\right) = \frac{1}{|\gamma|^2} \sum_{\xi \in \mathfrak{D}} \widehat{f}\left(\frac{\xi}{\gamma}\right) S(\alpha, \xi; \gamma), \quad (5.4)$$

where $S(\alpha, \beta; \gamma)$ denotes the Kloosterman sum over $\mathbb{Q}(i)$ that is given by:

$$S(\alpha, \beta; \gamma) = \sum_{\delta \in (\mathfrak{D}/\gamma\mathfrak{D})^*} e\left(\operatorname{Re}\left(\frac{\alpha\delta^* + \beta\delta}{\gamma}\right)\right). \quad (5.5)$$

PROOF. For (5.1) see, for example [35, Chapter 13, Section 6]. By replacing in (5.1) the function F by $G(x, y) = f(x + iy)e(\operatorname{Re}((x + iy)\tau))$ (and \widehat{F} by \widehat{G}), one obtains (5.2): this is justified, since one has $G(x, y) = F(x, y)e(\operatorname{Re}(\tau)x)e(-\operatorname{Im}(\tau)y)$,

from which it may be deduced that G lies in the Schwartz space if F does. Noting that

$$\sum_{\nu \equiv \alpha \pmod{\gamma\mathfrak{D}}} f(\nu) = \frac{1}{|\gamma|^2} \sum_{\beta \in \mathfrak{D}/\gamma\mathfrak{D}} e\left(\operatorname{Re}\left(\frac{\alpha\beta}{\gamma}\right)\right) \sum_{\nu} f(\nu) e\left(\operatorname{Re}\left(-\frac{\beta}{\gamma}\nu\right)\right),$$

we see (5.3) now follows by applying (5.2) to the inner sum on the right here. Finally:

$$\sum_{\nu} f(\nu) e\left(\operatorname{Re}\left(\alpha \frac{\nu^*}{\gamma}\right)\right) = \sum_{\delta \in (\mathfrak{D}/\gamma\mathfrak{D})^*} e\left(\operatorname{Re}\left(\alpha \frac{\delta^*}{\gamma}\right)\right) \sum_{\nu \equiv \delta \pmod{\gamma\mathfrak{D}}} f(\nu),$$

so that (5.4)–(5.5) is a corollary of (5.3). □

LEMMA 8. *Suppose that f and F are as in Lemma 7. For $z = x + iy$ with $x, y \in \mathbb{R}$, let*

$$(\Delta_{\mathbb{C}}f)(z) = (\Delta_{\mathbb{R} \times \mathbb{R}}F)(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x + iy).$$

Then $\Delta_{\mathbb{R} \times \mathbb{R}}F$ (the Laplacian of F) is a member of the Schwartz space. The functions f and $\Delta_{\mathbb{C}}f$ have Fourier transforms $\widehat{f}, \widehat{\Delta_{\mathbb{C}}f} : \mathbb{C} \rightarrow \mathbb{C}$ (defined as in Lemma 7) that are related to one another by:

$$|2\pi w|^2 \widehat{f}(w) = -\widehat{\Delta_{\mathbb{C}}f}(w) \quad \text{for } w \in \mathbb{C}. \tag{5.6}$$

For all $w \in \mathbb{C} - \{0\}$ and all $j \in \mathbb{N} \cup \{0\}$, one has

$$\begin{aligned} |\widehat{f}(w)| &= (2\pi|w|)^{-2j} \left| \widehat{\Delta_{\mathbb{C}}^j f}(w) \right| \leq (2\pi|w|)^{-2j} \left| \widehat{\Delta_{\mathbb{C}}^j f}(0) \right| = \\ &= (2\pi|w|)^{-2j} \int_{\mathbb{C}} |(\Delta_{\mathbb{C}}^j f)(z)| d_+ z. \end{aligned} \tag{5.7}$$

PROOF. This lemma is a restatement of [50, Lemma 3.8]: see there for a proof. □

Remark 5.1. Let α be a non-zero complex constant, and let S_{α} be the associated «rotation-dilatation operator» which maps any function f of a complex variable z to the function $g(z)$ such that $g(z) = f(\alpha z)$ for all $z \in \mathbb{C}$. Then, when f is as in Lemma 7, the function $G : \mathbb{R}^2 \rightarrow \mathbb{C}$ given by $G(x, y) = (S_{\alpha}f)(x + iy)$ is a member

of the Schwartz space, and a linear change of variables of integration shows that one has

$$\widehat{S_\alpha f}(w) = |\alpha|^{-2} \widehat{f}(w/\alpha) \quad \text{for } w \in \mathbb{C}, \tag{5.8}$$

where the Fourier transforms, $\widehat{S_\alpha f}$ and \widehat{f} , are as defined in Lemma 7. An immediate, and useful, consequence of (5.8) is that if $f(z) = f(|z|)$ for all $z \in \mathbb{C}$, then $\widehat{f}(w) = \widehat{f}(|w|)$ for all $w \in \mathbb{C}$.

LEMMA 9. *Let $\mathfrak{N} : \mathbb{C} \rightarrow [0, \infty)$ satisfy $\mathfrak{N}(z) = |z|^2$ for $z \in \mathbb{C}$. Let $\Upsilon : [0, \infty) \rightarrow \mathbb{C}$ be an infinitely differentiable function on $[0, \infty)$, and let the support of Υ be contained in the interval $[e^{-1}, e]$. Suppose moreover that $\eta_0 \in (0, 1]$, and that*

$$\Upsilon^{(j)}(x) \ll_j \eta_0^{-j} \quad \text{for all } x \in [e^{-1}, e], j \in \mathbb{N} \cup \{0\}. \tag{5.9}$$

Let $X > 0$, and let $C : \mathfrak{D} - \{0\} \rightarrow \mathbb{C}$ satisfy both

$$C(\xi) = 0 \quad \text{for all } \xi \in \mathfrak{D} \text{ such that } |\xi|^2/X^2 \in (0, 1/2] \cup [2, \infty) \tag{5.10}$$

and

$$C(i\xi) = C(\xi) \quad \text{for all } \xi \in \mathfrak{D} - \{0\}. \tag{5.11}$$

Let $D > 0$ and put

$$E_{D,C} = \frac{1}{2D^2} \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \Upsilon\left(\frac{|2d + it|^2}{D^2}\right) \left| \sum_{\xi \neq 0} C(\xi) \Lambda^d(\xi) |\xi|^{-2it} \right|^2 dt.$$

Then, for all $Q \in (0, 1/2]$ and all $j \in \mathbb{N}$, one has

$$E_{D,C} = \sum_{\substack{\xi_1 \neq 0 \\ \xi_2 \neq 0 \\ |\xi_1 - \xi_2| < QX}} C(\xi_1) \overline{C(\xi_2)} \widehat{\Upsilon \circ \mathfrak{N}}\left(\frac{D}{\pi} \log\left(\frac{\xi_1}{\xi_2}\right)\right) + O_j\left((\eta_0 Q D)^{-2j} \|C\|_1^2\right), \tag{5.12}$$

where the Fourier transform $\widehat{\Upsilon \circ \mathfrak{N}}$ is defined as in Lemma 7.

PROOF. Let σ denote the sum over ξ in the definition of $E_{D,C}$. By first expanding $|\sigma|^2$ as $\sigma \overline{\sigma}$, one can then integrate and sum (over d) term by term, so as to obtain:

$$2D^2 E_{D,C} = \sum_{\xi_1 \neq 0} \sum_{\xi_2 \neq 0} C(\xi_1) \overline{C(\xi_2)} F\left(\frac{1}{\pi i} \log\left(\frac{\xi_1}{\xi_2}\right)\right), \tag{5.13}$$

where (given that $\Lambda^d(\alpha)$ is as in (1.2)) one has

$$F(\alpha) = \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} \Upsilon \left(\frac{|2d + it|^2}{D^2} \right) e(\operatorname{Re}((2d + it)\bar{\alpha})) dt. \quad (5.14)$$

Suppose now that ξ_1 and ξ_2 are non-zero elements of \mathfrak{D} . Then, since $F(\alpha + 1/2) = F(\alpha)$, one has

$$F\left(\frac{1}{\pi i} \log\left(\frac{\xi_1}{\xi_2}\right)\right) = F\left(\frac{1}{\pi i} \log\left(\frac{\epsilon \xi_1}{\xi_2}\right)\right), \quad (5.15)$$

where ϵ is the unique unit of \mathfrak{D} for which

$$-\pi/4 < \operatorname{Arg}(\epsilon \xi_1 / \xi_2) \leq \pi/4. \quad (5.16)$$

Therefore, supposing now that

$$\alpha = \frac{1}{\pi i} \log\left(\frac{\epsilon \xi_1}{\xi_2}\right), \quad (5.17)$$

one has, by (5.16),

$$-\frac{1}{4} < \operatorname{Re}(\alpha) \leq \frac{1}{4}. \quad (5.18)$$

By (5.14),

$$\begin{aligned} F(\alpha) &= \sum_{d=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-1}^1 \Upsilon \left(\frac{|2d + 2ni + ui|^2}{D^2} \right) e(\operatorname{Re}((2d + 2ni + ui)\bar{\alpha})) du = \\ &= \int_{-1}^1 G(\alpha, u) e(\operatorname{Im}(\alpha)u) du, \end{aligned} \quad (5.19)$$

where

$$G(\alpha, u) = \sum_{\beta \in \mathfrak{D}} \Upsilon \left(\frac{|2\beta + iu|^2}{D^2} \right) e(\operatorname{Re}(2\bar{\alpha}\beta)).$$

For $u \in \mathbb{R}$, an application of (5.2) of Lemma 7 yields:

$$\begin{aligned}
 G(\alpha, u) &= \sum_{\gamma \in \mathfrak{D}} \int_{\mathbb{C}} \Upsilon \left(\frac{|2z + iu|^2}{D^2} \right) e(-\operatorname{Re}((\gamma - 2\bar{\alpha})z)) d_+ z = \\
 &= (D/2)^2 e(-\operatorname{Im}(\alpha)u) \sum_{\gamma \in \mathfrak{D}} \widehat{\Upsilon \circ \mathfrak{N}}((\gamma - 2\bar{\alpha})D/2) e(-\operatorname{Im}(\gamma)u/2) \quad (5.20)
 \end{aligned}$$

(where the second line follows by a change of variables of integration, as in Remark 5.1).

By the result (5.7) of Lemma 8, we have the upper bound

$$\left| \widehat{\Upsilon \circ \mathfrak{N}}(w) \right| \leq (2\pi|w|)^{-2j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^j \Upsilon(|x + iy|^2) \right|_{(x,y)=(x_1,y_1)} dx_1 dy_1,$$

for all $w \in \mathbb{C} - \{0\}$ and all $j \in \mathbb{N}$: note that, by reasoning similar to that which is used in the proof of the results (7.23)–(7.24) of Lemma 16 (below), it can be shown to follow from our hypotheses concerning Υ and η_0 (including, in particular, (5.9)) that the function $(x, y) \mapsto \Upsilon(|x + iy|^2)$ is in the Schwartz space, and that for all $k, \ell \in \mathbb{N} \cup \{0\}$, and all points $(x_1, y_1) \in \mathbb{R}^2$, one has

$$\frac{\partial^{k+\ell}}{\partial x^k \partial y^\ell} \Upsilon(|x + iy|^2) \Big|_{(x,y)=(x_1,y_1)} = \begin{cases} O_{k,\ell}(\eta_0^{-(k+\ell)}) & \text{if } e^{-1} \leq |x_1 + iy_1|^2 \leq e, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, for all $w \in \mathbb{C} - \{0\}$ and all $j \in \mathbb{N}$, we have:

$$\left| \widehat{\Upsilon \circ \mathfrak{N}}(w) \right| \leq (2\pi|w|)^{-2j} \int_{-\sqrt{e}}^{\sqrt{e}} \int_{-\sqrt{e}}^{\sqrt{e}} O_j(\eta_0^{-2j}) dx_1 dy_1 \ll_j (\eta_0|w|)^{-2j}. \quad (5.21)$$

The case $j = 2$ of (5.21) ensures uniform absolute convergence (for all $u \in \mathbb{R}$) of the sum over $\gamma \in \mathfrak{D}$ in (5.20), and so justifies both the substitution of the expression for $G(\alpha, u)$ obtained in (5.20) into (5.19), and the term by term integration, with respect to $u \in [-1, 1]$, of the series expansion of $G(\alpha, u)e(\operatorname{Im}(\alpha)u)$ brought about by that substitution. This term by term integration has the effect of annihilating

any term for which the index γ is not real (this being so by virtue of the fact that the condition of summation $\gamma \in \mathfrak{D}$ implies $\text{Im}(\gamma) \in \mathbb{Z}$). Moreover, the terms with $\gamma \in \mathfrak{D} \cap \mathbb{R} = \mathbb{Z}$ are trivial to integrate (being independent of u), and so, by (5.19) and (5.20), one obtains:

$$F(\alpha) = \frac{D^2}{2} \sum_{g=-\infty}^{\infty} \widehat{\Upsilon \circ \mathfrak{N}}((g/2 - \bar{\alpha})D).$$

Given (5.18), it follows by (5.21) that if $g \in \mathbb{Z} - \{0\}$, then $\widehat{\Upsilon \circ \mathfrak{N}}((g/2 - \bar{\alpha})D) \ll \ll_j (\eta_0 D |g|)^{-2j}$ for $j \in \mathbb{N}$. By applying this observation to the sum over $g \in \mathbb{Z}$ occurring in the expression just obtained for $F(\alpha)$, we find that

$$F(\alpha) = \frac{D^2}{2} \widehat{\Upsilon \circ \mathfrak{N}}(-\bar{\alpha} D) + O_j \left(\eta_0^{-2j} D^{2-2j} \right) \quad \text{for all } j \in \mathbb{N}. \tag{5.22}$$

Moreover, by (5.22) and (5.21) (again), it follows that if $\alpha \neq 0$ then

$$F(\alpha) \ll_j \left(|\alpha|^{-2j} + 1 \right) \eta_0^{-2j} D^{2-2j} \quad \text{for all } j \in \mathbb{N}. \tag{5.23}$$

The results (5.22) and (5.23) will be used to estimate the terms of the double sum in (5.13). Given (5.10), and (5.13), it is trivial that one needs these estimates only in the cases where

$$X/\sqrt{2} \leq |\xi_1|, |\xi_2| \leq \sqrt{2}X. \tag{5.24}$$

Choose now any Q such that $0 < Q \leq 1/2$. Taking, in turn, each pair $\xi_1, \xi_2 \in \mathfrak{D}$ satisfying (5.24), suppose that (5.22) is applied if and only if

$$|\epsilon \xi_1 - \xi_2| < QX, \tag{5.25}$$

while (5.23) is applied if and only if the condition (5.25) is not satisfied. Then, in the cases where the bound (5.23) is applied, one will have $|\alpha| > Q/6$: indeed, by (5.17), the condition $Q \leq 1/2$ and the inequalities $|\exp(z) - 1| \leq |z| \exp(|z|)$, $\pi \exp(\pi/12) < 3\sqrt{2}$ and (5.24), it follows that if one were to have $|\alpha| \leq Q/6$ then the condition (5.25) would be satisfied, and so (5.23) would not be being applied. Consequently, where it is applied, (5.23) shows:

$$F(\alpha) \ll_j \left((Q/6)^{-2j} + 1 \right) \eta_0^{-2j} D^{2-2j} \ll_j (\eta_0 Q)^{-2j} D^{2-2j}.$$

In the cases where it is instead (5.22) that is implied, one has (5.24) and (5.25), by which (given that $0 < Q \leq 1/2$) it follows that

$$|\epsilon\xi_1 - \xi_2| < |\xi_2|/\sqrt{2}.$$

Since the last inequality renders redundant the condition (5.16) that was imposed on the choice of unit $\epsilon \in \{1, i, -1, -i\}$, it therefore follows from the conclusions of the above paragraph that, by means of (5.13), (5.15), (5.17) and the estimates (5.22) and (5.23), one can deduce that

$$E_{D,C} = \frac{1}{4} \sum_{\epsilon^4=1} \sum_{\substack{\xi_1 \neq 0 \\ |\epsilon\xi_1 - \xi_2| < QX}} \sum_{\substack{\xi_2 \neq 0 \\ |\epsilon\xi_1 - \xi_2| < QX}} C(\xi_1) \overline{C(\xi_2)} \widehat{\Upsilon \circ \mathfrak{N}} \left(-\frac{1}{\pi i} \log \left(\frac{\epsilon\xi_1}{\xi_2} \right) D \right) + O_j \left(\|C\|_1^2 (\eta_0 Q D)^{-2j} \right),$$

for all $j \in \mathbb{N}$. The conclusion of the lemma now follows, via the substitution $\xi_1 = \epsilon^{-1}\xi$, the observation that the hypothesis (5.11) implies that $\sum_{\epsilon^4=1} C(\epsilon^{-1}\xi) = 4C(\xi)$, for $\xi \in \mathfrak{D} - \{0\}$, and the final point noted in Remark 5.1. \square

Remark 5.2. Let $0 < \eta \leq 1$, and let Ω_η be an infinitely differentiable function from $[0, \infty)$ to $[0, \infty)$ that has as its support some subset of the interval $[1, e^\eta]$, and that satisfies both

$$\Omega_\eta^{(j)}(x) \ll_j \eta^{-j} \quad (\text{for } x > 0 \text{ and } j = 0, 1, 2, \dots) \quad \text{and} \quad \frac{1}{\eta} \int_0^\infty \Omega_\eta(x) dx = 1$$

(such functions do exist: one example is the mapping $x \mapsto \|\Phi\|_1^{-1} \eta \kappa \Phi((x-1)\kappa - 1)$, where $\kappa = 2/(e^\eta - 1)$ and $\Phi : \mathbb{R} \rightarrow [0, e^{-1}]$ is given by the equation (3.100) of Remark 3.1, above). Suppose moreover that δ is a real number satisfying $0 < \delta \leq (4e)^{-1}$. Then, by multiplying both sides of the equation (5.12) by $(\delta\eta)^{-1} \Omega_\eta(Q^2/\delta)$, applying the substitution $Q = \sqrt{\delta_1}$, and then integrating both sides of the resulting equation with respect to the (positive valued) variable δ_1 , one obtains:

$$E_{D,C} = O_j \left(\delta^{-j} (\eta_0 D)^{-2j} \|C\|_1^2 \right) +$$

$$+ \sum_{\xi_1 \neq 0} \sum_{\xi_2 \neq 0} W_\eta \left(\frac{|\xi_1 - \xi_2|^2}{\delta X^2} \right) C(\xi_1) \overline{C(\xi_2)} \widehat{\Upsilon \circ \mathfrak{N}} \left(\frac{D}{\pi} \log \left(\frac{\xi_1}{\xi_2} \right) \right), \tag{5.26}$$

where, for $u \geq 0$,

$$W_\eta(u) = \frac{1}{\eta} \int_u^\infty \Omega_\eta(x) dx. \tag{5.27}$$

Moreover, given the stated properties of the function $\Omega_\eta(x)$, the function $W_\eta(u)$ here is real valued, monotonic decreasing and infinitely differentiable on $[0, \infty)$, and satisfies: $W_\eta(u) = 1$ if $0 \leq u \leq 1$; $W_\eta(u) = 0$ if $u \geq e^\eta$; $W_\eta^{(j)}(u) \ll_j \eta^{-j}$ for $u > 0$ and $j \in \mathbb{N} \cup \{0\}$.

LEMMA 10 (ESTIMATES FOR KLOOSTERMAN SUMS). *Let $\alpha, \beta, \gamma \in \mathfrak{D}$, with $\gamma \neq 0$, and let the Kloosterman sum $S = S(\alpha, \beta; \gamma)$ be given by the equation (5.5) of Lemma 7. Then*

$$|S| \leq \phi_{\mathfrak{D}}(\gamma) = |\gamma|^2 \prod_{(\pi_1) | (\gamma)} \left(1 - \frac{1}{|\pi_1|^2} \right) \leq |\gamma|^2, \tag{5.28}$$

where one has $\phi_{\mathfrak{D}}(\gamma) = |(\mathfrak{D}/\gamma\mathfrak{D})^*|$ (Euler’s function for Gaussian integers), and where the product is over distinct prime ideal factors $(\pi_1) = \pi_1\mathfrak{D}$ of the ideal $(\gamma) = \gamma\mathfrak{D} \subset \mathfrak{D}$ (and so is equal to 1 if and only if γ is a unit of \mathfrak{D}). One has also the (much deeper) «Weil–Estermann estimates»:

$$|S|^2 \leq 2^3 \tau_2^2(\gamma) |(\alpha, \beta, \gamma)\gamma|^2, \tag{5.29}$$

where $\tau_2(\gamma) = \sum_{\delta | \gamma} 1$ and (α, β, γ) is an arbitrary highest common factor of α, β and γ . Moreover, if $\gamma | \beta$ then one has the (elementary) «Ramanujan sum evaluation»:

$$\begin{aligned} S &= \frac{1}{4} \sum_{\nu | (\alpha, \gamma)} \mu_{\mathfrak{D}} \left(\frac{\gamma}{\nu} \right) |\nu|^2 = \mu_{\mathfrak{D}} \left(\frac{\gamma}{(\alpha, \gamma)} \right) |(\alpha, \gamma)|^2 \prod_{\substack{(\pi_1) | ((\alpha, \gamma)) \\ (\pi_1) | (\gamma/(\gamma, \alpha))}} \left(1 - \frac{1}{|\pi_1|^2} \right) = \\ &= \frac{\mu_{\mathfrak{D}}(\gamma/(\alpha, \gamma)) \phi_{\mathfrak{D}}(\gamma)}{\phi_{\mathfrak{D}}(\gamma/(\gamma, \alpha))}, \end{aligned} \tag{5.30}$$

where, as in (5.28), the product is over distinct prime ideals $(\pi_1) \subset \mathfrak{D}$, while

$$\mu_{\mathfrak{D}}(\kappa) = \begin{cases} 0 & \text{if there exists a Gaussian prime } \pi_1 \text{ such that } \pi_1^2 \mid \kappa, \\ (-1)^{\omega(\kappa)} & \text{otherwise,} \end{cases}$$

with $\omega(\kappa)$ denoting the number of prime ideals in the ring \mathfrak{D} that contain κ . The result (5.30) implies that

$$|S| \leq |(\alpha, \gamma)|^2 \quad \text{if } \gamma \mid \beta. \tag{5.31}$$

PROOF. The first two equalities of (5.30) follow from [49, Equations (5.35)] and points noted in the last few lines of the proof of [49, Lemma 5.7]. The definition (5.5) implies that $|S| \leq S(0, 0; \gamma) = \phi_{\mathfrak{D}}(\gamma)$, and so we have both the first inequality in (5.28) and, by virtue of the case $\alpha = \beta = 0$ of the first two equalities of (5.30), the equality in (5.28) also. As for the final inequality in (5.28), that is a trivial consequence of the fact that each factor $1 - |\pi_1|^{-2}$ occurring in the product in (5.28) must certainly satisfy $0 \leq 1 - |\pi_1|^{-2} \leq 1$. Similarly, the result (5.31) follows from the first two equalities in (5.30). Two applications of the equality in (5.28) are enough to give the final equality in (5.30).

