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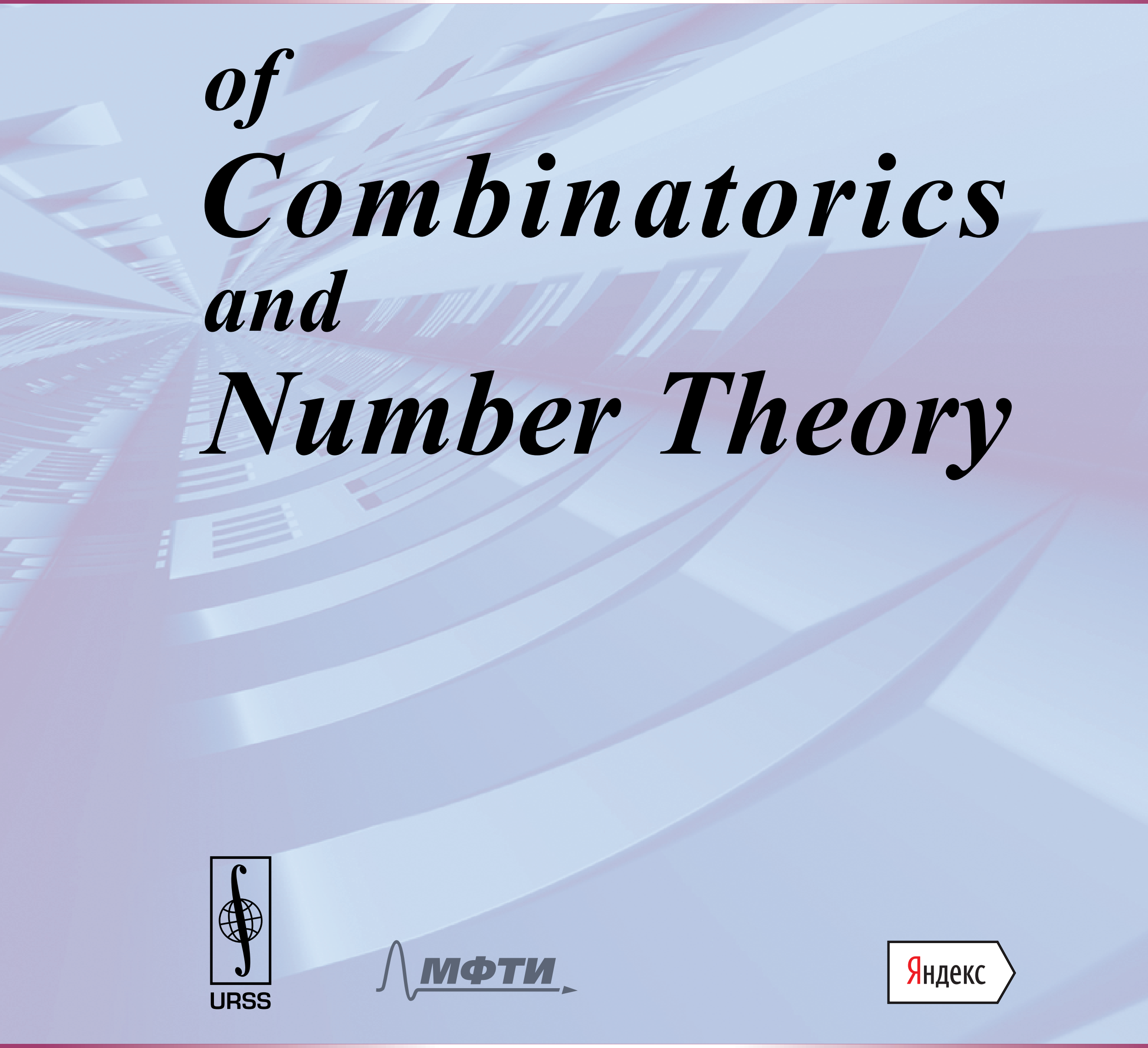

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Tight t-designs on two concentric spheres

Eiichi Bannai (Shanghai), Etsuko Bannai (Fukuoka)

Abstract:
In this paper we develop the general theory of tight Euclidean t-designs on two concentric spheres in $\mathbb{R}^n$. We describe all the useful informations towards the classifications of such objects. For odd t, they were classified completely up to $t \leq 9$ in our previous papers. In this paper, we show that for t odd and $t \geq 13$, there exist only finitely many such objects for each given t, by reducing the problem to the existence of integer solutions of certain diophantine equations. Namely, we show that dimension n is bounded by a certain function of t. Our method does not allow us to get the same conclusion for $t = 11$. So, we will describe many additional informations in the case of $t = 11$. We will also describe the information for even t. This case seems to be more difficult than the case of odd t.

Keywords: spherical design, Euclidean design, tight design, association scheme, coherent configuration, cubature formula, diophantine equation

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1. Introduction

A spherical t-design X is a finite set on the unit sphere S^{n-1} such that any integral of polynomial $f(x)$ of degree at most t on S^{n-1} is given by the average value of $f(x)$ over the finite set X. It was first defined by Delsarte—Goethals—Seidel [15]. (See Definition 2.1 below.) Euclidean t-designs are a two step generalization of the spherical designs, allowing a weight function $w : X \longrightarrow \mathbb{R}_{>0}$ and also allowing X

not necessarily to be on the sphere. Since X is a finite set, it must be on several spheres. The concept of Euclidean t-designs is due to Neumaier—Seidel [21] in combinatorics, but similar concepts were also considered in Mathematical statistics in the name of rotatable designs, and in numerical analysis or approximation theory, in the name of cubature formulas with respect to a spherical symmetric measure on $\mathbb{R}^n$.

Fisher type inequalities for spherical t-designs are generalized to Euclidean designs (See Theorem 2.1 below). See also Möller [18, 19], Neumaier—Seidel [21], Delsarte—Seidel [16], and Bannai—Bannai—Hirao—Sawa [9]. So, the concept of tight Euclidean t-designs were introduced as those attaining the Fisher type bound (see Definition 2.2 below). This lower bound depends on t, n as well as the number p of spheres which supports the finite set X. In the previous