The bound (5.29) is a corollary of the more general results [5, Proposition 9 and Theorem 10] of Bruggeman and Miatello. □

6. Beginning the proof of Theorem 2

In this section we show, in effect, that if \mathcal{E} is the term given by Equation (1.30) (in combination with the definitions in (1.21)–(1.29)) then $|\mathcal{E}|$ is either quite small (smaller, subject to the condition (1.18), than the bounds (1.31) and (1.32) would imply), or else is bounded above by a certain sum of Kloosterman sums (a multiple of the sum Y defined in Lemma 13). Note that the mean value on the left-hand side of Equation (1.30) has an obvious «rational integer analogue», which was studied by Deshouillers and Iwaniec: their work in [10, Section 2] and [8, Section 9.2] provided the model for the principal steps of this section.

We assume throughout this section (and the next) that the numbers $\varepsilon, \eta, K_0, K_1, K_2, M_1$ are as in Theorem 2, that the function $C : \mathfrak{D} - \{0\} \rightarrow \mathbb{C}$ and terms

$c(d, it)$ ($d \in \mathbb{Z}, t \in \mathbb{R}$) are as defined in (1.21) and (1.29), that T satisfies (1.18), and that N is given by the equation (1.24). The functions $w_0(x), w_1(x), w_2(x)$ and $a(\mu)$ are also assumed to be as in Theorem 2, except that (to make conditions of the form $x \neq 0, \mu \neq 0$ implicit) their respective domains are enlarged to include 0, by defining $a(0) = 0$ and $w_i(0) = 0$ for $i = 1, 2, 3$. All Fourier transforms $\widehat{f}(w)$ are as defined in Lemma 7. The Euclidean Laplace operator $\Delta_{\mathbb{C}}$ is as defined in Lemma 8. As in Lemma 9, we take $\mathfrak{N}(z)$ to denote the function on \mathbb{C} satisfying $\mathfrak{N}(z) = |z|^2$ for all $z \in \mathbb{C}$. We shall moreover assume that the functions Ω_η and W_η are as described in Remark 5.2 (were it necessary, we could explicitly define just such a pair of functions).

In cases where (in addition to (1.18) and (1.24)) one has $T = O(1)$, the results (1.30)–(1.32) of Theorem 2 can be verified by means of a direct (and quite trivial) upper bound estimate for the absolute value of the sum on the left-hand side of Equation (1.30), in combination with similarly direct and trivial upper bound estimates for $|\mathcal{D}_0|, |\mathcal{D}_1^*|$ and $|\mathcal{D}_2^*|$ (where $\mathcal{D}_0, \mathcal{D}_1^*$ and \mathcal{D}_2^* are as defined in (1.21)–(1.28)). Therefore, in completing our proof of Theorem 2, we may certainly suppose that

$$T \geq 2^{10} \pi^8. \tag{6.1}$$

See (7.52), below, for what motivates this particular choice of a lower bound for T .

LEMMA 11. *Let*

$$\Xi = T^\epsilon M_1 K_2 / K_1 \tag{6.2}$$

and, for $\psi, \psi', \nu, \xi \in \mathfrak{D}$, put

$$\begin{aligned} L(\psi, \psi'; \nu, \xi) &= |\psi\psi'|^2 \int_{\mathbb{C}} \int_{\mathbb{C}} w_0\left(\frac{|\psi\psi' z_1 z_2|^2}{T}\right) w_1(|\psi' z_2|^2) \overline{w_1(|\psi z_2|^2)} |z_2|^2 \times \\ &\times e(\operatorname{Re}(\nu z_1 + \xi z_2)) d_{+z_1} d_{+z_2}. \end{aligned} \tag{6.3}$$

Then

$$\begin{aligned} &\sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} |c(d, it)|^2 w_0\left(\frac{|2d + it|^2}{\pi^2 T}\right) dt = \\ &= 2\pi \widehat{w_0 \circ \mathfrak{N}}(0) T \|C\|_2^2 + (\pi/2) (\mathcal{D}^* + \mathcal{E}^*) + O_{\epsilon, \eta}\left(T^{5\epsilon+1/2} K_1 K_2 M_1 \|a\|_2^2\right), \end{aligned} \tag{6.4}$$

where

$$\mathcal{D}^* = \sum_{\varphi \neq 0} \sum_{\kappa_2} \sum_{\substack{\mu_1 \quad \kappa_4 \quad \mu_2 \\ (\kappa_2 \mu_1, \kappa_4 \mu_2) \sim \varphi}} w_2(|\kappa_2|^2) a(\mu_1) \overline{w_2(|\kappa_4|^2) a(\mu_2)} G_\varphi\left(\frac{\kappa_2 \mu_1}{\varphi}, \frac{\kappa_4 \mu_2}{\varphi}; 0\right), \tag{6.5}$$

and

$$\mathcal{E}^* = \sum_{\varphi \neq 0} \sum_{\kappa_2} \sum_{\substack{\mu_1 \quad \kappa_4 \quad \mu_2 \\ (\kappa_2 \mu_1, \kappa_4 \mu_2) \sim \varphi}} w_2(|\kappa_2|^2) a(\mu_1) w_2(|\kappa_4|^2) a(\mu_2) H_\varphi\left(\frac{\kappa_2 \mu_1}{\varphi}, \frac{\kappa_4 \mu_2}{\varphi}\right), \tag{6.6}$$

with

$$G_\varphi(\psi_1, \psi_2; \xi) = \sum_{\nu \neq 0} W_\eta\left(\frac{|\varphi \nu|^2}{N}\right) L(\psi_1, \psi_2; \nu, \xi) e\left(\operatorname{Re}\left(\frac{\psi_1^* \nu \xi}{\psi_2}\right)\right) \tag{6.7}$$

and

$$H_\varphi(\psi_1, \psi_2) = \sum_{\xi \neq 0} W_\eta\left(\frac{|\varphi \xi|^2}{\Xi}\right) G_\varphi(\psi_1, \psi_2; \xi). \tag{6.8}$$

PROOF. By (1.2), (1.29) and (1.21), one has

$$c(d, it) = \sum_{\xi \neq 0} C(\xi) \Lambda^d(\xi) |\xi|^{-2it}. \tag{6.9}$$

Since $|i\kappa|^2 = |\kappa|^2$ for $\kappa \in \mathfrak{D}$, and since $i\mathfrak{D} = \mathfrak{D}$, it is evident from (1.21) that $C(\xi)$ satisfies the condition (5.11) of Lemma 9. Moreover, by (1.19) and (1.20) (in which $0 < \eta \leq (\log 2)/3$), this $C(\xi)$ also satisfies, for $X = (K_1 K_2 M_1)^{1/2}$, the condition (5.10) of Lemma 9. Therefore Lemma 9 may be applied with $\Upsilon(x) = w_0(x)$, and with $C(\xi)$ as in (1.21). In particular, by (6.9) and the case $\eta_0 = \eta$, $X = (K_1 K_2 M_1)^{1/2}$, $D = \pi T^{1/2}$, $\delta = T^{\epsilon-1}$ of (5.26) (the corollary of Lemma 9 noted in Remark 5.2), and by (1.18)–(1.21), (1.24) and the Cauchy–Schwarz inequality, one has

$$\sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} |c(d, it)|^2 w_0\left(\frac{|2d + it|^2}{\pi^2 T}\right) dt = O_{j,\eta}\left(T^{7/2-j\epsilon} \|a\|_2^2\right) +$$

$$+ 2\pi T \sum_{\xi_1} \sum_{\xi_2} W_\eta \left(\frac{|\xi_1 - \xi_2|^2}{N} \right) C(\xi_1) \overline{C(\xi_2)} \widehat{w_0 \circ \mathfrak{N}} \left(\sqrt{T} \log \left(\frac{\xi_1}{\xi_2} \right) \right), \quad (6.10)$$

for all $j \in \mathbb{N}$.

Taking $j = [(A + 4)/\varepsilon] + 1$ (for an arbitrary $A \geq 0$) one may replace the O -term in (6.10) by $O_{\varepsilon, \eta, A}(T^{-A} \|a\|_2^2)$. The terms of the sum on the right in (6.10) are «diagonal» when $\xi_1 = \xi_2$, and otherwise are «off-diagonal». Let $C(\xi_1)$ and $C(\xi_2)$ in (6.10) be expressed, through (1.21), in terms of variables of summation $\kappa_1, \kappa_2, \mu_1$ and $\kappa_3, \kappa_4, \mu_2$ satisfying $\kappa_1 \kappa_2 \mu_1 = \xi_1$ and $\kappa_3 \kappa_4 \mu_2 = \xi_2$. Then, by separating diagonal from off-diagonal terms, and, for each pair $\varphi, \nu \in \mathfrak{D} - \{0\}$, grouping together off-diagonal terms with both

$$(\kappa_2 \mu_1, \kappa_4 \mu_2) \sim \varphi$$

and

$$\kappa_1 \kappa_2 \mu_1 - \kappa_3 \kappa_4 \mu_2 = \xi_1 - \xi_2 = \varphi \nu,$$

one may rewrite the case $j = [(A + 4)/\varepsilon] + 1$ of (6.10) as

$$\sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} |c(d, it)|^2 w_0 \left(\frac{|2d + it|^2}{\pi^2 T} \right) dt = 2\pi \widehat{w_0 \circ \mathfrak{N}}(0) T \|C\|_2^2 + (\pi/2) \mathcal{D}' + O_{\varepsilon, \eta, A} \left(T^{-A} \|a\|_2^2 \right), \quad (6.11)$$

where

$$\begin{aligned} \mathcal{D}' &= \sum_{\varphi \neq 0} \sum_{\nu \neq 0} W_\eta \left(\frac{|\varphi \nu|^2}{N} \right) \times \\ &\times \sum_{\substack{\kappa_2 \mu_1 \kappa_4 \mu_2 \\ (\kappa_2 \mu_1, \kappa_4 \mu_2) \sim \varphi}} w_2 \left(|\kappa_2|^2 \right) a(\mu_1) \overline{w_2 \left(|\kappa_4|^2 \right) a(\mu_2)} U_\nu \left(\frac{\kappa_2 \mu_1}{\varphi}, \frac{\kappa_4 \mu_2}{\varphi} \right), \end{aligned} \quad (6.12)$$

with

$$U_\nu(\psi_1, \psi_2) = T \sum_{\substack{\kappa_1 \kappa_3 \\ \psi_1 \kappa_1 - \psi_2 \kappa_3 = \nu}} \widehat{w_0 \circ \mathfrak{N}} \left(\sqrt{T} \log \left(\frac{\psi_1 \kappa_1}{\psi_2 \kappa_3} \right) \right) w_1 \left(|\kappa_1|^2 \right) \overline{w_1 \left(|\kappa_3|^2 \right)} \quad (6.13)$$

(note that we have here simplified the first term on the right-hand side Equation (6.11) by making use of the equality $W_\eta(0) = 1$, which is one of the properties of the function W_η that are mentioned below (5.27)). Within the sum in (6.13) one has

$$\kappa_3 = \frac{\psi_1 \kappa_1 - \nu}{\psi_2} = \left(1 - \frac{\nu}{\psi_1 \kappa_1}\right) \frac{\psi_1 \kappa_1}{\psi_2},$$

where, given how (1.19), (1.20) and the properties of $W_\eta(u)$ noted below (5.27) effectively restrict the ranges of each of $|\varphi\nu|$, $|\kappa_2\mu_1|$, $|\kappa_4\mu_2|$ and $|\kappa_1|$ in the sums occurring in (6.12) and (6.13), it may be supposed that

$$\psi_2 \neq 0 \quad \text{and} \quad \left| \frac{\nu}{\psi_1 \kappa_1} \right|^2 \leq \frac{e^{4\eta N}}{K_1 K_2 M_1} < 3T^{\epsilon-1} < \frac{1}{5}.$$

By this, the Taylor expansions of $\log z$ and $\exp(w)$ about $z = 1$, $w = 0$, the definition of the Fourier transform, and (1.19), for $i \in \{0, 1\}$, one finds that, for the relevant (constrained) values of $\varphi, \nu, \psi_1, \psi_2, \kappa_1 \in \mathfrak{D}$,

$$\begin{aligned} \widehat{w_0 \circ \mathfrak{N}} \left(\sqrt{T} \log \left(\frac{\psi_1 \kappa_1}{\psi_2 \kappa_3} \right) \right) \overline{w_1(|\kappa_3|^2)} &= \\ &= \widehat{w_0 \circ \mathfrak{N}} \left(\sqrt{T} \frac{\nu}{\psi_1 \kappa_1} \right) \overline{w_1 \left(\left| \frac{\psi_1}{\psi_2} \kappa_1 \right|^2 \right)} + O\left(\eta^{-1} T^{(\epsilon-1)/2}\right). \end{aligned}$$

Therefore, by (6.13) and the case $i = 1$ of (1.19), one has in (6.12):

$$U_\nu(\psi_1, \psi_2) = \widetilde{U}_\nu(\psi_1, \psi_2) + V_\nu(\psi_1, \psi_2), \tag{6.14}$$

where

$$V_\nu(\psi_1, \psi_2) \ll \eta^{-1} T^{(\epsilon+1)/2} \sum_{\substack{\kappa_1 \equiv \psi_1^* \nu \pmod{\psi_2 \mathfrak{D}} \\ e^{-\eta} \leq |\kappa_1|^2 / K_1 \leq e^\eta}} 1 \tag{6.15}$$

and

$$\widetilde{U}_\nu(\psi_1, \psi_2) = T \sum_{\kappa_1 \equiv \psi_1^* \nu \pmod{\psi_2 \mathfrak{D}}} \widehat{w_0 \circ \mathfrak{N}} \left(\frac{\sqrt{T} \nu}{\psi_1 \kappa_1} \right) w_1(|\kappa_1|^2) \overline{w_1 \left(\left| \frac{\psi_1}{\psi_2} \kappa_1 \right|^2 \right)}, \tag{6.16}$$

with $\psi_1^* \in \mathfrak{D}$ being an arbitrary solution of the congruence $\psi_1 \psi_1^* \equiv 1 \pmod{\psi_2 \mathfrak{D}}$.

Using the definition of $\widehat{w_0 \circ \mathfrak{N}}(s)$ as an integral, it follows by a change of variable of integration, and a change in the order of summation and integration, that (6.16) may be rewritten as:

$$\widetilde{U}_\nu(\psi_1, \psi_2) = \widehat{F}_{\psi_1, \psi_2, \nu}(\nu), \quad (6.17)$$

where

$$F_{\psi_1, \psi_2, \nu}(z) = \sum_{\kappa \equiv \psi_1^* \nu \pmod{\psi_2 \Omega}} f_{\psi_1, \psi_2, z}(\kappa), \quad (6.18)$$

with

$$f_{\psi_1, \psi_2, z}(\kappa) = |\psi_1 \kappa|^2 w_1(|\kappa|^2) \overline{w_1\left(\left|\frac{\psi_1}{\psi_2} \kappa\right|^2\right)} w_0\left(\frac{|z \psi_1 \kappa|^2}{T}\right). \quad (6.19)$$

By (6.19), (1.19) and (for example) [49, Lemma 9.4], the functions $w_0(x)$ and $w_1(x)$ (extended to have $w_0(0) = w_1(0) = 0$) are such that the function $G : \mathbb{R}^2 \rightarrow \mathbb{C}$ given by $G(x, y) = f_{\psi_1, \psi_2, z}(x + iy)$ lies in the Schwartz space. Therefore it follows from (6.18) and the result (5.3) of Lemma 7 that

$$F_{\psi_1, \psi_2, \nu}(z) = \frac{1}{|\psi_2|^2} \sum_{\xi \in \Omega} \widehat{f}_{\psi_1, \psi_2, z}\left(\frac{\xi}{\psi_2}\right) e\left(\operatorname{Re}\left(\frac{\psi_1^* \nu \xi}{\psi_2}\right)\right) \quad (z \in \mathbb{C}). \quad (6.20)$$

Note that when ψ_1, ψ_2, z are known (and so to be treated as constants) it then follows by (6.19) and (1.19) that one has $f_{\psi_1, \psi_2, z}(\kappa) = h(|\kappa|^2)$, where the function $h : [0, \infty) \rightarrow \mathbb{C}$ is infinitely differentiable, satisfies

$$h^{(k)}(x) = \begin{cases} O_k\left(K_1 |\psi_1|^2 (\eta x)^{-k}\right) & \text{if } e^{-\eta} K_1 \leq x \leq e^\eta K_1, \\ 0 & \text{otherwise,} \end{cases} \quad (6.21)$$

for all $x > 0$ and all $k \in \mathbb{N} \cup \{0\}$, and is, moreover, identically zero unless it is the case that

$$e^{-2\eta} T/K_1 \leq |z \psi_1|^2 \leq e^{2\eta} T/K_1. \quad (6.22)$$

As an operator on smooth functions of a complex variable $s = u + iv$ the Laplacian $\Delta_{\mathbb{C}}$ satisfies:

$$\Delta_{\mathbb{C}} = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = 4 \frac{\partial}{\partial s} \frac{\partial}{\partial \bar{s}}$$

where

$$\frac{\partial}{\partial s} = \frac{1}{2} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{s}} = \frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

Therefore, and since (as was found above) $f_{\psi_1, \psi_2, z}(s) = h(|s|^2) = h(\bar{s}s)$, one may determine that

$$\left(\Delta_{\mathbb{C}}^j f_{\psi_1, \psi_2, z} \right) (s) = \sum_{r=0}^j \frac{4^j (j!)^2 |s|^{2j-2r} h^{(2j-r)}(|s|^2)}{(r!)((j-r)!)^2} \quad \text{for } s \in \mathbb{C}, j \in \mathbb{N} \cup \{0\}.$$

By this, (6.21) and the bound (5.7) of Lemma 8, it follows that if $\xi \neq 0$, then, for all $j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \widehat{f}_{\psi_1, \psi_2, z} \left(\frac{\xi}{\psi_2} \right) &\ll \left| \frac{\xi}{\psi_2} \right|^{-2j} \int_{\mathbb{C}} \left| \left(\Delta_{\mathbb{C}}^j f_{\psi_1, \psi_2, z} \right) (s) \right| d_+ s = \\ &= |\psi_2|^{2j} |\xi|^{-2j} O_j \left(K_1^2 |\psi_1|^2 \left(\eta K_1^{1/2} \right)^{-2j} \right) \ll_{\eta, j} K_1^{2-j} |\psi_1|^2 |\psi_2|^{2j} |\xi|^{-2j}. \end{aligned}$$

Given (1.19), and given how ψ_1 and ψ_2 arise in (6.12)–(6.13), it is here sufficient to treat cases in which one has $|\psi_i|^2 \asymp |\varphi|^{-2} K_2 M_1$ for $i = 1, 2$. In such a case the above bound for $\widehat{f}_{\psi_1, \psi_2, z}(\xi/\psi_2)$ implies that if $X_1 \asymp |\varphi|^{-2} \Xi$, where Ξ is given by (6.2), then, for $j \geq 2$, one has

$$\frac{1}{|\psi_2|^2} \sum_{|\xi|^2 \geq X_1} \left| \widehat{f}_{\psi_1, \psi_2, z} \left(\frac{\xi}{\psi_2} \right) \right| \ll_{\eta, j} K_1^{2-j} \left(\frac{K_2 M_1}{|\varphi|^2} \right)^j X_1^{1-j} \asymp_j T^{-j\epsilon} K_1^2 X_1.$$

By the last estimates, combined with (6.20) and (1.18), one obtains:

$$\begin{aligned} F_{\psi_1, \psi_2, \nu}(z) &= \frac{1}{|\psi_2|^2} \sum_{\xi \in \mathfrak{D}} W_{\eta} \left(\frac{|\varphi \xi|^2}{\Xi} \right) \widehat{f}_{\psi_1, \psi_2, z} \left(\frac{\xi}{\psi_2} \right) e \left(\operatorname{Re} \left(\frac{\psi_1^* \nu \xi}{\psi_2} \right) \right) + \\ &+ O_{\eta, j} \left(T^{1-j\epsilon} |\varphi|^{-2} \Xi \right), \end{aligned} \tag{6.23}$$

with the function $W_\eta(u)$ being as described below (5.27).

Since $f_{\psi_1, \psi_2, z}(s)$ (and hence also its Fourier transform) is identically zero unless ψ_1 and z satisfy (6.22), it follows from (6.17), (6.20), (6.2) and (6.23) that, when $|\psi_i|^2 \asymp |\varphi|^{-2} K_2 M_1$ (for $i = 1, 2$), one will have:

$$\begin{aligned} \tilde{U}_\nu(\psi_1, \psi_2) &= \sum_\xi W_\eta \left(\frac{|\varphi \xi|^2}{\Xi} \right) \widehat{g}_{\psi_1, \psi_2, \xi}(\nu) e \left(\operatorname{Re} \left(\frac{\psi_1^* \nu \xi}{\psi_2} \right) \right) + \\ &+ O_{\eta, j} \left(K_1^{-2} T^{2-(j-1)\varepsilon} \right) \quad \text{for } j \geq 2, \end{aligned} \tag{6.24}$$

where

$$g_{\psi_1, \psi_2, \xi}(z) = \frac{1}{|\psi_2|^2} \widehat{f}_{\psi_1, \psi_2, z} \left(\frac{\xi}{\psi_2} \right) = \int_{\mathbb{C}} f_{\psi_1, \psi_2, z}(\psi_2 z_2) e(-\operatorname{Re}(\xi z_2)) d_{+z_2},$$

so that, by (6.19),

$$\widehat{g}_{\psi_1, \psi_2, \xi}(\nu) = \int_{\mathbb{C}} \int_{\mathbb{C}} f_{\psi_1, \psi_2, z_1}(\psi_2 z_2) e(-\operatorname{Re}(\xi z_2 + \nu z_1)) d_{+z_2} d_{+z_1} = L(\psi_1, \psi_2; \nu, \xi), \tag{6.25}$$

with $L(\psi, \psi'; \nu, \xi)$ defined as in (6.3).

By (6.12), (6.14), (6.24) (with $j = [2/\varepsilon] + 2$) and (6.25), it follows that

$$\mathcal{D}' = \mathcal{D}^* + \mathcal{E}^* + \mathcal{E}' + O_{\eta, \varepsilon}(\mathcal{E}''), \tag{6.26}$$

where \mathcal{D}^* and \mathcal{E}^* are as defined in (6.5)–(6.8), while, by (1.18)–(1.20), in combination with the properties of $W_\eta(u)$ (described below (5.27)), (1.24), (6.15), (6.1), the bound (2.13) and the hypotheses that $\varepsilon \in (0, 1/6]$ and $\eta \in (0, (\log 2)/3]$, one has:

$$\begin{aligned} \mathcal{E}' &= \sum_{\varphi \neq 0} \sum_{\nu \neq 0} W_\eta \left(\frac{|\varphi \nu|^2}{N} \right) \times \\ &\times \sum_{\kappa_2} \sum_{\mu_1} \sum_{\kappa_4} \sum_{\mu_2} \overline{w_2(|\kappa_2|^2) a(\mu_1) w_2(|\kappa_4|^2) a(\mu_2)} V_\nu \left(\frac{\kappa_2 \mu_1}{\varphi}, \frac{\kappa_4 \mu_2}{\varphi} \right) \ll \\ &\ll \sum_{\varphi \neq 0} \sum_{\nu \neq 0} \sum_{\substack{K_2 e^{-\eta} \leq |\kappa_2|^2, |\kappa_4|^2 \leq e^\eta K_2 \\ |\varphi \nu|^2 \leq e^\eta N}} \sum_{\substack{M_1 e^{-\eta} \leq |\mu_1|^2, |\mu_2|^2 \leq e^\eta M_1 \\ (\kappa_2 \mu_1, \kappa_4 \mu_2) \sim \varphi}} |a(\mu_1) a(\mu_2)| \times \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{\substack{K_1 e^{-\eta} \leq |\kappa_1|^2 \leq e^\eta K_1 \\ \kappa_1 \kappa_2 \mu_1 \equiv \varphi \nu \pmod{\kappa_4 \mu_2 \mathfrak{D}}} } \eta^{-1} T^{\varepsilon+1/2} \ll \\
 & \ll \eta^{-1} T^{\varepsilon+1/2} \sum_{\kappa_1} \sum_{\kappa_2} \sum_{\mu_1} \sum_{\kappa_3} \sum_{\kappa_4} \sum_{\mu_2} (|a(\mu_1)|^2 + |a(\mu_2)|^2) \ll \\
 & \quad 1/2 \leq |\kappa_1 \kappa_2 \mu_1|^2 / (K_1 K_2 M_1) \leq 2 \quad 0 < |\kappa_1 \kappa_2 \mu_1 - \kappa_3 \kappa_4 \mu_2|^2 \leq 2N \\
 & \ll \eta^{-1} T^{\varepsilon+1/2} \sum_{e^{-\eta} M_1 \leq |\mu|^2 \leq e^\eta M_1} |a(\mu)|^2 \sum_{|\alpha|^2 \asymp K_1 K_2} \tau_2(\alpha) \sum_{\substack{\beta \neq 0 \\ 0 < |\beta - \mu \alpha|^2 \ll N}} \tau_3(\beta) \ll \\
 & \ll \eta^{-1} T^{\varepsilon+1/2} \sum_{\mu} |a(\mu)|^2 O_\varepsilon((K_1 K_2)^{1+\varepsilon}) \sum_{0 < |\delta|^2 \ll N} O_\varepsilon((K_1 K_2 M_1)^\varepsilon) = \\
 & = O_\varepsilon(\eta^{-1} T^{\varepsilon+1/2} (K_1 K_2)^{1+2\varepsilon} M_1^\varepsilon N) \sum_{\mu} |a(\mu)|^2 \ll_{\varepsilon, \eta} T^{5\varepsilon-1/2} K_1^2 K_2^2 M_1 \|a\|_2^2 \quad (6.27)
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{E}'' &= \sum_{\varphi \neq 0} \sum_{\nu \neq 0} W_\eta \left(\frac{|\varphi \nu|^2}{N} \right) \sum_{\substack{\kappa_2 \quad \mu_1 \quad \kappa_4 \quad \mu_2 \\ (\kappa_2 \mu_1, \kappa_4 \mu_2) \sim \varphi}} |w_2(|\kappa_2|^2) a(\mu_1) w_2(|\kappa_4|^2) a(\mu_2)| K_1^{-2} \ll \\
 & \ll K_1^{-2} \sum_{0 < |\nu|^2 \ll N} \sum_{|\kappa_2|^2 \asymp K_2} \sum_{|\kappa_4|^2 \asymp K_2} \sum_{|\mu_1|^2 \asymp M_1} \sum_{|\mu_2|^2 \asymp M_1} (|a(\mu_1)|^2 + |a(\mu_2)|^2) \ll \\
 & \ll K_1^{-2} N K_2^2 M_1 \sum_{\mu} |a(\mu)|^2 = T^{\varepsilon-1} K_1^{-1} K_2^3 M_1^2 \|a\|_2^2. \quad (6.28)
 \end{aligned}$$

Given that (1.18) holds, the result (6.4) of the lemma now follows by virtue of the equations (6.11) and (6.26), and the bounds noted in (6.27) and (6.28). \square

LEMMA 12. Let \mathcal{D}^* be given by the equations (6.5), (6.7) and (6.3) of Lemma 11. Then, for all $A > 0$, one has

$$\mathcal{D}^* = \mathcal{D}_1^* + \mathcal{D}_2^* + O_{\eta, \varepsilon, A} \left(T^{-A} K_2^2 M_1 \|a\|_2^2 \right), \quad (6.29)$$

where \mathcal{D}_1^* and \mathcal{D}_2^* are as defined in the equations (1.23)–(1.28) of Theorem 2.

PROOF. We begin by reformulating (at the cost of having to introduce a small error term) the factor of form $G_\varphi(\psi_1, \psi_2; 0)$ occurring in the summand on the right-hand

side of Equation (6.5). In doing so, we may assume that

$$e^{-2\eta} K_2 M_1 \leq |\varphi \psi_i|^2 \leq e^{2\eta} K_2 M_1 \quad \text{for } i = 1, 2 \tag{6.30}$$

(for, by (1.19) and (1.20), the coefficient of $G_\varphi(\kappa_2 \mu_1 / \varphi, \kappa_4 \mu_2 / \varphi; 0)$ in (6.5) will otherwise be zero). Now observe that the properties of $W_\eta(u)$ (outlined after (5.27)) ensure that (6.7) implies

$$\left| G_\varphi(\psi_1, \psi_2; 0) - \sum_{\nu \neq 0} L(\psi_1, \psi_2; \nu, 0) \right| \leq \sum_{\substack{\nu \\ N < |\varphi \nu|^2}} |L(\psi_1, \psi_2; \nu, 0)|. \tag{6.31}$$

Here, upon using a change of variable of integration to rewrite the case $\xi = 0$ of (6.3), one finds that

$$L(\psi_1, \psi_2; \nu, 0) = T \int_{\mathbb{C}} \widehat{w_0 \circ \mathfrak{N}} \left(\frac{\sqrt{T\nu}}{\psi_1 \psi_2 z_2} \right) w_1(|\psi_2 z_2|^2) \overline{w_1(|\psi_1 z_2|^2)} d_+ z_2 \tag{6.32}$$

$(\nu \in \mathfrak{D}).$

By (1.19), one has $|w_1(|\psi_2 z_2|^2)| > 0$ only when $|\psi_2 z_2|^2 \asymp K_1$; by this, in combination with the result (5.7) of Lemma 8, the case $i = 0$ of (1.19), the hypothesis that $K_0 = 1$, the inequalities (6.30) and the definition (1.24), it follows that when $|w_1(|\psi_2 z_2|^2)| > 0$ one has

$$\widehat{w_0 \circ \mathfrak{N}} \left(\frac{\sqrt{T\nu}}{\psi_1 \psi_2 z_2} \right) \ll_j \left| \frac{\eta \sqrt{T\nu}}{\psi_1 \psi_2 z_2} \right|^{-2j} \ll_{\eta, j} \left(\frac{T|\varphi \nu|^2}{K_1 K_2 M_1} \right)^{-j} = \left(\frac{T^\epsilon |\varphi \nu|^2}{N} \right)^{-j} \tag{6.33}$$

$(0 \neq \nu \in \mathfrak{D}, \quad j \in \mathbb{N}).$

Therefore, given (6.32) and (1.19) (and subject to (6.30) holding), one has:

$$L(\psi_1, \psi_2; \nu, 0) \ll_{\eta, j} \frac{|\varphi|^2 T K_1}{K_2 M_1} \left(\frac{T^\epsilon |\varphi \nu|^2}{N} \right)^{-j} \quad \text{for all } j \in \mathbb{N}.$$

By applying these estimates to the right-hand side of (6.31) one finds that, for $j \geq 2$,

$$\begin{aligned} \left| G_\varphi(\psi_1, \psi_2; 0) - \sum_{\nu \neq 0} L(\psi_1, \psi_2; \nu, 0) \right| &\ll_{\eta, j} \frac{|\varphi|^2 T K_1}{K_2 M_1} \left(\frac{T^\varepsilon |\varphi|^2}{N} \right)^{-j} \left(\frac{N}{|\varphi|^2} \right)^{1-j} = \\ &= \frac{K_1 N T^{1-j\varepsilon}}{K_2 M_1} = K_1^2 T^{-(j-1)\varepsilon}. \end{aligned}$$

By this one obtains (given (1.18), and subject to (6.30) holding):

$$G_\varphi(\psi_1, \psi_2; 0) = \sum_{\nu \neq 0} L(\psi_1, \psi_2; \nu, 0) + O_{\eta, \varepsilon, A}(T^{-A}) \quad \text{for } A > 0. \quad (6.34)$$

Since (6.33) and (1.19) imply that the sum

$$\sum_{\nu \neq 0} \widehat{w_0 \circ \mathfrak{N}}(T^{1/2} \nu \psi_1^{-1} \psi_2^{-1} z_2^{-1}) w_1(|\psi_2 z_2|^2) \overline{w_1(|\psi_1 z_2|^2)}$$

is uniformly convergent for all $z_2 \in \mathbb{C}$ such that $|z_2|^2 \asymp |\psi_2|^{-2} K_1$, one can deduce from (6.32) and (6.34) (via a change in the order of summation and integration) that, for $A > 0$,

$$\begin{aligned} G_\varphi(\psi_1, \psi_2; 0) + L(\psi_1, \psi_2; 0, 0) &= \\ = T \int_{\mathbb{C}} \left(\sum_{\nu} \widehat{w_0 \circ \mathfrak{N}} \left(\frac{\sqrt{T} \nu}{\psi_1 \psi_2 z_2} \right) \right) w_1(|\psi_2 z_2|^2) \overline{w_1(|\psi_1 z_2|^2)} d_+ z_2 + O_{\eta, \varepsilon, A}(T^{-A}). \end{aligned} \quad (6.35)$$

Here, by means of the identity noted in (5.8), we are able to apply Poisson summation (i. e. the case $\tau = 0$ of the result (5.2) of Lemma 7) so as to obtain:

$$\begin{aligned} \sum_{\nu} \widehat{w_0 \circ \mathfrak{N}} \left(\frac{\sqrt{T} \nu}{\psi_1 \psi_2 z_2} \right) &= \left| \frac{\psi_1 \psi_2 z_2}{\sqrt{T}} \right|^2 \sum_{\alpha} (w_0 \circ \mathfrak{N}) \left(\frac{\psi_1 \psi_2 z_2 \alpha}{\sqrt{T}} \right) = \\ &= \frac{|\psi_1 \psi_2 z_2|^2}{T} \sum_{\alpha} w_0 \left(\frac{|\psi_1 \psi_2 z_2 \alpha|^2}{T} \right). \end{aligned}$$

By this, (1.19) and the hypothesis $K_0 = 1$, it follows that the integral on the right-hand side of Equation (6.35) is trivially equal to zero if $\max\{|\psi_1|^2, |\psi_2|^2\} > e^{2\eta} T K_1^{-1}$. Therefore, given the inequalities in (6.30), it is certainly the case that the integral

in (6.35) is zero whenever φ satisfies $|\varphi|^2 < e^{-4\eta T^{-1}} K_1 K_2 M_1$. This means that when $0 < A < \infty$ one has

$$G_\varphi(\psi_1, \psi_2; 0) = -L(\psi_1, \psi_2; 0, 0) + O_{\eta, \varepsilon, A}(T^{-A}) \quad \text{if } 0 < |\varphi|^2 < e^{-4\eta T^{-\varepsilon}} N. \tag{6.36}$$

Note now that, due to the presence of the factor $W_\eta(|\varphi\nu|^2/N)$ in (6.7), it is effectively a condition of the summation over φ in (6.5) that φ must satisfy

$$|\varphi|^2 \leq e^\eta N. \tag{6.37}$$

Consequently it is only when $|\varphi|^2$ lies in the interval $[e^{-4\eta T^{-\varepsilon}} N, e^\eta N]$ that one has to refrain from using the result (6.36), and make do with (6.34) instead. The combination of (6.34) and (6.36) completes our reformulation of $G_\varphi(\psi_1, \psi_2; 0)$.

Following a change of the variable of integration, the case $\nu = 0$ of (6.32) gives:

$$L(\psi_1, \psi_2; 0, 0) = \widehat{w_0 \circ \mathfrak{N}}(0) T |\varphi|^2 \int_{\mathbb{C}} w_1(|\varphi\psi_2 z|^2) \overline{w_1(|\varphi\psi_1 z|^2)} d_+ z \quad (0 \neq \varphi \in \mathfrak{D}).$$

Given that (subject to (6.30) holding) one has (6.34) and (6.36), and given the last equation (above), the observation (6.37) and the hypothesis (1.19), it follows (by virtue of the arithmetic-geometric mean inequality) that the sum \mathcal{D}^* defined in (6.5) satisfies

$$\mathcal{D}^* = \mathcal{D}''_1 + \mathcal{D}''_2 + O_{\eta, \varepsilon, A} \left(T^{-A} K_2^2 M_1 \sum_{\mu} |a(\mu)|^2 \right) \quad (0 \leq A < \infty), \tag{6.38}$$

with

$$\begin{aligned} \mathcal{D}''_1 = & 4 \sum_{\substack{\beta_1 \neq 0 \\ |(\beta_1, \beta_2)|^2 < e^{-4\eta T^{-\varepsilon}} N}} \sum_{\beta_2 \neq 0} \left(\sum_{\substack{\kappa_2 \\ \kappa_2 \mu_1 = \beta_1}} \sum_{\mu_1} w_2(|\kappa_2|^2) a(\mu_1) \right) \times \\ & \overline{\left(\sum_{\substack{\kappa_4 \\ \kappa_4 \mu_2 = \beta_2}} \sum_{\mu_2} w_2(|\kappa_4|^2) a(\mu_2) \right)} \left(-\widehat{w_0 \circ \mathfrak{N}}(0) \right) \times \\ & \times T |(\beta_1, \beta_2)|^2 \int_{\mathbb{C}} w_1(|\beta_2 z|^2) \overline{w_1(|\beta_1 z|^2)} d_+ z \end{aligned}$$

and

$$\mathcal{D}_2'' = \sum_{e^{-4\eta}T^{-\varepsilon}N \leq |\varphi|^2 \leq e^{\eta}N} \sum_{\substack{\kappa_2 \mu_1 \kappa_4 \mu_2 \\ (\kappa_2\mu_1, \kappa_4\mu_2) \sim \varphi}} w_2(|\kappa_2|^2) a(\mu_1) \overline{w_2(|\kappa_4|^2) a(\mu_2)} \lambda\left(\frac{\kappa_2\mu_1}{\varphi}, \frac{\kappa_4\mu_2}{\varphi}\right), \tag{6.39}$$

where

$$\lambda(\psi_1, \psi_2) = \sum_{\nu \neq 0} L(\psi_1, \psi_2; \nu, 0). \tag{6.40}$$

Given the equality $\widehat{w_0 \circ \mathfrak{N}}(0) = \pi \int_0^\infty w_0(x) dx$ (for which see (7.72), below), it may be deduced from the above that in (6.38) one has

$$\mathcal{D}_1'' = \mathcal{D}_1^*, \tag{6.41}$$

with \mathcal{D}_1^* given by the equations (1.25) and (1.23) of Theorem 2.

We now have only to attend to the term \mathcal{D}_2'' defined in (6.39)–(6.40). Each term appearing in the sum on the right of (6.40) may be expressed, via the equation (6.32), as an integral; upon applying the result (5.6) of Lemma 8 (twice) to the factor $\widehat{w_0 \circ \mathfrak{N}}(T^{1/2}\nu\psi_1^{-1}\psi_2^{-1}z_2^{-1})$ in the integrand in (6.32), and then recalling the definition of the Fourier transform, one finds (thereby) that

$$\begin{aligned} L(\psi_1, \psi_2; \nu, 0) &= \frac{|\psi_1\psi_2|^4}{|2\pi\nu|^4 T} \int_{\mathbb{C}} \int_{\mathbb{C}} \left(\Delta_{\mathbb{C}}^2(w_0 \circ \mathfrak{N}) \right)(s) \times \\ &\quad \times e\left(-\operatorname{Re}\left(\frac{\sqrt{T}\nu s}{\psi_1\psi_2 z_2}\right)\right) d_+ s w_1(|\psi_2 z_2|^2) \overline{w_1(|\psi_1 z_2|^2)} |z_2|^4 d_+ z_2. \end{aligned}$$

Substituting here $z_2 = -\varphi z$, and then summing over non-zero $\nu \in \mathfrak{D}$, one obtains (given (6.40)):

$$\begin{aligned} \lambda(\psi_1, \psi_2) &= \frac{|\varphi|^6 |\psi_1\psi_2|^4}{(2\pi)^4 T} \int_{\mathbb{C}} \int_{\mathbb{C}} \left(\Delta_{\mathbb{C}}^2(w_0 \circ \mathfrak{N}) \right)(s) w_1(|\varphi\psi_2 z|^2) \times \\ &\quad \times \overline{w_1(|\varphi\psi_1 z|^2)} |z|^4 p\left(\frac{\sqrt{T}s}{\varphi\psi_1\psi_2 z}\right) d_+ s d_+ z, \end{aligned}$$

where $p(\alpha)$ is as defined in Theorem 2, Equation (1.26). Since $p(i\alpha) = p(\alpha)$, and since

$$\begin{aligned} \left(\Delta_{\mathbb{C}}^2(w_0 \circ \mathfrak{N})\right)(s) &= \left(\Delta_{\mathbb{C}}^2(w_0 \circ \mathfrak{N})\right)(|s|) = \\ &= \left(\frac{\partial^2}{\partial r^2} + r^{-1} \frac{\partial}{\partial r}\right)^2 w_0(r^2) \Big|_{r=|s|} = \left(4x \frac{d^2}{dx^2} + 4 \frac{d}{dx}\right)^2 w_0(x) \Big|_{x=|s|^2}, \end{aligned}$$

it follows by (6.39) and the expression just obtained for $\lambda(\psi_1, \psi_2)$ that one has

$$\begin{aligned} \mathcal{D}_2'' &= \frac{1}{4\pi^2 T} \sum_{\beta_1 \neq 0} \sum_{\beta_2 \neq 0} |(\beta_1, \beta_2)|^2 |[\beta_1, \beta_2]|^4 \\ &\quad e^{-4\eta T - \varepsilon N \leq |(\beta_1, \beta_2)|^2 \leq e^\eta N} \\ &\times \left(\sum_{\substack{\kappa_2 \\ \kappa_2 \mu_1 = \beta_1}} \sum_{\mu_1} w_2(|\kappa_2|^2) a(\mu_1) \right) \overline{\left(\sum_{\substack{\kappa_4 \\ \kappa_4 \mu_2 = \beta_2}} \sum_{\mu_2} w_2(|\kappa_4|^2) a(\mu_2) \right)} \\ &\times \int_{\mathbb{C}} \int_{\mathbb{C}} \left(\Delta_{\mathbb{C}}^2(w_0 \circ \mathfrak{N})\right)(s) p\left(\frac{\sqrt{T}s}{[\beta_1, \beta_2]z}\right) d_+ s w_1(|\beta_2 z|^2) \overline{w_1(|\beta_1 z|^2)} |z|^4 d_+ z \\ &= \mathcal{D}_2^*, \end{aligned} \tag{6.42}$$

where \mathcal{D}_2^* is as defined in Theorem 2, Equations (1.23), (1.24) and (1.26)–(1.28). By (6.38), (6.41) and (6.42), the result (6.29) follows \square

LEMMA 13. *Suppose that*

$$Z = T^\varepsilon M_1, \tag{6.43}$$

and that Ξ and \mathcal{E}^* are given by the equations (6.2), (6.3) and (6.6)–(6.8) of Lemma 11. Then either

$$\mathcal{E}^* \ll_{\eta, \varepsilon} T^{6\varepsilon} K_1 K_2^2 M_1 \|a\|_2^2,$$

or else, for some $\varphi, \Phi_1, \varphi_1, \Phi_2, \varphi_2 \in \mathfrak{D}$ and some $\mathbf{z} \in \mathbb{C}^3$, one has:

$$0 < |\varphi|^2 \leq e^\eta N, \tag{6.44}$$

$$\Phi_1 \varphi_1 = \Phi_2 \varphi_2 = \varphi, \tag{6.45}$$

$$|z_1|^2 \ll \frac{|\varphi|^2 T^\varepsilon}{N}, \quad |z_2|^2 \ll \frac{|\varphi|^2 T^\varepsilon}{\Xi}, \quad |z_3|^2 \ll \frac{|\Phi_1|^2 T^\varepsilon}{Z} \tag{6.46}$$

and

$$\mathcal{E}^* \ll_\varepsilon T^{1+\varepsilon} |\varphi z_2 z_3|^2 |Y|,$$

where

$$\begin{aligned} Y = & \sum_{\substack{\rho_1 \quad \rho_2 \\ (\rho_1, \varphi_1) \sim 1 \sim (\rho_2, \varphi_2)}} a(\Phi_1 \rho_1) \overline{a(\Phi_2 \rho_2)} \sum_{\nu \neq 0} \sum_{\xi \neq 0} W_\eta \left(\frac{|\varphi \nu|^2}{N} \right) W_\eta \left(\frac{|\varphi \xi|^2}{\Xi} \right) e(\operatorname{Re}(z_1 \nu + z_2 \xi)) \times \\ & \times \sum_{\zeta \neq 0} W_\eta \left(\frac{|\Phi_1 \zeta|^2}{Z} \right) e(\operatorname{Re}(z_3 \zeta)) \sum_{\varpi \neq 0} g_{\rho_1, \rho_2}(|\varpi|^2) S(\nu \xi \rho_1^*, \zeta; \rho_2 \varpi) \end{aligned} \tag{6.47}$$

with

$$\begin{aligned} g_{\rho_1, \rho_2}(x) = & g_{\rho_1, \rho_2}(\varphi_1, \varphi_2; z_1, z_2, z_3; x) = \\ = & \frac{|z_1 z_2 z_3 \rho_1|^2 |\rho_2|^4 x^2}{T} w_0 \left(\frac{|z_1 z_2 z_3 \rho_1|^2 |\rho_2|^4 x^2}{T} \right) w_1(|z_2 \rho_2|^2 x) \\ & \times \overline{w_1(|z_2 z_3 \rho_1 \rho_2|^2 x)} w_2(|\varphi_1 z_3 \rho_2|^2 x) \overline{w_2(|\varphi_2|^2 x)} \end{aligned} \tag{6.48}$$

and with $S(\alpha, \beta; \gamma)$ denoting the Kloosterman sum defined by the equation (5.5) of Lemma 7.

PROOF. We first clarify the effect of the condition $(\kappa_2 \mu_1, \kappa_4 \mu_2) \sim \varphi$ in (6.6), by defining new \mathfrak{D} -valued variables $\Phi_1, \Phi_2, \varphi_1, \varphi_2$, which are rendered dependent by being required to satisfy

$$\Phi_1 \sim (\mu_1, \varphi), \quad \Phi_2 \sim (\mu_2, \varphi)$$

and the condition (6.45). These relations imply that

$$\mu_1 = \Phi_1 \rho_1 \quad \text{and} \quad \mu_2 = \Phi_2 \rho_2,$$

where $\rho_1, \rho_2 \in \mathfrak{D}$ are such that

$$(\rho_1, \varphi_1) \sim (\rho_2, \varphi_2) \sim 1. \tag{6.49}$$

It follows that if $(\kappa_2\mu_1, \kappa_4\mu_2) \sim \varphi$, then $\varphi_1 \mid \kappa_2$ and $\varphi_2 \mid \kappa_4$, and so, for a unique pair $\varpi_1, \varpi_2 \in \mathfrak{D}$, one has

$$\kappa_2 = \varphi_1\varpi_1 \quad \text{and} \quad \kappa_4 = \varphi_2\varpi_2.$$

Given (6.45) and the last two equations above, one has (simultaneously)

$$(\kappa_2\mu_1, \kappa_4\mu_2) \sim \varphi, \quad (\mu_1, \varphi) \sim \Phi_1 \quad \text{and} \quad (\mu_2, \varphi) \sim \Phi_2$$

if and only if the dependent variables $\varphi_1, \varphi_2, \rho_1, \rho_2, \varpi_1, \varpi_2$ satisfy both (6.49) and

$$(\varpi_1\rho_1, \varpi_2\rho_2) \sim 1.$$

Using the above to rewrite the right-hand side of equation (6.6), one obtains:

$$\begin{aligned} \mathcal{E}^* &= \frac{1}{16} \sum_{\varphi \neq 0} \sum_{\substack{\Phi_1 \\ \Phi_1\varphi_1 = \varphi = \Phi_2\varphi_2}} \sum_{\varphi_1} \sum_{\Phi_2} \sum_{\varphi_2} \sum_{\substack{\rho_1 \\ \rho_2}} \sum_{(\rho_1, \varphi_1) \sim 1 \sim (\rho_2, \varphi_2)} a(\Phi_1\rho_1) \overline{a(\Phi_2\rho_2)} \sum_{\varpi_2} \overline{w_2(|\varphi_2\varpi_2|^2)} \times \\ &\times \sum_{\varpi_1} w_2(|\varphi_1\varpi_1|^2) H_\varphi(\rho_1\varpi_1, \rho_2\varpi_2), \end{aligned} \tag{6.50}$$

where $H_\varphi(\psi_1, \psi_2)$ is as defined in (6.7) and (6.8) and is (therefore) equal to zero whenever $\psi_1, \psi_2 \in \mathfrak{D}$ fail to satisfy the condition $(\psi_1, \psi_2) \sim 1$.

Since $W_\eta(u)$ is equal to zero for $u \geq e^\eta$, the definitions (6.7) and (6.8) imply that $H_\varphi(\psi_1, \psi_2)$ is (trivially) equal to zero whenever one has $|\varphi|^2 \geq e^\eta N$. This, combined with the relations

$$\sum_{\varphi \neq 0} \frac{\tau_2^2(\varphi)}{|\varphi|^{2+2\varepsilon}} \ll_\varepsilon \sum_{\varphi \neq 0} \frac{1}{|\varphi|^{2+\varepsilon}} \ll_\varepsilon 1$$

enables us to deduce from (6.50) that, for some $\varphi, \Phi_1, \varphi_1, \Phi_2, \varphi_2 \in \mathfrak{D}$ satisfying (6.44) and (6.45), one has

$$\mathcal{E}^* \ll_\varepsilon |\varphi|^{2+2\varepsilon} \left| \sum_{\substack{\rho_1 \\ \rho_2}} \sum_{(\rho_1, \varphi_1) \sim 1 \sim (\rho_2, \varphi_2)} a(\Phi_1\rho_1) \overline{a(\Phi_2\rho_2)} \sum_{\varpi_2} \overline{w_2(|\varphi_2\varpi_2|^2)} Q(\rho_1, \rho_2\varpi_2) \right|, \tag{6.51}$$

where

$$Q(\rho, \psi) = \sum_{\varpi_1} w_2(|\varphi_1 \varpi_1|^2) H_\varphi(\rho \varpi_1, \psi). \tag{6.52}$$

Accordingly, we assume henceforth (as we may) that $\varphi, \Phi_1, \varphi_1, \Phi_2$ and φ_2 are given Gaussian integers satisfying (6.44) and (6.45), and that what is stated in (6.51) and (6.52) is valid.

By (6.52), (6.8), (6.7) and (6.3) it follows that, for $\rho, \psi \in \mathfrak{D} - \{0\}$, one has

$$Q(\rho, \psi) = \sum_{\nu \neq 0} \sum_{\xi \neq 0} W_\eta \left(\frac{|\varphi \nu|^2}{N} \right) W_\eta \left(\frac{|\varphi \xi|^2}{\Xi} \right) \times \int_{\mathbb{C}} \int_{\mathbb{C}} R(\nu \xi, \rho; \psi; z_1, z_2) e(\operatorname{Re}(\nu z_1 + \xi z_2)) d_+ z_1 d_+ z_2, \tag{6.53}$$

where

$$R(\beta, \rho; \psi; z_1, z_2) = \sum_{\varpi_1 \in \mathfrak{D}} r_{\rho, \psi, z_1, z_2}(\varpi_1) e\left(\operatorname{Re}\left(\frac{\beta \rho^* \varpi_1^*}{\psi}\right)\right), \tag{6.54}$$

with

$$r_{\rho, \psi, z_1, z_2}(s) = w_1(|z_2 \psi|^2) |z_2 \psi \rho s|^2 w_0\left(\frac{|z_1 z_2 \psi \rho s|^2}{T}\right) \overline{w_1(|z_2 \rho s|^2)} w_2(|\varphi_1 s|^2) \tag{6.55}$$

$(s \in \mathbb{C}).$

Therefore, given (1.19), one may apply the result (5.4) of Lemma 7 to the sum on the right-hand side of Equation (6.54), so as to obtain:

$$R(\beta, \rho; \psi; z_1, z_2) = \frac{1}{|\psi|^2} \sum_{\zeta \in \mathfrak{D}} \widehat{r}_{\rho, \psi, z_1, z_2}\left(\frac{\zeta}{\psi}\right) S(\beta \rho^*, \zeta; \psi). \tag{6.56}$$

The condition (1.19) applies to each of the functions w_0, w_1 and w_2 , and so, as is evident from a comparison of (6.55) with the definition (6.19) of the function $f_{\psi_1, \psi_2, z}(\kappa)$ considered in our proof of Lemma 11, one may bound $\widehat{r}_{\rho, \psi, z_1, z_2}(\zeta/\psi)$ through steps similar to the steps described in the paragraph below (6.22) (where we have estimated the Fourier transform $\widehat{f}_{\psi_1, \psi_2, z}$). Indeed, in place of what was noted (concerning the function $f_{\psi_1, \psi_2, z}$) in (6.21) and (6.22), we are now able to note

that, for each fixed choice of ρ, ψ, z_1 and z_2 , one has $r_{\rho, \psi, z_1, z_2}(z) = h(|z|^2)$ (say), where the function $h : [0, \infty) \rightarrow \mathbb{C}$ is such that, for $x > 0$ and $k \in \mathbb{N} \cup \{0\}$, one has

$$h^{(k)}(x) = \begin{cases} O_k \left(K_1 |\psi|^2 (\eta x)^{-k} \right) & \text{if } e^{-\eta} |\varphi_1|^{-2} K_2 \leq x \leq e^{\eta} |\varphi_1|^{-2} K_2, \\ 0 & \text{otherwise,} \end{cases}$$

and where h is identically equal to zero unless one has both

$$e^{-2\eta T/K_1} \leq |\psi z_1|^2 \leq e^{2\eta T/K_1} \quad \text{and} \quad e^{-\eta} K_1 \leq |\psi z_2|^2 \leq e^{\eta} K_1. \quad (6.57)$$

Hence, by means of the result (5.7) of Lemma 8, one finds that if $\zeta \neq 0$ then, for all $j \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \left| \widehat{r}_{\rho, \psi, z_1, z_2} \left(\frac{\zeta}{\psi} \right) \right| &\ll |\psi|^{2j} |\zeta|^{-2j} O_j \left(|\varphi_1|^{-2} K_2 K_1 |\psi|^2 \left(\eta |\varphi_1|^{-1} K_2^{1/2} \right)^{-2j} \right) \ll_{\eta, j} \\ &\ll_{\eta, j} K_1 \left(|\varphi_1|^{-2} K_2 \right)^{1-j} |\psi|^{2j+2} |\zeta|^{-2j}. \end{aligned}$$

By using the last bound above, and estimating the Kloosterman sums in (6.56) trivially (i. e. by the result (5.28) of Lemma 10), one finds that

$$\sum_{|\zeta|^2 \geq X} \frac{\left| \widehat{r}_{\rho, \psi, z_1, z_2}(\zeta/\psi) S(\beta \rho^*, \zeta; \psi) \right|}{|\psi|^2} \ll_{\eta, j} \frac{K_1 |\psi|^4}{T^{(j-1)\varepsilon}} \quad (X \asymp T^\varepsilon K_2^{-1} |\varphi_1 \psi|^2 \text{ and } j \geq 2). \quad (6.58)$$

Since our principal concern (for the remainder of this proof) is with the sum over ρ_1, ρ_2 and ϖ_2 on the right-hand side of Equation (6.51), it therefore follows by (1.19), (1.20) and (6.45) that the sum $Q(\rho, \psi)$ defined in (6.52) requires our attention only in cases where $|\psi|^2 \asymp |\varphi|^{-2} K_2 M_1 \leq K_2 M_1$ and $|\rho|^2 \asymp |\Phi_1|^{-2} M_1 \leq M_1$. In all such cases it follows by (6.56), (6.58), (6.45) and (1.18) that one has, in (6.53),

$$\begin{aligned} R(\beta, \rho; \psi; z_1, z_2) &= \sum_{\zeta} W_{\eta} \left(\frac{|\Phi_1 \zeta|^2}{Z} \right) |\psi|^{-2} \widehat{r}_{\rho, \psi, z_1, z_2} \left(\frac{\zeta}{\psi} \right) S(\beta \rho^*, \zeta; \psi) \\ &\quad + O_{\eta, j} \left(\frac{T^{2-(j-1)\varepsilon} |\psi|^2}{|\rho|^2} \right), \end{aligned} \quad (6.59)$$

with $W_\eta : [0, \infty) \rightarrow [0, 1]$ as described below (5.27), and with $Z = T^\varepsilon M_1$ (as in (6.43)). Moreover, by a change of variable of integration,

$$|\psi|^{-2} \widehat{r}_{\rho, \psi, z_1, z_2} \left(\frac{\zeta}{\psi} \right) = \int_{\mathbb{C}} r_{\rho, \psi, z_1, z_2} (-\psi z_3) e(\operatorname{Re}(\zeta z_3)) d_+ z_3.$$

Upon substituting the last expression for $|\psi|^{-2} \widehat{r}_{\rho, \psi, z_1, z_2}(\zeta/\psi)$ into the case $j = [2/\varepsilon] + 3$ of (6.59), and then substituting the resulting estimate for $R(\beta, \rho; \psi; z_1, z_2)$ into (6.53) (with due application of the properties of the function $W_\eta(u)$ set out below (5.27), while also bearing in mind what was noted in, and just above, (6.57)), one obtains:

$$Q(\rho, \psi) = \sum_{\nu \neq 0} \sum_{\xi \neq 0} \sum_{\zeta} W_\eta \left(\frac{|\varphi \nu|^2}{N} \right) W_\eta \left(\frac{|\varphi \xi|^2}{\Xi} \right) W_\eta \left(\frac{|\Phi_1 \zeta|^2}{Z} \right) \times \\ \times \mathcal{I}(\rho, \psi; \nu, \xi, \zeta) S(\nu \xi \rho^*, \zeta; \psi) + O_{\eta, \varepsilon}(|\varphi|^{-4} N \Xi |\rho \psi|^{-2} T^{1-\varepsilon}), \quad (6.60)$$

with, for $\mathbf{s} \in \mathbb{C}^3$,

$$\mathcal{I}(\rho, \psi; \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3) = \mathcal{I}(\rho, \psi; \mathbf{s}) = \\ = \int_{\mathbb{C}} \int_{\mathbb{C}} \int_{\mathbb{C}} r_{\rho, \psi, z_1, z_2} (-\psi z_3) e(\operatorname{Re}(\mathbf{s} \cdot \mathbf{z})) d_+ z_1 d_+ z_2 d_+ z_3 \quad (6.61)$$

where $\mathbf{z} = (z_1, z_2, z_3)$, and where, by (6.55),

$$r_{\rho, \psi, z_1, z_2} (-\psi z_3) = \left| \rho \psi^2 z_2 z_3 \right|^2 w_0 \left(\frac{|\rho \psi^2 z_1 z_2 z_3|^2}{T} \right) w_1 \left(|z_2 \psi|^2 \right) \times \\ \times \overline{w_1 \left(|\rho \psi z_2 z_3|^2 \right)} w_2 \left(|\varphi_1 \psi z_3|^2 \right). \quad (6.62)$$

By (6.57), (6.62) and (1.19), it follows that

$$|r_{\rho, \psi, z_1, z_2} (-\psi z_3)| > 0 \text{ only if } |z_1|^2 \asymp \frac{T}{|\psi|^2 K_1}, |z_2|^2 \asymp \frac{K_1}{|\psi|^2} \text{ and } |z_3|^2 \asymp \frac{1}{|\rho|^2}. \quad (6.63)$$

By (6.61)–(6.63) and (1.19) (again), one has

$$\mathcal{I}(\rho, \psi; \mathbf{s}) \ll |\rho \psi|^{-2} T K_1 \quad (\rho, \psi \in \mathfrak{D} - \{0\}, \mathbf{s} \in \mathbb{C}^3). \quad (6.64)$$

The result (5.31) of Lemma 10 implies that, for all $\nu, \xi, \rho, \psi \in \mathfrak{D} - \{0\}$ such that $(\rho, \psi) \sim 1$, one has

$$|S(\nu\xi\rho^*, 0; \psi)| \leq |(\nu\xi\rho^*, \psi)|^2 = |(\nu\xi, \psi)|^2.$$

Therefore, given (6.64) and the properties of $W_\eta(\nu)$ summarised below (5.27), it follows that the total contribution to the sum in (6.60) arising from terms with $\zeta = 0$ is

$$O\left(\sum_{\substack{\nu \neq 0 \\ |\varphi\nu|^2 \ll N}} \sum_{\substack{\xi \neq 0 \\ |\varphi\xi|^2 \ll \Xi}} |\rho\psi|^{-2} TK_1 |(\nu\xi, \psi)|^2\right) \ll |\rho\psi|^{-2} TK_1 \sum_{0 < |\lambda|^2 \ll N\Xi/|\varphi|^4} \tau_2(\lambda) |(\lambda, \psi)|^2 \ll_\varepsilon \ll_\varepsilon |\rho\psi|^{-2} TK_1 |\varphi|^{-4} (N\Xi)^{1+\varepsilon} \tau_2(\psi). \quad (6.65)$$

Moreover, given (1.24), (6.1), (6.2) and the hypotheses that $K_1 \geq 1$, $K_2 \geq 1$ and $M_1 \geq 1$, it follows that the final upper bound in (6.65) is greater than what appears inside the brackets of the O -term in (6.60). The result (6.60) therefore yields the estimate

$$Q(\rho, \psi) = \sum_{\nu \neq 0} \sum_{\xi \neq 0} \sum_{\zeta \neq 0} W_\eta\left(\frac{|\varphi\nu|^2}{N}\right) W_\eta\left(\frac{|\varphi\xi|^2}{\Xi}\right) W_\eta\left(\frac{|\Phi_1\zeta|^2}{Z}\right) \times \\ \times \mathcal{I}(\rho, \psi; \nu, \xi, \zeta) S(\nu\xi\rho^*, \zeta; \psi) + O_{\eta, \varepsilon}\left(|\rho\psi|^{-2} TK_1 |\varphi|^{-4} (N\Xi)^{1+\varepsilon} \tau_2(\psi)\right),$$

which, when substituted into (6.51), gives

$$\mathcal{E}^* \ll_\varepsilon |\varphi|^{2+2\varepsilon} |\mathcal{E}^b + \mathcal{E}^\sharp|, \quad (6.66)$$

where

$$\mathcal{E}^b = \sum_{\substack{\rho_1 \\ (\rho_1, \varphi_1) \sim 1}} \sum_{\substack{\rho_2 \\ (\rho_2, \varphi_2) \sim 1}} a(\Phi_1\rho_1) \overline{a(\Phi_2\rho_2)} \times \\ \times \sum_{\varpi_2} \overline{w_2(|\varphi_2\varpi_2|^2)} \sum_{\nu \neq 0} \sum_{\xi \neq 0} W_\eta\left(\frac{|\varphi\nu|^2}{N}\right) W_\eta\left(\frac{|\varphi\xi|^2}{\Xi}\right) \times$$

$$\times \sum_{\zeta \neq 0} W_\eta \left(\frac{|\Phi_1 \zeta|^2}{Z} \right) \mathcal{I}(\rho_1, \rho_2 \varpi_2; \nu, \xi, \zeta) S(\nu \xi \rho_1^*, \zeta; \rho_2 \varpi_2), \quad (6.67)$$

and

$$\begin{aligned} \mathcal{E}^\sharp &= \sum_{\substack{\rho_1 \\ (\rho_1, \varphi_1) \sim 1}} \sum_{\substack{\rho_2 \\ (\rho_2, \varphi_2) \sim 1}} a(\Phi_1 \rho_1) \overline{a(\Phi_2 \rho_2)} \sum_{\varpi_2} \overline{w_2(|\varphi_2 \varpi_2|^2)} \times \\ &\times O_{\eta, \varepsilon} \left(|\varphi|^{-4} (N \Xi)^{1+\varepsilon} T K_1 |\rho_1 \rho_2 \varpi_2|^{-2} \tau_2(\rho_2 \varpi_2) \right) \ll_{\eta, \varepsilon} \\ &\ll_{\eta, \varepsilon} |\varphi|^{-4} (N \Xi)^{1+\varepsilon} T K_1 \times \\ &\times \sum_{\rho_1} \sum_{\rho_2} \frac{|a(\Phi_1 \rho_1) a(\Phi_2 \rho_2)|}{|\rho_1 \rho_2|^2} \sum_{\varpi_2} \left| w_2(|\varphi_2 \varpi_2|^2) \right| |\varpi_2|^{-2} \tau_2(\rho_2 \varpi_2) \ll_\varepsilon \\ &\ll_\varepsilon |\varphi|^{-4} (N \Xi)^{1+\varepsilon} T K_1 \|a\|_2^2 \left(\sum_{\substack{\rho \\ |\Phi_1 \rho|^2 \asymp M_1}} \frac{1}{|\rho|^4} \right)^{1/2} \left(\sum_{\substack{\rho_2 \\ |\Phi_2 \rho_2|^2 \asymp M_1}} \frac{1}{|\rho_2|^4} \right)^{1/2} (M_1 K_2)^\varepsilon \ll \\ &\ll |\varphi|^{-4} |\Phi_1 \Phi_2| M_1^{-1} (M_1 K_2)^\varepsilon (N \Xi)^{1+\varepsilon} T K_1 \|a\|_2^2 \ll |\varphi|^{-2} T^{5\varepsilon} K_1 K_2^2 M_1 \|a\|_2^2 \quad (6.68) \end{aligned}$$

(with the last four estimates following by virtue of the estimate (2.13) and the Cauchy–Schwarz inequality, given that we have $\varepsilon \in (0, 1/6]$, $K_1, K_2, M_1 \in [1, \infty)$, $\varphi, \Phi_1, \varphi_1, \Phi_2, \varphi_2 \in \mathfrak{D} - \{0\}$ and (1.18)–(1.20), (1.24), (6.2) and (6.45)).

By using (6.61) and (6.62) we are able to reformulate (6.67) as the equation

$$\mathcal{E}^b = \int_{\mathbb{C}^*} \int_{\mathbb{C}^*} \int_{\mathbb{C}^*} \mathcal{F}(z_1, z_2, z_3) d_\times z_1 d_\times z_2 d_\times z_3, \quad (6.69)$$

in which we have $d_\times z = |z|^{-2} d_+ z$ and, for $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{C}^3$,

$$\begin{aligned} \mathcal{F}(z_1, z_2, z_3) &= \sum_{\substack{\rho_1 \\ (\rho_1, \varphi_1) \sim 1}} \sum_{\substack{\rho_2 \\ (\rho_2, \varphi_2) \sim 1}} a(\Phi_1 \rho_1) \overline{a(\Phi_2 \rho_2)} \sum_{\nu \neq 0} \sum_{\xi \neq 0} \sum_{\zeta \neq 0} W_\eta \left(\frac{|\varphi \nu|^2}{N} \right) W_\eta \left(\frac{|\varphi \xi|^2}{\Xi} \right) \times \\ &\times W_\eta \left(\frac{|\Phi_1 \zeta|^2}{Z} \right) e(\operatorname{Re}(\mathbf{z} \cdot (\nu, \xi, \zeta))) \sum_{\varpi_2} f(\mathbf{z}; \rho_1, \rho_2, \varpi_2) S(\nu \xi \rho_1^*, \zeta; \rho_2 \varpi_2), \quad (6.70) \end{aligned}$$

with

$$f(\mathbf{z}; \rho_1, \rho_2, \varpi) = |z_1 z_2 z_3|^2 r_{\rho_1, \rho_2 \varpi, z_1, z_2}(-\rho_2 \varpi z_3) \overline{w_2(|\varphi_2 \varpi|^2)}. \tag{6.71}$$

By (6.71), (6.63) and (1.19), one has $f(\mathbf{z}; \rho_1, \rho_2, \varpi_2) \neq 0$ on the right-hand side of Equation (6.70) only when it is the case that $|\varpi_2|^2 \asymp |\varphi_2|^{-2} K_2$ and

$$|z_1|^2 \asymp \frac{T}{|\rho_2 \varpi_2|^2 K_1} \asymp \frac{|\varphi_2|^2 T}{|\rho_2|^2 K_1 K_2}, \quad |z_2|^2 \asymp \frac{K_1}{|\rho_2 \varpi_2|^2} \asymp \frac{|\varphi_2|^2 K_1}{|\rho_2|^2 K_2} \quad \text{and} \quad |z_3|^2 \asymp \frac{1}{|\rho_1|^2}.$$

Furthermore, the hypothesis (1.20) implies that $a(\Phi_1 \rho_1) \overline{a(\Phi_2 \rho_2)} = 0$ unless $|\rho_i|^2 \asymp |\Phi_i|^{-2} M_1$ for $i = 1, 2$. Given what has just been noted, and given the condition (6.45) and the definitions of N, Ξ, Z and $\mathcal{F}(z_1, z_2, z_3)$ in (1.24), (6.2), (6.43) and (6.70), it follows that, for $z_1, z_2, z_3 \in \mathbb{C}$, one has $\mathcal{F}(z_1, z_2, z_3) \neq 0$ only if $|z_1|^2 \asymp T^\varepsilon |\varphi|^2 N^{-1}$, $|z_2|^2 \asymp T^\varepsilon |\varphi|^2 \Xi^{-1}$ and $|z_3|^2 \asymp T^\varepsilon |\Phi_1|^2 Z^{-1}$; since one can (moreover) verify that the function $\mathcal{F}(z_1, z_2, z_3)$ defined by (6.70) is continuous on \mathbb{C}^3 , it therefore follows from (6.69) that one has:

$$\mathcal{E}^b \ll |\mathcal{F}(z_1, z_2, z_3)| \quad \text{for some } (z_1, z_2, z_3) \in (\mathbb{C}^*)^3 \tag{6.72}$$

such that the conditions in (6.46) are satisfied.

We observe that, by (1.18), (1.24), and the hypotheses that $K_1, K_2, M_1 \geq 1$ and $0 < \varepsilon \leq 1/6$, it is certainly the case that $N \ll T$. Therefore, given that φ satisfies (6.44), it follows from (6.66), (6.68) and (6.72) that, for some $(z_1, z_2, z_3) \in \mathbb{C}^3$ satisfying (6.46), one has

$$\mathcal{E}^* \ll_\varepsilon \max \left\{ O_{\eta, \varepsilon} \left(T^{6\varepsilon} K_1 K_2^2 M_1 \|a\|_2^2 \right), |\varphi|^2 T^\varepsilon |\mathcal{F}(z_1, z_2, z_3)| \right\}. \tag{6.73}$$

By (6.70), (6.71), (6.62) and the case $i = 2$ of (1.19), we have $\mathcal{F}(z_1, z_2, z_3) = T|z_2 z_3|^2 Y$, where Y is as defined in the equations (6.47) and (6.48). Recall moreover that, in all of our discussion since obtaining the estimate (6.51)–(6.52), we are able to assume a fixed choice of $\varphi, \varphi_1, \Phi_1, \varphi_2, \Phi_2 \in \mathfrak{D}$ such that both of the conditions (6.44) and (6.45) are satisfied. The results of the lemma are therefore an immediate corollary of our finding that (6.73) holds for some $(z_1, z_2, z_3) \in \mathbb{C}^3$ satisfying (6.46). □

7. Completing the proof of Theorem 2

In this section we first work to establish certain upper bounds for the absolute value of Y (the sum of Kloosterman sums defined by (6.47)–(6.48), in Lemma 13). Our proof of these bounds is principally an exercise in the application of a theorem proved in [49]: for the convenience of the reader, this theorem [49, Theorem 11] is reproduced below (as Lemma 14). There are, however, certain extreme cases in which this method of obtaining bounds for Y fails, because of a limitation on the scope of application of Lemma 14 that is an unfortunate artefact of our own construction, rather than being an essential feature of the method of proof used in [49] (for more details of this matter, see Remark 7.1, below). In these extreme cases we obtain an adequate bound for Y through the application of Lemma 15, below: although the bound in (7.12) is weaker than that in (7.8), it has the advantage of being valid under more general conditions.

Lemma 16 and Lemma 17 are technical in nature (they enable us to ascertain that Lemma 14 and Lemma 15 are applicable, where needed). Bounds for the sum Y are obtained in Lemma 18 (it is in the proof of this lemma that Lemma 14 and Lemma 15 are applied). At the end of the section we complete the proof of Theorem 2 by showing that the results (1.30)–(1.32) may be deduced from the bounds (7.33) and (7.34) of Lemma 18 and the three lemmas of Section 6.

Our assumptions throughout this final section are those detailed in the second and third paragraphs of Section 6.

LEMMA 14. *Let ϑ be the real absolute constant defined in (1.10)–(1.11), let $\Psi_0, \Psi_1, \Psi_2, \Psi_3 \geq 1 \geq \delta > 0$, and let $\varepsilon_1 > 0$. Let D be a complex valued function with domain $\mathfrak{D} - \{0\}$; for $h = 1, 2, 3$, let $A_h : \mathbb{C} \rightarrow \mathbb{C}$ be a smooth function such that, for $j, k \in \mathbb{N} \cup \{0\}$ and $x, y \in \mathbb{R}$, one has*

$$(\delta|x+iy|)^{j+k} \frac{\partial^{j+k}}{\partial x^j \partial y^k} A_h(x+iy) = \begin{cases} O_{j,k}(1) & \text{if } \Psi_h/2 < |x+iy|^2 < \Psi_h, \\ 0 & \text{otherwise.} \end{cases} \quad (7.1)$$

Let $P, Q, R, S \geq 1$ and $X > 0$ satisfy

$$Q = RS \geq \max \left\{ \sqrt{\Psi_0}, \sqrt{\Psi_1} \right\} \quad (7.2)$$

and

$$X = \frac{PS\sqrt{R}}{4\pi^2\sqrt{\Psi_0\Psi_1}} \geq 2, \tag{7.3}$$

and let b be a complex-valued function with domain

$$\mathcal{B}(R, S) = \left\{ (\rho, \sigma) \in \mathfrak{D} \times \mathfrak{D} : R/2 < |\rho|^2 \leq R, S/2 < |\sigma|^2 \leq S \text{ and } (\rho, \sigma) \sim 1 \right\}. \tag{7.4}$$

For each pair $(\rho, \sigma) \in \mathcal{B}(R, S)$, let $g_{\rho, \sigma} : (0, \infty) \rightarrow \mathbb{C}$ be an infinitely differentiable function such that, for $j \in \mathbb{N} \cup \{0\}$ and $x > 0$, one has

$$g_{\rho, \sigma}^{(j)}(x) = \begin{cases} O_j \left((\eta x)^{-j} \right) & \text{if } P/2 < x < P, \\ 0 & \text{otherwise.} \end{cases} \tag{7.5}$$

Put

$$Y_{\dagger} = \sum_{(\rho, \sigma) \in \mathcal{B}(R, S)} b(\rho, \sigma) \sum_{\Psi_0/4 < |\psi_0|^2 \leq \Psi_0} D(\psi_0) \sum_{\Psi_1/2 < |\psi_1|^2 < \Psi_1} A_1(\psi_1) G_{\rho, \sigma}(\psi_0, \psi_1), \tag{7.6}$$

where

$$G_{\rho, \sigma}(\psi_0, \psi_1) = \sum_{0 \neq \varpi \in \mathfrak{D}} g_{\rho, \sigma} \left(|\varpi|^2 \right) S(\rho^* \psi_0, \psi_1; \varpi \sigma), \tag{7.7}$$

with $S(\alpha, \beta; \gamma)$ being the Kloosterman sum that is defined in Equation (5.5), and with $\rho^* \in \mathfrak{D}$ satisfying $\rho^* \rho \equiv 1 \pmod{\varpi \sigma \mathfrak{D}}$ (so that ϖ , the variable of summation in (7.7), is implicitly constrained to satisfy the condition $(\varpi, \rho) \sim 1$). Then

$$Y_{\dagger}^2 \ll_{\varepsilon_1, \eta} \left(1 + \frac{X^2}{(1 + Q\Psi_0^{-1})(1 + Q\Psi_1^{-1})^2\Psi_1} \right)^{\vartheta} \times \\ \times (\Psi_0 + Q)(\Psi_1 + Q) \delta^{-11} Q^{\varepsilon_1} \|b\|_2^2 \|D\|_2^2 \Psi_1 P^2 S \log^2(X). \tag{7.8}$$

If it is moreover the case that

$$\Psi_0 = \Psi_2 \Psi_3, \tag{7.9}$$

and that

$$D(\psi) = \sum_{\psi'|\psi} A_2(\psi') A_3\left(\frac{\psi}{\psi'}\right) \quad (0 \neq \psi \in \mathfrak{D}), \tag{7.10}$$

then one has also:

$$Y_{\dagger}^2 \ll_{\varepsilon_1, \eta} \left(\left(1 + \frac{X^2}{(\Psi_2 + \Psi_3)(1 + Q\Psi_1^{-1})^2\Psi_1} \right)^\vartheta \Psi_0 + \left(1 + \frac{X^2}{Q^2\Psi_0^{-1}(1 + Q\Psi_1^{-1})^2\Psi_1} \right)^\vartheta Q \right) \times \\ \times (\Psi_1 + Q)\delta^{-22}Q^{1+\varepsilon_1} \|b\|_\infty^2 \Psi_0\Psi_1 P^2 S \log^2(X). \tag{7.11}$$

PROOF. This lemma is essentially just a reformulation of [49, Theorem 11]. It is however worth mentioning that, since η may be arbitrarily small, the condition (7.5) is weaker than the corresponding condition in the statement of [49, Theorem 11] (i. e. in [49, Condition (1.4.9)] one has $O_j(x^{-j})$, in place of the term $O_j((\eta x)^{-j})$ which occurs in (7.5)). To justify this weaker condition we observe that if η is assigned any fixed numerical value from the interval $(0, (\log 2)/3]$ (that being the range of values permitted by our current assumptions) then η effectively becomes an absolute positive constant, and so, subject to that assignment having been made, the condition (7.5) will in fact imply that one has $g_{\rho, \sigma}^{(j)}(x) \ll_j x^{-j}$ for all $x \in (P/2, P)$ and all $j \in \mathbb{N} \cup \{0\}$. We therefore obtain bounds which, except in respect of the dependence of the relevant implicit constants upon η , are equivalent to the bounds in [49, (1.4.12) and (1.4.14)]. □

Remark 7.1. The condition $RS \geq \max\{\sqrt{\Psi_0}, \sqrt{\Psi_1}\}$ occurring in (7.2) is somewhat artificial: it is not of any significance as regards the method of proof of [49, Theorem 11], but without it we would have had to include certain additional terms in the upper bounds (7.8) and (7.11) for Y_{\dagger}^2 . At the time of writing [49] we had thought that the condition (7.2) (which corresponds to the condition [49, (1.4.6)]) would not hinder in any way our intended future use of Lemma 14 in proving Theorem 2 of the present paper. However we subsequently found that the constraint $RS \geq \max\{\sqrt{\Psi_0}, \sqrt{\Psi_1}\}$ does prevent the application of Lemma 14 in respect of certain «extreme» cases that arise in our proof of Theorem 2; we therefore regret our earlier decision to simplify [49, Theorem 11] through the inclusion of the condition « $RS \geq \max\{\sqrt{N}, \sqrt{L}\}$ » which appears in [49, (1.4.6)]. Rather than to show now how (7.8) or (7.11) should be modified when one has $RS < \max\{\sqrt{\Psi_0}, \sqrt{\Psi_1}\}$, we

instead prefer an ad hoc solution to problem of the above mentioned deficiency of Lemma 14; our chosen solution makes use of the following lemma.

LEMMA 15. *Let $\vartheta, \varepsilon_1, \delta, \Psi_0, \Psi_1, A_1, D, P, Q, R, S, X, b, \mathcal{B}(R, S)$ and the family $(g_{\rho, \sigma})_{(\rho, \sigma) \in \mathcal{B}(R, S)}$ be such that, if one excludes the both the condition $RS \geq \max\{\sqrt{\Psi_0}, \sqrt{\Psi_1}\}$, occurring in (7.2), and those of the hypotheses of Lemma 14 that are concerned with « Ψ_2 », or « Ψ_3 », or « A_2 », or « A_3 », then all of the remaining hypotheses of that lemma are satisfied. Suppose moreover that Y_{\dagger} is as defined in (7.6)–(7.7). Then*

$$Y_{\dagger}^2 \ll_{\varepsilon_1, \eta} (\Psi_0 \Psi_1)^{\varepsilon_1} \|b\|_2^2 \|D\|_2^2 \Psi_1 P^2 S (\Psi_0 + Q) (\Psi_1 + Q) X^{2\theta} \log^2(X). \tag{7.12}$$

PROOF. By essentially the same steps as those through which the result [49, (9.75)–(9.76)] was arrived at (within the proof of [49, Theorem 11]), we find here that either

$$Y_{\dagger} \ll_{\varepsilon_1} \eta^{-5} \log(X) P S R^{1/2} \|D\|_2 \|A_1\|_2 \times \sum_{(\rho, \sigma) \in \mathcal{B}(R, S)} |b(\rho, \sigma)| \left(1 + \frac{\Psi_0^{(1+\varepsilon_1)/2}}{|\rho\sigma|}\right) \left(1 + \frac{\Psi_1^{(1+\varepsilon_1)/2}}{|\rho\sigma|}\right), \tag{7.13}$$

or else

$$Y_{\dagger} \ll \eta^{-2} \log(X) P S R^{1/2} \int_{-\infty}^{\infty} \frac{U(t) dt}{(1 + |t|)^2} \tag{7.14}$$

where

$$U(t) = \sum_{(\rho, \sigma) \in \mathcal{B}(R, S)} |b(\rho, \sigma)| \sum_{\substack{(\Gamma_0(\rho\sigma)) \\ V \\ \nu_V > 0}} X^{\nu_V} \times \left| \sum_{\frac{\Psi_1}{2} < |\psi_1|^2 < \Psi_1} \frac{A_1(\psi_1) c_V^{\infty}(\psi_1; \nu_V, 0)}{|\psi_1|^{it}} \sum_{\frac{\Psi_0}{4} < |\psi_0|^2 \leq \Psi_0} \frac{\overline{D(\psi_0)} c_V^{1/\sigma}(\psi_0; \nu_V, 0)}{|\psi_0|^{-it}} \right|, \tag{7.15}$$

with the summation to which the superscript « $(\Gamma_0(\rho\sigma))$ » attaches being over cuspidal irreducible subspaces V of $L^2(\Gamma_0(\rho\sigma) \backslash SL(2, \mathbb{C}))$, while the «spectral parameter» ν_V and modified Fourier coefficients $c_V^{\infty}(\psi; \nu_V, 0)$, $c_V^{1/\sigma}(\psi; \nu_V, 0)$ ($0 \neq \psi \in \mathfrak{D}$) are as

defined in [49, Section 1.1]. We have, therefore, just two cases to consider: the case in which (7.13) holds and the case in which what is stated in (7.14)–(7.15) holds.

If (7.13) holds then, by the definition (7.4), the Cauchy–Schwarz inequality and the case $h = 1, j = k = 0$ of the hypothesis (7.1), one has

$$Y_{\dagger}^2 \ll_{\varepsilon_1, \eta} \log^2(X) P^2 S^2 R \|D\|_2^2 \Psi_1 \left(1 + \frac{\Psi_0^{1+\varepsilon_1}}{RS} \right) \left(1 + \frac{\Psi_1^{1+\varepsilon_1}}{RS} \right) RS \|b\|_2^2,$$

and so (given that we have $X \in [2, \infty)$, $RS = Q$ and, by (1.13), $\vartheta \in [0, 2/9]$) the result (7.12) follows.

We now have only to consider the case in which what is stated in (7.14)–(7.15) holds. By the discussion in [49, Section 1.1] leading up to the point noted in [49, (1.1.11)], it follows that in (7.15) one has $1 - \nu_V^2 = \lambda_V$ (say), where λ_V is some positive eigenvalue of the operator $-\Delta_3 : \mathfrak{D}_{\Gamma_0(\rho\sigma)} \rightarrow L^2(\Gamma_0(\rho\sigma) \backslash \mathbb{H}_3)$ (the notation just used being that introduced between (1.8) and (1.9)). In particular, by (1.12) one has $1 - \nu_V^2 \geq 1 - \vartheta^2$ in the sum occurring in (7.15), and so (given the explicit condition $\nu_V > 0$ attached to that sum) each of the relevant spectral parameters ν_V must satisfy $\nu_V \leq \vartheta$. Therefore, since $X \geq 2$, and since (by (1.13)) one has $\vartheta \leq 2/9 < 1$, it follows by way of the triangle inequality from (7.15) that, for all $t \in \mathbb{R}$, we have:

$$\begin{aligned} |U(t)| &\leq X^\vartheta \sum_{j=0}^1 \sum_{(\rho, \sigma) \in \mathcal{B}(R, S)} |b(\rho, \sigma)| \times \\ &\times \sum_{\substack{V \\ |p_V|, |\nu_V| \leq 1}}^{(\Gamma_0(\rho\sigma))} \left| \sum_{\frac{\Psi_1}{2} < |\psi_1|^2 < \Psi_1} A_1(\psi_1) |\psi_1|^{-it} c_V^\infty(\psi_1; \nu_V, p_V) \sum_{\frac{\Psi_0}{2^{j+1}} < |\psi_0|^2 \leq \frac{\Psi_0}{2^j}} \overline{D(\psi_0)} |\psi_0|^{it} c_V^{1/\sigma}(\psi_0; \nu_V, p_V) \right|. \end{aligned}$$

Hence, by an application of the Cauchy–Schwarz inequality, we are able to deduce that

$$|U(t)| \leq X^\vartheta \sum_{j=0}^1 \sum_{(\rho, \sigma) \in \mathcal{B}(R, S)} |b(\rho, \sigma)| U_{\rho, \sigma}^{1/2} V_{\rho, \sigma, j}^{1/2} \quad (t \in \mathbb{R}), \tag{7.16}$$

where

$$U_{\rho, \sigma} = U_{\rho, \sigma}(t) = \sum_{\substack{V \\ |p_V|, |\nu_V| \leq 1}}^{(\Gamma_0(\rho\sigma))} \left| \sum_{\frac{\Psi_1}{2} < |\psi_1|^2 < \Psi_1} A_1(\psi_1) |\psi_1|^{-it} c_V^\infty(\psi_1; \nu_V, p_V) \right|^2,$$

$$V_{\rho,\sigma,j} = V_{\rho,\sigma,j}(t) = \sum_{\substack{V \\ |\nu_V|, |\nu_V| \leq 1}}^{\Gamma_0(\rho\sigma)} \left| \sum_{\substack{\frac{\Psi_0}{2^{j+1}} < |\psi_0|^2 \leq \frac{\Psi_0}{2^j}}} \overline{D(\psi_0)} |\psi_0|^{it} c_V^{1/\sigma}(\psi_0; \nu_V, p_V) \right|^2.$$

By (7.4) and the case « $P = K = 1$ » of [50, Theorem 1] one has, in (7.16),

$$\begin{aligned} U_{\rho,\sigma} &\ll \left(1 + O_{\varepsilon_1} \left(\frac{\Psi_1^{1+\varepsilon_1}}{|\rho\sigma|^2} \right) \right) \|A_1\|_2^2 \quad \text{and} \\ V_{\rho,\sigma,j} &\ll \left(1 + O_{\varepsilon_1} \left(\frac{\Psi_0^{1+\varepsilon_1}}{|\rho\sigma|^2} \right) \right) \|D\|_2^2. \end{aligned} \tag{7.17}$$

Since one has $\int_{-\infty}^{\infty} (1 + |t|)^{-2} dt = 2 < \infty$, it follows that from (7.16), (7.17) and the assumed upper bound (7.14) one may deduce the upper bound

$$\begin{aligned} Y_{\dagger} &\ll_{\varepsilon_1, \eta} X^{\vartheta} \log(X) P S R^{1/2} \|D\|_2 \|A_1\|_2 \times \\ &\times \sum_{(\rho,\sigma) \in \mathcal{B}(R,S)} |b(\rho, \sigma)| \left(1 + \frac{\Psi_0^{(1+\varepsilon_1)/2}}{|\rho\sigma|} \right) \left(1 + \frac{\Psi_1^{(1+\varepsilon_1)/2}}{|\rho\sigma|} \right). \end{aligned} \tag{7.18}$$

By a calculation similar to that employed in dealing with the case in which (7.13) holds, it follows from (7.18) that we obtain the result (7.12). □

LEMMA 16. *Let Υ_{η} be the function defined on the interval $[0, \infty)$ by:*

$$\Upsilon_{\eta}(u) = W_{\eta}(u) - W_{\eta}(e^{\eta}u) \quad (u \geq 0). \tag{7.19}$$

Then Υ_{η} is real valued and infinitely differentiable on $[0, \infty)$; the support of Υ_{η} is contained in the interval $[e^{-\eta}, e^{\eta}]$, and one has both

$$\Upsilon_{\eta}^{(j)}(u) \ll_j \eta^{-j} \quad (u \geq 0 \text{ and } j = 0, 1, 2, \dots) \tag{7.20}$$

and

$$\sum_{0 \leq h < 1 + \eta^{-1} \log B} \Upsilon_{\eta} \left(\frac{|\beta|^2}{B e^{-h\eta}} \right) = W_{\eta} \left(\frac{|\beta|^2}{B} \right) \quad (0 \neq \beta \in \mathfrak{D} \text{ and } 0 < B < \infty). \tag{7.21}$$

Moreover, if $B_1 \in (0, \infty)$ and $\gamma \in \mathbb{C}$, and if A is the complex function given by

$$A(z) = \Upsilon_\eta \left(\frac{|z|^2}{B_1} \right) e(\operatorname{Re}(\gamma z)) \quad (z \in \mathbb{C}), \tag{7.22}$$

then A is smooth on \mathbb{C} and, for

$$\delta = \frac{1}{\eta^{-1} + B_1^{1/2} |\gamma|} \tag{7.23}$$

and all $j, k \in \mathbb{N} \cup \{0\}$, one has:

$$(\delta |x + iy|)^{j+k} \frac{\partial^{j+k}}{\partial x^j \partial y^k} A(x + iy) = \begin{cases} O_{j,k}(1) & \text{if } B_1 e^{-\eta} < |x + iy|^2 < B_1 e^\eta, \\ 0 & \text{otherwise,} \end{cases} \tag{7.24}$$

at all points $(x, y) \in \mathbb{R}^2$.

PROOF. We omit the proofs of the results stated in, or above, (7.20): those results are straightforward consequences of our hypothesis that W_η is as described in Remark 5.2. In order to prove (7.21) we first note that, given (7.19), one can show by induction that

$$\sum_{h=0}^{H-1} \Upsilon_\eta \left(e^{h\eta} u \right) = W_\eta(u) - W_\eta \left(e^{H\eta} u \right) \quad (u \geq 0 \text{ and } H \in \mathbb{N}). \tag{7.25}$$

Then we observe that if $B > 0$ and $\beta \in \mathfrak{D} - \{0\}$ then $|\beta|^2/B \geq 1/B$, and so (given that $W_\eta(u) = \Upsilon_\eta(u) = 0$ for $u \geq e^\eta$) one has $\Upsilon_\eta(e^{h\eta} |\beta|^2/B) = W_\eta(e^{h\eta} |\beta|^2/B) = 0$ for all $h \in [1 + \eta^{-1} \log B, \infty)$. The result (7.21) therefore follows from the case $H = 1 + [\eta^{-1} \log B]$, $u = |\beta|^2/B$ of the equality in (7.25).

In our proofs of the remaining results of the lemma we may assume that $B_1 > 0$, that $\gamma \in \mathbb{C}$, and that $A : \mathbb{C} \rightarrow \mathbb{C}$ is the function defined in (7.22). Since Υ_η is infinitely differentiable on $[0, \infty)$, and since the support of Υ_η is a compact subset of $(0, \infty)$, it therefore follows by (for example) the case $t = 0$ of [49, Lemma 9.4] that the function $z \rightarrow \Upsilon_\eta(B_1^{-1}|z|^2)$ is smooth and compactly supported in $\mathbb{C} - \{0\}$. Moreover, for $x, y \in \mathbb{R}$, we have $e(\operatorname{Re}((x + iy)\gamma)) = \exp(2\pi i(ux - vy)) = \exp(2\pi iux) \times \exp(-2\pi ivy)$, where $u = \operatorname{Re}(\gamma)$, $v = \operatorname{Im}(\gamma)$; note that, since $\exp(z)$ is a holomorphic function on \mathbb{C} , the mappings $z \mapsto \exp(2\pi iu \operatorname{Re}(z))$ and $z \mapsto \exp(-2\pi iv \operatorname{Im}(z))$

are smooth functions on \mathbb{C} . Therefore, given that any product of two smooth functions is smooth, we may conclude that $A(z)$ (as defined in (7.22)) is a product of two functions that are smooth on \mathbb{C} , and so is itself smooth on \mathbb{C} .

Since the support of Υ_η is contained in the interval $[e^{-\eta}, e^\eta]$, and since Υ_η is infinitely differentiable (and so continuous) on $[0, \infty)$, it follows that one has $\Upsilon_\eta(u) = 0$ for all $u \in [0, e^{-\eta}] \cup [e^\eta, \infty)$. Therefore, given the definition (7.22), we find that for all $z \in \mathbb{C}$ one has $A(z) \neq 0$ only if $B_1 e^{-\eta} < |z|^2 < B_1 e^\eta$. This last observation implies part of what is asserted in (7.23)–(7.24); in order to prove the remaining part of (7.23)–(7.24), we need only show that if δ is given by the equation (7.23) then one has the upper bounds

$$\frac{\partial^{j+k}}{\partial x^j \partial y^k} A(x + iy) \ll_{j,k} (\delta |x + iy|)^{-(j+k)} \quad (j, k \in \mathbb{N} \cup \{0\})$$

at all points $(x, y) \in \mathbb{R}^2$ satisfying

$$B_1 e^{-\eta} < |x + iy|^2 < B_1 e^\eta.$$

Since A is a smooth complex function, and since, when x and y are real variables and z is the dependent complex variable $x + iy$, one has

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \frac{\partial}{\partial y} = i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right),$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial \operatorname{Re}(z)} - i \frac{\partial}{\partial \operatorname{Im}(z)} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial \operatorname{Re}(z)} + i \frac{\partial}{\partial \operatorname{Im}(z)} \right),$$

it therefore follows from the preceding discussion that we may complete the proof of the lemma by showing that, for all $m, n \in \mathbb{N} \cup \{0\}$, one has

$$\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} A(z) \ll_{m,n} (\delta |z|)^{-(m+n)} \quad (B_1 e^{-\eta} < |z|^2 < B_1 e^\eta), \quad (7.26)$$

with δ as in (7.23) (and with z understood to be the complex variable having real part x and imaginary part y). Accordingly, we assume henceforth that m and n are non-negative integers, and that δ is the positive real number given by the equation (7.23).

Given how the operators $\partial/\partial z$ and $\partial/\partial \bar{z}$ act upon holomorphic or anti-holomorphic functions (see, for example, [49, Equations (5.21)]), it follows from the definition (7.22) (by way of a short calculation in which the identity $e(\operatorname{Re}(\gamma z)) = \exp(\pi i \gamma z) \exp(\pi i \bar{\gamma} \bar{z})$ is utilized) that one has

$$\begin{aligned} \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} A(z) &= \sum_{r=0}^m \sum_{s=0}^n \binom{m}{r} \binom{n}{s} \times \\ &\times (\pi i \gamma)^r (\pi i \bar{\gamma})^s e(\operatorname{Re}(\gamma z)) \left(\left(\frac{\partial}{\partial z} \right)^{m-r} \left(\frac{\partial}{\partial \bar{z}} \right)^{n-s} \Upsilon_\eta \left(\frac{|z|^2}{B_1} \right) \right). \end{aligned} \tag{7.27}$$

Moreover, by a direct application of the definitions of the operators $\partial/\partial z$, $\partial/\partial \bar{z}$, one can show that

$$\begin{aligned} &\left(\frac{\partial}{\partial z} \right)^{m'} \left(\frac{\partial}{\partial \bar{z}} \right)^{n'} \Upsilon_\eta \left(\frac{|z|^2}{B_1} \right) = \\ &= \sum_{0 \leq t \leq \min\{m', n'\}} \frac{1}{t!} \binom{m'}{t} \binom{n'}{t} B_1^{t-(m'+n')} (\bar{z})^{m'-t} z^{n'-t} \Upsilon_\eta^{(m'+n'-t)} \left(B_1^{-1} |z|^2 \right), \end{aligned}$$

whenever m' and n' are non-negative integers. By this last result, combined with (7.27) and the hypothesis (7.20), it follows that at all points $z \in \mathbb{C}$ such that $0 < |z|^2 < B_1 e^\eta$ one has:

$$\begin{aligned} \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} A(z) &\ll_{m,n} \sum_{r=0}^m \sum_{s=0}^n \binom{m}{r} \binom{n}{s} |\gamma|^{r+s} \sum_{0 \leq t \leq \min\{m-r, n-s\}} \frac{|z|^{(m-r)+(n-s)-2t}}{(\eta B_1)^{(m-r)+(n-s)-t}} < \\ &< e^{(m+n)\eta} \sum_{r=0}^m \sum_{s=0}^n \binom{m}{r} \binom{n}{s} |\gamma|^{r+s} |z|^{(r+s)-(m+n)} \sum_{t=0}^\infty \eta^{r+s+t-(m+n)} = \\ &= e^{(m+n)\eta} (\eta |z|)^{-(m+n)} (1 + \eta |\gamma z|)^{m+n} (1 - \eta)^{-1} \leq \\ &\leq (1 - \eta)^{-1} e^{(m+n)\eta} (\eta |z|)^{-(m+n)} (1 + \eta |\gamma| B_1^{1/2} e^{\eta/2})^{m+n} \leq \\ &\leq (1 - \eta)^{-1} e^{(3/2)(m+n)\eta} (\delta |z|)^{-(m+n)}. \end{aligned}$$

Since $0 < \eta \leq (\log 2)/3 < 1$, the above bounds imply that (7.26) holds, and so are sufficient to complete the proof of the lemma. □

LEMMA 17. Let $\rho_1, \rho_2, \varphi_1, \varphi_2 \in \mathfrak{D} - \{0\}$, let $\mathbf{z} \in \mathbb{C}^3$, and let the function $g : (0, \infty) \rightarrow \mathbb{C}$ satisfy

$$g(x) = g_{\rho_1, \rho_2}(\varphi_1, \varphi_2; z_1, z_2, z_3; x) \quad (0 < x < \infty),$$

where $g_{\rho_1, \rho_2}(\varphi_1, \varphi_2; z_1, z_2, z_3; x)$ is defined by the equation (6.48) of Lemma 13. Then g is infinitely differentiable on $(0, \infty)$, and, for $j \in \mathbb{N} \cup \{0\}$ and $x > 0$, one has:

$$g^{(j)}(x) = \begin{cases} O_j((\eta x)^{-j}) & \text{if } P/2 < x < P, \\ 0 & \text{otherwise,} \end{cases} \tag{7.28}$$

where

$$P = 2^{1/2} |\varphi_2|^{-2} K_2. \tag{7.29}$$

PROOF. We observe that one of the factors of the product on the right-hand side of equation (6.48) is the complex conjugate of $w_2(|\varphi_2|^2 x)$. Therefore, by our hypothesis that (1.19) holds for $i = 2$, it follows that, for $x \in (0, \infty)$, one has $g(x) \neq 0$ only if $e^{-\eta} |\varphi_2|^{-2} K_2 \leq x \leq e^\eta |\varphi_2|^{-2} K_2$; this shows, since $1 < e^\eta \leq 2^{1/3}$, that if P is given by (7.29) then one has $g(x) = 0$ for $x \in (0, 2^{-6/7} P) \cup (2^{-1/7} P, \infty)$, and so one obtains the result (7.28) for all $x \in (0, P/2] \cup [P, \infty)$ and all $j \in \mathbb{N} \cup \{0\}$.

Similarly, in light of the occurrence of the factor $w_0(T^{-1} |z_1 z_2 z_3 \rho_1|^2 |\rho_2|^4 x^2)$ on the right-hand side of equation (6.48), and given that we have $K_0 = 1$, we may infer from (1.19) that

$$g(x) = 0 \quad \text{for all } x \in (0, \infty) \quad \text{satisfying} \quad \frac{|z_1 z_2 z_3 \rho_1|^2 |\rho_2|^4 x^2}{T} > 2^{1/2}. \tag{7.30}$$

We postpone making use of this observation until a later stage of this proof.

Since w_0, w_1, w_2 and the mapping $x \mapsto x^2$ are all infinitely differentiable complex valued functions on the interval $(0, \infty)$, it follows from the stated hypotheses of the lemma and the definition (6.48) that g is an infinitely differentiable complex valued function on the same interval. Indeed, by Leibniz’s rule for higher order derivatives of products, one has, for $j = 0, 1, 2, \dots$, the equality

$$g^{(j)}(x) = \sum_{\substack{j_1 \geq 0 \\ \dots \\ j_6 \geq 0 \\ j_1 + \dots + j_6 = j}} \dots \sum \frac{j!}{(j_1!) \dots (j_6!)} \left(\frac{|z_1 z_2 z_3 \rho_1|^2 |\rho_2|^4}{T} \frac{d^{j_1}}{dx^{j_1}} x^2 \right) \times$$

$$\begin{aligned}
 & \times \left(\frac{d^{j_2}}{dx^{j_2}} w_0 \left(\frac{|z_1 z_2 z_3 \rho_1|^2 |\rho_2|^4 x^2}{T} \right) \right) \left(\frac{d^{j_3}}{dx^{j_3}} w_1 \left(|z_2 \rho_2|^2 x \right) \right) \times \\
 & \times \left(\frac{d^{j_4}}{dx^{j_4}} w_1 \left(|z_2 z_3 \rho_1 \rho_2|^2 x \right) \right) \left(\frac{d^{j_5}}{dx^{j_5}} w_2 \left(|\varphi_1 z_3 \rho_2|^2 x \right) \right) \times \\
 & \times \left(\frac{d^{j_6}}{dx^{j_6}} w_2 \left(|\varphi_2|^2 x \right) \right) \tag{7.31}
 \end{aligned}$$

at all points x of the interval $(0, \infty)$.

Given what has already been shown, the proof of the lemma will be complete if we can show next that $g^{(j)}(x) \ll_j (\eta x)^{-j}$ whenever one has $j \in \mathbb{N} \cup \{0\}$ and $x \in (0, \infty)$. We begin this task by noting that it follows from the hypothesis (1.19) that, when c is a non-negative real constant, when $i \in \{1, 2\}$, and when $j \in \mathbb{N} \cup \{0\}$, one has

$$\frac{d^j}{dx^j} w_i(cx) \ll_j (\eta x)^{-j} \quad (x > 0). \tag{7.32}$$

Moreover, it can be proved by induction that for each $j \in \mathbb{N} \cup \{0\}$ one has the identity

$$\frac{d^j}{dy^j} w_0(y^2) = \sum_{j/2 \leq k \leq j} \alpha(j, k) w_0^{(k)}(y^2) y^{2k-j} \quad (y > 0),$$

where α is the function on $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ determined by the recurrence relation

$$\alpha(j, k) = \begin{cases} (2k + 1 - j)\alpha(j - 1, k) + 2\alpha(j - 1, k - 1) & \text{if } 0 < j/2 \leq k \leq j, \\ 1 & \text{if } j = k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It therefore follows from the hypothesis (1.19) that, when c is a non-negative real constant, and when $j \in \mathbb{N} \cup \{0\}$, one has

$$\frac{d^j}{dx^j} w_0(cx^2) = \sum_{j/2 \leq k \leq j} \alpha(j, k) w_0^{(k)}(cx^2) c^k x^{2k-j} = \sum_{j/2 \leq k \leq j} O_j(\eta^{-k} x^{-j}) \ll_j (\eta x)^{-j} \quad (x > 0).$$

By this last estimate, combined with (7.30), (7.31) and (7.32), we are able to deduce that

$$g^{(j)}(x) = \sum_{\substack{j_1 \geq 0 \\ j_1 + \dots + j_6 = j}} \dots \sum_{\substack{j_6 \geq 0 \\ j_1 + \dots + j_6 = j}} O_j(x^{-j_1}) \prod_{\ell=2}^6 O_{j_\ell}((\eta x)^{-j_\ell}) \ll_j (\eta x)^{-j} \max\{\eta^{j_1} : j_1 \geq 0\} = (\eta x)^{-j}$$

whenever $j \in \mathbb{N} \cup \{0\}$ and $x > 0$. □

LEMMA 18. *Let Ξ and Z be as defined in (6.2) and (6.43), respectively. Let $\varphi, \Phi_1, \varphi_1, \Phi_2, \varphi_2 \in \mathfrak{D}$ and $\mathbf{z} \in \mathbb{C}^3$ be such that the conditions (6.44)–(6.46) of Lemma 13 are satisfied, and let Y be as defined in (6.47)–(6.48). Then one has both*

$$Y \ll_{\eta, \varepsilon} \left(\frac{T^{8\varepsilon} K_2^2 M_1^2 \|a\|_2^2}{|\varphi|^4 |\Phi_1|^2} \right) \left(\frac{M_1^2}{T^{1/2}} + \left(\frac{M_1^{2-(3/2)\vartheta}}{T^{(1-\vartheta)/2}} \right) \left(\frac{K_2}{T^{1/2}} \right)^\vartheta \right) \tag{7.33}$$

and

$$Y \ll_{\eta, \varepsilon} \left(\frac{T^{10\varepsilon} K_2^2 M_1^3 \|a\|_\infty^2}{|\varphi|^4 |\Phi_1|^2} \right) \left(\frac{M_1^2}{T^{1/2}} + \left(\frac{M_1^2}{T^{1/2}} \right)^{1-\vartheta} \left(\left(\frac{K_1}{T^{1/2}} \right)^{\vartheta/2} \left(\frac{K_2}{T^{1/2}} \right)^{1-(\vartheta/2)} + \left(\frac{K_2}{T^{1/2} M_1^{1/2}} \right)^\vartheta \right) \right), \tag{7.34}$$

where ϑ is the real absolute constant defined in (1.10)–(1.11).

PROOF. Given the definition (6.47), it follows by virtue of the identity (7.21) of Lemma 16 that we have

$$Y = \sum_{\substack{\mathbf{h} \in \mathbb{Z}^3 \\ 0 \leq h_i < H_i \ (i=1,2,3)}} Y_\eta(h_1, h_2, h_3), \tag{7.35}$$

where:

$$\begin{aligned} H_1 &= 1 + \eta^{-1} \log \left(|\Phi_1|^{-2} Z \right), \\ H_2 &= 1 + \eta^{-1} \log \left(|\varphi|^{-2} \Xi \right), \\ H_3 &= 1 + \eta^{-1} \log \left(|\varphi|^{-2} N \right) \end{aligned} \tag{7.36}$$

and

$$\begin{aligned}
 Y_\eta(h_1, h_2, h_3) &= \\
 &= \sum_{\substack{\rho_1 \\ (\rho_1, \varphi_1) \sim 1}} \sum_{\substack{\rho_2 \\ (\rho_2, \varphi_2) \sim 1}} a(\Phi_1 \rho_1) \overline{a(\Phi_2 \rho_2)} \sum_{\nu} \sum_{\xi} \sum_{\zeta} \Upsilon_\eta\left(\frac{|\varphi \nu|^2}{e^{-h_3 \eta N}}\right) \Upsilon_\eta\left(\frac{|\varphi \xi|^2}{e^{-h_2 \eta \Xi}}\right) \times \\
 &\quad \times \Upsilon_\eta\left(\frac{|\Phi_1 \zeta|^2}{e^{-h_1 \eta Z}}\right) e(\operatorname{Re}(\mathbf{z} \cdot (\nu, \xi, \zeta))) \sum_{\varpi \neq 0} g_{\rho_1, \rho_2}(|\varpi|^2) S(\nu \xi \rho_1^*, \zeta; \rho_2 \varpi), \quad (7.37)
 \end{aligned}$$

with $S(\alpha, \beta; \gamma)$, $g_{\rho_1, \rho_2}(x)$ and $\Upsilon_\eta(u)$ as given by (5.5), (6.48) and (7.19), respectively. By (6.43)–(6.45), (6.1), (6.2), (1.18), (1.24) and the hypotheses of Theorem 2 concerning ε , η and K_1 , it follows from the definitions in (7.36) that the parameters H_1 , H_2 and H_3 occurring in (7.35) satisfy

$$1 + H_i \leq O\left(\eta^{-1} \log(T)\right) \quad (i = 1, 2, 3). \quad (7.38)$$

Therefore either it is the case that the sum on the right-hand side of Equation (7.35) is empty (i. e. void of terms), or else it is the case that, for some $\mathbf{h} \in \mathbb{Z}^3$ satisfying

$$0 \leq h_i < H_i \quad (i = 1, 2, 3), \quad (7.39)$$

one has

$$Y \ll \left(\eta^{-1} \log(T)\right)^3 |Y_\eta(h_1, h_2, h_3)|. \quad (7.40)$$

Only the latter of these two cases need concern us in what follows: for in the former case one has $Y = 0$, so that the results (7.33) and (7.34) of the lemma are (in that case) certainly valid. Therefore we shall assume henceforth that $\mathbf{h} = (h_1, h_2, h_3)$ is a given element of \mathbb{Z}^3 satisfying the conditions in (7.39), and that this \mathbf{h} is such that the bound (7.40) holds.

We shall complete this proof with the aid of certain bounds for the sum $Y_\eta(h_1, h_2, h_3)$ that we have defined in (7.37); the bounds in question will be obtained through applications of Lemma 14 and Lemma 15. Having that in mind, we now put

$$\Psi_1 = e^{(1-h_1)\eta} |\Phi_1|^{-2} Z, \quad \Psi_2 = e^{(1-h_2)\eta} |\varphi|^{-2} \Xi, \quad \Psi_3 = e^{(1-h_3)\eta} |\varphi|^{-2} N, \quad (7.41)$$

$$A_t(s) = \Upsilon_\eta \left(\frac{|s|^2}{e^{-\eta} \Psi_t} \right) e(\operatorname{Re}(sz_{4-t})) \quad (s \in \mathbb{C}, t = 1, 2, 3), \tag{7.42}$$

$$\Psi_0 = \Psi_2 \Psi_3, \quad D(\psi) = \sum_{\psi'|\psi} A_2(\psi') A_3\left(\frac{\psi}{\psi'}\right) \quad (0 \neq \psi \in \mathfrak{D}), \tag{7.43}$$

$$R = e^\eta |\Phi_1|^{-2} M_1 \quad \text{and} \quad S = e^\eta |\Phi_2|^{-2} M_1, \tag{7.44}$$

and we take $\mathcal{B}(R, S) \subset \mathfrak{D} \times \mathfrak{D}$ to be defined by (7.4) and (7.44); for $(\rho_1, \rho_2) = (\rho, \sigma) \in \mathcal{B}(R, S)$, we put

$$b(\rho, \sigma) = \begin{cases} a(\Phi_1 \rho) \overline{a(\Phi_2 \sigma)} & \text{if } (\rho, \varphi_1) \sim (\sigma, \varphi_2) \sim 1, \\ 0 & \text{otherwise,} \end{cases} \tag{7.45}$$

and we take g_{ρ_1, ρ_2} to be (as in (7.37)) the function defined on the interval $(0, \infty)$ by the equation (6.48).

Let Y_\dagger be the sum of complex terms defined by the equations (7.6), (7.7) and (7.4) of Lemma 14, in conjunction with (7.41)–(7.45), (6.48) and (5.5). By (7.37), the hypothesis (1.20), the result (7.24) of Lemma 16 and the hypothesis that $0 < \eta \leq (\log 2)/3$, it follows that we have

$$Y_\dagger = Y_\eta(h_1, h_2, h_3). \tag{7.46}$$

We seek next to verify as many as possible of the hypotheses of Lemma 14, assuming that $\Psi_0, \Psi_1, \Psi_2, \Psi_3, R, S$, the set $\mathcal{B}(R, S)$, the functions A_1, A_2, A_3, D, b and $g_{\rho, \sigma}$ ($(\rho, \sigma) \in \mathcal{B}(R, S)$) and the sum Y_\dagger are as we have indicated in the last two paragraphs above (and also bearing in mind our hypotheses throughout the present section, and the additional hypotheses of the lemma).

By (6.46), (7.39) and (7.41), we have $e^{-\eta} \Psi_t |z_{4-t}|^2 \ll e^{-h_t \eta} T^\epsilon \leq T^\epsilon$, for $t = 1, 2, 3$. Therefore it follows by the results (7.22)–(7.24) of Lemma 16 that the complex functions A_1, A_2 and A_3 defined in (7.42) are smooth on \mathbb{C} , and are such

that, for $t = 1, 2, 3$ and all $j, k \in \mathbb{N} \cup \{0\}$, one has

$$\left(\frac{|x + iy|}{\eta^{-1} + T^{\varepsilon/2}}\right)^{j+k} \frac{\partial^{j+k}}{\partial x^j \partial y^k} A_t(x + iy) = \begin{cases} O_{j,k}(1) & \text{if } e^{-2\eta}\Psi_t < |x + iy|^2 < \Psi_t, \\ 0 & \text{otherwise,} \end{cases} \tag{7.47}$$

at all points $(x, y) \in \mathbb{R}^2$. Independently of the point just noted, it is shown by Lemma 17 that, for $(\rho_1, \rho_2) = (\rho, \sigma) \in (\mathfrak{D} - \{0\})^2$, the function $g_{\rho_1, \rho_2} : (0, \infty) \rightarrow \mathbb{C}$ defined by Equation (6.48) is infinitely differentiable on $(0, \infty)$, and is furthermore such that, for all $j \in \mathbb{N} \cup \{0\}$ and all $x > 0$, one has

$$g_{\rho_1, \rho_2}^{(j)}(x) = \begin{cases} O_j \left((\eta x)^{-j} \right) & \text{if } 2^{-1/2} |\varphi_2|^{-2} K_2 < x < 2^{1/2} |\varphi_2|^{-2} K_2, \\ 0 & \text{otherwise.} \end{cases} \tag{7.48}$$

By (7.36), (7.39), (7.41) and (7.43), we have $\Psi_1, \Psi_2, \Psi_3 \geq 1$ and $\Psi_0 \geq 1$. Since the definition (7.4) implies that $\mathcal{B}(R, S)$ is the empty set if either $R < 1$ or $S < 1$, and since the definition (7.6) trivially implies that Y_{\dagger} equals zero if $\mathcal{B}(R, S) = \emptyset$, we may therefore assume it is also the case that

$$R, S \geq 1. \tag{7.49}$$

We put now

$$\delta = (\eta^{-1} + T^{\varepsilon/2})^{-1}, \quad \varepsilon_1 = \varepsilon, \quad P = 2^{1/2} |\varphi_2|^{-2} K_2 \tag{7.50}$$

and (as hypothesised in Lemma 14)

$$Q = RS \quad \text{and} \quad X = \frac{PS\sqrt{R}}{4\pi^2\sqrt{\Psi_0\Psi_1}}. \tag{7.51}$$

Given that $0 < \varepsilon \leq 1/6$, it follows by (7.51), (7.50), (7.43), (7.44), (7.41), (6.45), (1.24), (6.2), (6.43), (7.39) and (6.1) that we have

$$4\pi^2 X = \frac{2^{1/2} |\varphi_2|^{-2} K_2 S \sqrt{R}}{\sqrt{\Psi_1 \Psi_2 \Psi_3}} =$$

$$\begin{aligned}
 &= \frac{2^{1/2} |\varphi_2 \Phi_2|^{-2} K_2 M_1^{3/2}}{\sqrt{e^{-(h_1+h_2+h_3)\eta} |\varphi|^{-4} Z \Xi N}} = 2^{1/2} e^{(h_1+h_2+h_3)\eta/2} T^{(1-3\epsilon)/2} \geq \\
 &\geq 2^{1/2} T^{1/4} \geq 8\pi^2.
 \end{aligned} \tag{7.52}$$

Since $0 < \eta \leq (\log 2)/3 < (\log 2)/2 < 1$, it follows from (7.47), (7.48), (7.49), (7.52) and the observations accompanying those points that, if one excludes the condition $RS \geq \max\{\sqrt{\Psi_0}, \sqrt{\Psi_1}\}$ occurring in (7.2), then the remaining hypotheses of Lemma 14 (including the conditions (7.9) and (7.10) attached to the result (7.11)) are satisfied by the choice of $D, A_1, A_2, A_3, \mathcal{B}(R, S), b, \delta, \epsilon_1, g_{\rho, \sigma} ((\rho, \sigma) \in \mathcal{B}(R, S)), \Psi_0, \Psi_1, \Psi_2, \Psi_3, P, Q, R, S$ and Y_{\dagger} declared in (7.41)–(7.45), between (7.45) and (7.46), and in (7.50) and (7.51). It therefore follows by Lemma 15 that we have the bound (7.12) for Y_{\dagger}^2 . Similarly, it follows by Lemma 14 that either we have $\max\{\sqrt{\Psi_0}, \sqrt{\Psi_1}\} > RS$, or else it is the case that the bounds (7.8) and (7.11) for Y_{\dagger}^2 are valid. We conclude from this that there are only two cases requiring further consideration: one of these being the case in which both of the bounds (7.8) and (7.11) hold; the other being the case in which both the bound (7.12) and the inequality $\max\{\sqrt{\Psi_0}, \sqrt{\Psi_1}\} > RS$ hold.

In order to facilitate our use of the results (7.8), (7.11) and (7.12) of Lemma 14 and Lemma 15, we note here that, by (7.49), (7.44), (7.51), (7.52), (1.18), (7.38), (7.39), (7.41), (6.43), (1.24), (7.43), (6.2) and (6.45), it follows that

$$|\Phi_i|^2 \leq e^{\eta} M_1 < 2M_1 \quad (i = 1, 2), \tag{7.53}$$

$$Q \asymp |\Phi_1 \Phi_2|^{-2} M_1^2 \ll T, \quad 2 \leq X \ll e^{(h_1+h_2+h_3)\eta/2} T^{(1-3\epsilon)/2} = T^{O(1)}, \tag{7.54}$$

$$\Psi_1 \ll e^{-h_1\eta} |\Phi_1|^{-2} T^{\epsilon} M_1 < T^{\epsilon} R \leq T^{\epsilon} Q, \quad \Psi_3 \ll e^{-h_3\eta} |\varphi|^{-2} T^{\epsilon-1} K_1 K_2 M_1 \tag{7.55}$$

and

$$\Psi_0 = \Psi_2 \Psi_3 \ll e^{-(h_2+h_3)\eta} |\varphi|^{-4} T^{2\epsilon-1} K_2^2 M_1^2 \ll |\varphi|^{-4} T^{2\epsilon} M_1^2 \ll T^{2\epsilon} Q \ll T^2. \tag{7.56}$$

By (6.44), (1.24) and (1.18), we also have

$$|\varphi|^2 \ll T^{\epsilon-1} K_1 K_2 M_1 \ll T^{\epsilon-(1/2)} K_2 M_1 \ll T^{\epsilon} M_1. \tag{7.57}$$

It moreover follows from (7.45), (7.43), (7.56) and the case $j = k = 0$ of (7.47) that we have:

$$\|b\|_2^2 \leq \sum_{(\rho, \sigma) \in \mathcal{D} \times \mathcal{D}} |a(\Phi_1 \rho) a(\Phi_2 \sigma)|^2 \leq \|a\|_2^4, \quad \|b\|_\infty \leq \|a\|_\infty^2 \tag{7.58}$$

and

$$\begin{aligned} \|D\|_2^2 &= \sum_{\psi_0 \neq 0} \left| \sum_{\substack{|\psi_2|^2 \asymp \Psi_2 \\ \psi_2 \psi_3 = \psi_0}} \sum_{\substack{|\psi_3|^2 \asymp \Psi_3}} O(1) \right|^2 \ll \\ &\ll \sum_{|\psi_2|^2 \asymp \Psi_2} \sum_{\substack{|\psi_3|^2 \asymp \Psi_3 \\ \psi_2' / \psi_2 = \psi_3 / \psi_3'}} \sum \sum 1 \ll \\ &\ll \Psi_2 \Psi_3 (1 + \log(\min\{\Psi_2, \Psi_3\})) \ll_\varepsilon \Psi_0 T^\varepsilon. \end{aligned} \tag{7.59}$$

By (7.44), (7.50), (7.54), (7.55), (7.56) and (6.45), we find furthermore that

$$\Psi_0 \Psi_1 P^2 S Q^2 \ll e^{-(h_1+h_2+h_3)\eta} |\varphi|^{-8} |\Phi_1|^{-6} |\Phi_2|^{-2} T^{3\varepsilon-1} K_2^4 M_1^8. \tag{7.60}$$

If both of the bounds (7.8) and (7.11) hold then, by (7.54), (7.55), (7.56) and the first two parts of (7.50), in combination with (7.58), (7.59), (7.60), (7.39), (6.45), (6.1) and (1.13), it follows that one has both

$$\begin{aligned} Y_{\dagger}^2 &\ll_{\varepsilon, \eta} Q^\varepsilon \|b\|_2^2 \|D\|_2^2 \Psi_1 P^2 S \log^2(X) \left(1 + \frac{X^2 \Psi_0 \Psi_1}{Q^3}\right)^\vartheta Q^2 T^{3\varepsilon} (\eta^{-1} + T^{\varepsilon/2})^{11} \ll_{\varepsilon, \eta} \\ &\ll_{\varepsilon, \eta} T^{10\varepsilon} \|b\|_2^2 \|D\|_2^2 \Psi_1 P^2 S Q^2 \left(1 + \frac{T^{1-3\varepsilon} |\varphi|^{-4} T^{2\varepsilon-1} K_2^2 M_1^2 |\Phi_1|^{-2} T^\varepsilon M_1}{|\Phi_1 \Phi_2|^{-6} M_1^6}\right)^\vartheta \ll_\varepsilon \\ &\ll_\varepsilon T^{11\varepsilon} \|a\|_2^4 \Psi_0 \Psi_1 P^2 S Q^2 \left(1 + \frac{|\Phi_1 \Phi_2|^3 K_2^2}{M_1^3}\right)^\vartheta \ll \\ &\ll T^{14\varepsilon-1} \|a\|_2^4 |\varphi|^{-8} |\Phi_1|^{-4} K_2^4 M_1^8 \left(1 + \frac{K_2^2}{M_1^3}\right)^\vartheta \end{aligned} \tag{7.61}$$

and

$$Y_{\dagger}^2 \ll_{\varepsilon, \eta} Q^{1+\varepsilon} \|b\|_\infty^2 \Psi_0 \Psi_1 P^2 S \log^2(X) \times$$

$$\begin{aligned}
 & \times \left(\left(1 + \frac{X^2 \Psi_1}{Q^2 \Psi_2} \right)^\vartheta \Psi_0 + \left(1 + \frac{X^2 \Psi_0 \Psi_1}{Q^4} \right)^\vartheta Q \right) Q T^\varepsilon \left(\eta^{-1} + T^{\varepsilon/2} \right)^{22} \ll_{\varepsilon, \eta} \\
 & \ll_{\varepsilon, \eta} T^{14\varepsilon} \|b\|_\infty^2 \Psi_0 \Psi_1 P^2 S Q^2 \times \\
 & \times \left(\left(\frac{X^2 \Psi_0 \Psi_1 \Psi_3}{Q^2} \right)^\vartheta \Psi_0^{1-2\vartheta} + \left(1 + \frac{X^2 \Psi_0 \Psi_1}{Q^4} \right)^\vartheta Q T^{2\varepsilon} \right) \ll_\varepsilon \\
 & \ll_\varepsilon T^{14\varepsilon} \|a\|_\infty^4 |\varphi|^{-8} |\Phi_1|^{-6} |\Phi_2|^{-2} T^{3\varepsilon-1} K_2^4 M_1^8 \times \\
 & \times \left(\left(\frac{|\Phi_1|^2 |\Phi_2|^4 T^{\varepsilon-1} K_1 K_2^3}{|\varphi|^6} \right)^\vartheta \left(T^{2\varepsilon-1} K_2^2 M_1^2 \right)^{1-2\vartheta} + \right. \\
 & \left. + \left(1 + \frac{|\Phi_1|^6 |\Phi_2|^8 K_2^2}{|\varphi|^4 M_1^5} \right)^\vartheta |\Phi_1 \Phi_2|^{-2} M_1^2 T^{2\varepsilon} \right) \leq \\
 & \leq T^{19\varepsilon-1} \|a\|_\infty^4 |\varphi|^{-8} |\Phi_1|^{-4} K_2^4 M_1^8 \times \\
 & \times \left(T^{\vartheta-1} K_1^\vartheta K_2^{2-\vartheta} M_1^{2-4\vartheta} + \left(1 + \frac{K_2^2}{M_1^5} \right)^\vartheta M_1^2 \right). \tag{7.62}
 \end{aligned}$$

By (7.40) and (7.46), the bounds (7.61) and (7.62) imply the bounds for Y that are stated in (7.33) and (7.34). Therefore, given the the conclusions of the paragraph immediately below (7.52), and bearing in mind the conditions subject to which (7.61) and (7.62) were obtained, it will suffice for the completion of the proof of the lemma that we show that, if it is the case that both the inequality $\max\{\sqrt{\Psi_0}, \sqrt{\Psi_1}\} > RS$ and the bound (7.12) hold, then it must also be the case that the bounds (7.33) and (7.34) hold. Accordingly, we assume henceforth that the bound (7.12) holds, and that we have

$$\max \left\{ \sqrt{\Psi_0}, \sqrt{\Psi_1} \right\} > RS. \tag{7.63}$$

By (7.63), (7.51), (7.54), (7.55), (7.56) and (7.39), we have $\max\{|\varphi|^{-2} T^\varepsilon M_1, |\Phi_1|^{-1} T^{\varepsilon/2} M_1^{1/2}\} \gg |\Phi_1 \Phi_2|^{-2} M_1^2$. It therefore follows, given (6.1), (6.45) and (7.53), that either

$$M_1 \ll \left(\frac{|\Phi_1 \Phi_2|^2}{|\varphi|^2} \right) T^\varepsilon \leq \left(\frac{|\Phi_1 \Phi_2|^2}{\max \left\{ |\Phi_1|^2, |\Phi_2|^2 \right\}} \right) T^\varepsilon, \tag{7.64}$$

or else

$$M_1^{3/2} \ll |\Phi_1| |\Phi_2|^2 T^{\epsilon/2} \ll \min \left\{ M_1^{1/2} |\Phi_2|^2 T^\epsilon, |\Phi_1| M_1 T^{\epsilon/2} \right\}. \tag{7.65}$$

If it is the case that (7.64) holds, then it follows that one has $M_1 T^{-\epsilon} \ll \min\{|\Phi_1|^2, |\Phi_2|^2\}$; the same conclusion follows if it is instead (7.65) which holds. Hence, and by (6.45), we are certain to have:

$$T^{-\epsilon} M_1 \ll |\Phi_i|^2 \leq |\varphi|^2, \quad \text{for } i = 1, 2. \tag{7.66}$$

In view of (7.57), we find furthermore that the bounds in (7.66) imply that

$$K_2 \gg T^{(1/2)-2\epsilon}. \tag{7.67}$$

We remark that, in view of the bounds for $|\varphi|^2$ in (7.57), our conclusion in (7.66) shows that the case that we are now considering is extreme, in the sense that φ , which first occurs (within the proof of Lemma 11) as a common factor of two independent variables, is almost as large in modulus as it can possibly be.

Given that the bound (7.12) holds, and that we have also $\epsilon_1 = \epsilon$ and what is stated in (6.1), (7.54)–(7.56) and (7.58)–(7.60), it follows that

$$\begin{aligned} Y_{\dagger}^2 &\ll_{\epsilon, \eta} \left(T^{3\epsilon} Q^2 \right)^\epsilon \|b\|_2^2 \|D\|_2^2 \Psi_1 P^2 S Q^2 T^{3\epsilon} X^{2\vartheta} \log^2(T) \ll_{\epsilon} \\ &\ll_{\epsilon} T^{7\epsilon} \|a\|_2^4 \Psi_0 \Psi_1 P^2 S Q^2 X^{2\vartheta} \ll \\ &\ll T^{7\epsilon} \|a\|_2^4 e^{-(h_1+h_2+h_3)\eta} |\varphi|^{-8} |\Phi_1|^{-6} |\Phi_2|^{-2} T^{3\epsilon-1} K_2^4 M_1^8 \left(T e^{(h_1+h_2+h_3)\eta} \right)^\vartheta \leq \\ &\leq |\varphi|^{-8} |\Phi_1|^{-6} |\Phi_2|^{-2} T^{10\epsilon+\vartheta-1} K_2^4 M_1^8 \|a\|_2^4. \end{aligned} \tag{7.68}$$

By (7.68), (7.67), (7.66), (6.1) and the bound $\vartheta \leq 2/9$ of Kim and Shahidi, we obtain:

$$\begin{aligned} Y_{\dagger}^2 &\ll_{\epsilon, \eta} |\varphi|^{-8} |\Phi_1|^{-4} (T^{-\epsilon} M_1)^{-2} T^{10\epsilon-1} \left(T^{2\epsilon} K_2 \right)^{2\vartheta} K_2^4 M_1^8 \|a\|_2^4 < \\ &< |\varphi|^{-8} |\Phi_1|^{-4} T^{13\epsilon-1} K_2^{4+2\vartheta} M_1^6 \|a\|_2^4. \end{aligned} \tag{7.69}$$

Since we certainly have here $M_1^6 \leq M_1^4 M_1^{4-3\vartheta}$, the combination of (7.69) with (7.40) and (7.46) yields the bound (7.33) for Y .

By the hypothesis (1.20), we have furthermore

$$\|a\|_2^4 \leq 64e^{2\eta} M_1^2 \|a\|_\infty^4 < 128M_1^2 \|a\|_\infty^4. \tag{7.70}$$

Since we have $M_1^6 M_1^2 = M_1^8 \leq M_1^{10-5\vartheta}$, the combination of (7.69), (7.70), (7.46) and (7.40) yields the bound (7.34) for Y .

Our work in the last four paragraphs has shown that if both the inequality (7.63) and the bound (7.12) hold, then so do the bounds stated in (7.33) and (7.34); it follows that, for the reasons given in the paragraph immediately below (7.62), the proof of the lemma is now complete. \square

The concluding steps of the proof of Theorem 2. Since T satisfies (6.1), it follows from Lemma 11 and Lemma 12 that we certainly have

$$\begin{aligned} \sum_{d=-\infty}^{\infty} \int_{-\infty}^{\infty} |c(d, it)|^2 w_0 \left(\frac{|2d + it|^2}{\pi^2 T} \right) dt &= 2\pi \widehat{w_0 \circ \mathfrak{N}}(0) T \|C\|_2^2 + \\ &+ (\pi/2) (\mathcal{D}_1^* + \mathcal{D}_2^* + \mathcal{E}^*) + O_{\eta,\varepsilon} \left(\left(T^{5\varepsilon+1/2} K_1 K_2 + K_2^2 \right) M_1 \|a\|_2^2 \right) \text{ (say),} \end{aligned} \tag{7.71}$$

where

$$\widehat{w_0 \circ \mathfrak{N}}(0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_0(x^2 + y^2) dx dy = \int_0^{\infty} w_0(r^2) d(\pi r^2) = \pi \int_0^{\infty} w_0(x) dx, \tag{7.72}$$

while the complex numbers \mathcal{D}_1^* and \mathcal{D}_2^* are defined by the equations (1.23)–(1.28), and the complex number \mathcal{E}^* is given by the equations (6.6)–(6.8), (6.2) and (6.3) (it being assumed that N is defined by (1.24)). Moreover, since (6.46), (6.43) and (6.2) imply that one has

$$T^{1+\varepsilon} |\varphi z_2 z_3|^2 \ll \frac{|\varphi|^4 |\Phi_1|^2 T^{1+3\varepsilon}}{\Xi Z} = \frac{|\varphi|^4 |\Phi_1|^2 T^{1+\varepsilon} K_1}{M_1^2 K_2},$$

it follows by Lemma 13 and Lemma 18 that the term \mathcal{E}^* occurring in Equation (7.71) satisfies both

$$\mathcal{E}^* \ll_{\eta,\varepsilon} T^{6\varepsilon} K_1 K_2^2 M_1 \|a\|_2^2 + \left(\frac{M_1^2}{T^{1/2}} + \left(\frac{M_1^{2-(3/2)\vartheta}}{T^{(1-\vartheta)/2}} \right) \left(\frac{K_2}{T^{1/2}} \right)^\vartheta \right) T^{1+9\varepsilon} K_1 K_2 \|a\|_2^2 \tag{7.73}$$

and

$$\begin{aligned} \mathcal{E}^* &\ll_{\eta,\varepsilon} T^{6\varepsilon} K_1 K_2^2 M_1 \|a\|_2^2 + \left(\frac{M_1^2}{T^{1/2}} + \left(\frac{M_1^2}{T^{1/2}} \right)^{1-\vartheta} \right) \times \\ &\times \left(\left(\frac{K_1}{T^{1/2}} \right)^{\vartheta/2} \left(\frac{K_2}{T^{1/2}} \right)^{1-(\vartheta/2)} + \left(\frac{K_2}{T^{1/2} M_1^{1/2}} \right)^\vartheta \right) T^{1+11\varepsilon} K_1 K_2 M_1 \|a\|_\infty^2. \end{aligned} \tag{7.74}$$

By (6.1) and the hypotheses of the theorem ((1.18) in particular), we have also:

$$\begin{aligned} \left(T^{5\varepsilon+1/2} K_1 K_2 + K_2^2 + T^{6\varepsilon} K_1 K_2^2 \right) M_1 \|a\|_2^2 &\ll T^{6\varepsilon+1/2} K_1 K_2 M_1 \|a\|_2^2 \leq \\ &\leq T^{9\varepsilon+1/2} K_1 K_2 M_1^2 \|a\|_2^2 = \\ &= \left(M_1^2 T^{-1/2} \right) \left(T^{1+9\varepsilon} K_1 K_2 \|a\|_2^2 \right). \end{aligned} \tag{7.75}$$

Moreover, as noted within the proof of Lemma 18 (in (7.70)), we have

$$\|a\|_2^2 \ll M_1 \|a\|_\infty^2. \tag{7.76}$$

By (7.71)–(7.73) and (7.75), the term \mathcal{E} determined by the equations (1.21)–(1.30) satisfies

$$\begin{aligned} \mathcal{E} &= O_{\eta,\varepsilon} \left(\left(T^{5\varepsilon+1/2} K_1 K_2 + K_2^2 \right) M_1 \|a\|_2^2 \right) + (\pi/2) \mathcal{E}^* \ll_{\eta,\varepsilon} \\ &\ll_{\eta,\varepsilon} \left(\frac{M_1^2}{T^{1/2}} + \left(\frac{M_1^{2-(3/2)\vartheta}}{T^{(1-\vartheta)/2}} \right) \left(\frac{K_2}{T^{1/2}} \right)^\vartheta \right) T^{1+9\varepsilon} K_1 K_2 \|a\|_2^2. \end{aligned}$$

We therefore have the bound stated in (1.31), which is the first result of the theorem. Similarly, given (1.30) and (1.22), and given that ε is positive, it follows from (7.71), (7.72), (7.74)–(7.76) and (6.1) that the bound (1.32) holds. □

Remark 7.2. In the proof of Theorem 2 just concluded, we have avoided making any direct use of the «Weil—Estermann bound» bound for Kloosterman sums that is stated in (5.29) (it should nevertheless be noted that we have used the bound (5.29) in proving the results of [50, Theorem A and Theorem 1], upon which the proofs of Lemma 14 and Lemma 15 of the present paper are dependent). The sum of

Kloosterman sums Y defined in (6.47) may, of course, be estimated by means of a direct application of the result in (5.29). This approach to the estimation of Y leads one, via Lemma 11 (in a slightly sharper revised form), Lemma 12, Lemma 13 and (7.72), to the upper bound

$$\mathcal{E} \ll_{\eta,\varepsilon} T^{6\varepsilon} K_1 K_2^{5/2} M_1^{5/2} \|a\|_2^2, \tag{7.77}$$

where \mathcal{E} is the final term in the equation (1.30) of Theorem 2 (it being assumed here that all the hypotheses of that theorem are satisfied).

Recall that, in addition to (7.77), we have also the bounds (1.31) and (1.32) for \mathcal{E} . Yet another bound for the term \mathcal{E} may be by obtained by using both the inequality $xy \leq (x^2 + y^2)/2$ and the elementary bound $|\{\delta \in \mathfrak{D} : 0 < |\delta| \leq r\}| \leq 8r^2$ to estimate that part of the sum on the right-hand side of Equation (6.10) in which one has $\xi_1 \neq \xi_2$; by this method one finds that, subject to the hypotheses of Theorem 2, the term \mathcal{E} in (1.30) must satisfy

$$\mathcal{E} = \begin{cases} O_{\eta,\varepsilon} \left(T^{2\varepsilon} K_1^2 K_2^2 M_1 \|a\|_2^2 \right) & \text{if } T^{\varepsilon-1} K_1 K_2 M_1 > e^{-\eta}, \\ O_{\eta,\varepsilon,A} \left(T^{-A} \|a\|_2^2 \right) & \text{otherwise,} \end{cases} \tag{7.78}$$

where A denotes an arbitrarily large positive constant.

Note that the only cases in which (7.77) is not weaker than (7.78) are those in which one has both

$$K_1 K_2 M_1 \gg T^{1-\varepsilon} \quad \text{and} \quad K_1^2 \gg T^{8\varepsilon} K_2 M_1^3. \tag{7.79}$$

Moreover, when the conditions in both (7.79) and (1.18) are satisfied, the bound for \mathcal{E} in (7.77) is weaker than that which is implied by (1.31). We therefore conclude that, in every case in which the hypotheses of Theorem 2 are satisfied, the combination of (1.31) with the elementary result (7.78) provides a bound for \mathcal{E} that is as at least as strong as that in (7.77).

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Bibliography

1. **R. C. Baker, G. Harman, J. Pintz**, *The difference between consecutive primes, II*, Proc. Lond. Math. Soc. (3) **83** (2001), 532–562.
2. **R. Balasubramanian**, *On the frequency of Titchmarsh's phenomenon for $\zeta(s)$, IV*, Hardy–Ramanujan J. **9** (1986), 1–10.
3. **S. Bettin, V. Chandee, M. Radziwiłł**, *The mean square of the product of the Riemann zeta function with Dirichlet polynomials*, preprint submitted for publication.
4. **V. Blomer, G. Harcos, P. Michel**, *A Burgess-like subconvexity bound for twisted L-functions*, with and appendix by Z. Mao, Forum Math. **19** (2007), 61–106.
5. **R. W. Bruggeman, R. J. Miatello**, *Estimates of Kloosterman sums for groups of real rank one*, Duke Math. J. **80** (1995), 105–137.
6. **R. W. Bruggeman, Y. Motohashi**, *Sum formula for Kloosterman sums and fourth moment of the Dedekind zeta-function over the Gaussian number field*, Funct. Approx. Comment. Math. **31** (2003), 23–92.
7. **H. Davenport**, *Multiplicative Number Theory*, Graduate Texts in Mathematics, No. 74, 2nd ed., revised by H. L. Montgomery, Springer-Verlag, New York, 1980.
8. **J.-M. Deshouillers, H. Iwaniec**, *Kloosterman sums and Fourier coefficients of cusp forms*, Invent. Math. **70** (1982), 219–288.
9. **J.-M. Deshouillers, H. Iwaniec**, *Power mean-values of the Riemann zeta-function*, Mathematika **29** (1982), 202–212.
10. **J.-M. Deshouillers, H. Iwaniec**, *Power mean-values for Dirichlet's polynomials and the Riemann zeta-function, II*, Acta Arith. **43** (1984), 305–312.
11. **W. Duke**, *Some problems in multidimensional analytic number theory*, Acta Arith. **52** (1989), 203–228.

12. **W. Duke, J. B. Friedlander, H. Iwaniec**, *A quadratic divisor problem*, *Invent. Math.* **115** (1994), 209–217.
13. **J. Elstrodt, F. Grunewald, J. Mennicke**, *Groups Acting on Hyperbolic Space: Harmonic Analysis and Number Theory*, Springer Monographs in Mathematics, Springer, Berlin, 1997.
14. **J. A. Gaggero Jara**, *Asymptotic Mean Square of the Product of the Second Power of the Riemann Zeta Function and a Dirichlet Polynomial*, Ph. D. Thesis, University of Rochester, NY, 1997.
15. **G. Harcos**, *Uniform approximate functional equation for principal L -functions*, *Int. Math. Res. Not.* **18** (2002), 923–932.
16. **G. Harcos**, *Uniform approximate functional equation for principal L -functions (Erratum)*, *Int. Math. Res. Not.* **13** (2004), 659–660.
17. **G. H. Hardy, E. M. Wright**, *An Introduction to the Theory of Numbers*, 5th ed., reprinted (with corrections), Oxford University Press, 1984.
18. **G. Harman**, *Prime-Detecting Sieves*, London Mathematical Society Monographs. Vol. 33, Princeton University Press, Princeton, 2007.
19. **G. Harman, A. Kumchev, P. A. Lewis**, *The distribution of prime ideals of imaginary quadratic fields*, *Trans. Amer. Math. Soc.* **356** (2003), 599–620.
20. **E. Hecke**, *Eine neue art von zetafunktionen und ihre beziehungen zur verteilung der primzahlen (erste mitteilung)*, *Math. Z.* **1** (1918), 357–376.
21. **E. Hecke**, *Eine neue art von zetafunktionen und ihre beziehungen zur verteilung der primzahlen (zweite mitteilung)*, *Math. Z.* **6** (1920), 11–51.
22. **C. P. Hughes, M. P. Young**, *The twisted fourth moment of the Riemann zeta function*, *J. Reine Angew. Math.* **641** (2010), 203–236.
23. **M. N. Huxley**, *The large sieve inequality for algebraic number fields. II: means of moments of Hecke zeta-functions*, *Proc. Lond. Math. Soc. (3)* **21** (1970), 108–128.
24. **A. E. Ingham**, *The Distribution of Prime Numbers*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 30, reissued as a volume in the Cambridge Mathematical Library series, with a foreward by R. C. Vaughan, Cambridge University Press, Cambridge, 1990.
25. **A. Ivić**, *An approximate functional equation for a class of Dirichlet series*, *J. Anal.* **3** (1995), 241–252.
26. **A. Ivić**, *The Riemann Zeta-Function: Theory and Applications*, reprinted with a list of errata, Dover, Mineola, 2003.
27. **H. Iwaniec**, *On mean values for Dirichlet's polynomials and the Riemann zeta-function*, *J. London Math. Soc. (2)* **22** (1980), 39–45.

28. **H. Iwaniec, E. Kowalski**, *Analytic Number Theory*, American Mathematical Society Colloquium Publications. Vol. 53, American Mathematical Society, Providence, Rhode Island, 2004.
29. **M. Jutila**, *A variant of the circle method*, in «Sieve Methods, Exponential Sums and their Applications in Number Theory: Proceedings of a symposium held in Cardiff, July 1995», G. R. H. Greaves, G. Harman, M. N. Huxley (eds.), London Mathematical Society Lecture Note Series, No. 237, Cambridge University Press, Cambridge, 1997, 245–254.
30. **R. M. Kaufman**, *An estimate of Hecke's L -functions of the Gaussian field on the line $\text{Re } s = 1/2$* , Dokl. Akad. Nauk. BSSR **22** (1978), 25–28 (in Russian).
31. **R. M. Kaufman**, *Estimate of the Hecke L -functions on the critical line*, in «Analytic Number Theory and the Theory of Functions. Part 2», Zap. Nauchn. Sem. LOMI **91**, «Nauka», Leningrad. (1979), 40–51 (in Russian, with a summary in English).
32. **H. H. Kim**, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2* , with Appendix 1 by D. Ramakrishnan, Appendix 2 by Kim P. Sarnak, J. Amer. Math. Soc. **16** (2003), 139–183.
33. **H. H. Kim**, *On local L -functions and normalized intertwining operators*, Canad. J. Math. **57** (2005), 535–597.
34. **H. H. Kim, F. Shahidi**, *Cuspidality of symmetric powers with applications*, Duke Math. J. **112** (2002), 177–197.
35. **S. Lang**, *Real Analysis*, 2nd ed., Addison–Wesley, Reading, Massachusetts, 1983.
36. **A. F. Lavrik**, *An approximate functional equation for the Hecke zeta function of an imaginary quadratic field*, Mat. Zametki **2** (1967), 475–482 (in Russian).
37. **P. A. Lewis**, *Finding information about Gaussian primes using analytic number theory sieve methods*, Ph. D. thesis, Cardiff University, 2002.
38. **H. Lokvenec-Guleska**, *Sum formula for SL_2 over imaginary quadratic number fields*, doctoral thesis, University of Utrecht, 2004 (includes summaries in Dutch and Macedonian).
39. **Y. Motohashi**, *Spectral Theory of the Riemann Zeta-Function*, Cambridge Tracts in Mathematics, No. 127, Cambridge University Press, Cambridge, 1997.
40. **Y. Motohashi**, *The Riemann Zeta-Function and Hecke Congruence Subgroups. II*, 2007, preprint available as arXiv:0709.2590.
41. **H. Rademacher**, *On the Phragmén-Lindelöf theorem and some applications*, Math. Z. **72** (1959), 192–204.
42. **K. Ramachandra**, *On the frequency of Titchmarsh's phenomenon for $\zeta(s)$, I*, J. London Math. Soc. (2) **8** (1974), 683–690.

43. **K. Ramachandra**, *On the Mean-Value and Omega-Theorems for the Riemann Zeta-Function*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, No. 85, Nairoso Publishing House, Bombay, 1995.
44. **P. Sarnak**, *Fourth moments of gossencharakteren zeta functions*, *Comm. Pure and Applied Math.* **38** (1985), 167–178.
45. **A. Selberg**, *On the estimation of Fourier coefficients of modular forms*, in «Theory of Numbers: Proceedings of the Symposium in Pure Mathematics Held at the California Institute of Technology, Padadena, California, November, 21, 22, 1963», A. L. Whiteman (editor), *Proceedings of Symposia in Pure Mathematics. Vol. VIII*, American Mathematical Society, Providence, Rhode Island, 1965, pp. 1–15.
46. **P. Söhne**, *An upper bound for Hecke zeta-functions with groessencharacters*, *J. Number Theory* **66** (1997), 225–250.
47. **E. C. Titchmarsh**, *The Theory of Functions*, 2nd ed., reprinted with corrections, Oxford University Press, Oxford, 1983.
48. **N. Watt**, *Kloosterman sums and a mean value for Dirichlet polynomials*, *J. Number Theory* **53** (1995), 179–210.
49. **N. Watt**, *Weighted spectral large-sieve inequalities for Hecke congruence subgroups of $SL(2, \mathbb{Z}[i])$* , *Funct. Approx. Comment. Math.* **48** (2013), 213–376.
50. **N. Watt**, *Spectral large sieve inequalities for Hecke congruence subgroups of $SL(2, \mathbb{Z}[i])$* , *J. Number Theory* **140** (2014), 349–424.
51. **E. T. Whittaker**, **G. N. Watson**, *A Course of Modern Analysis*, 4th ed., reprinted, Cambridge University Press, New York, 1973.

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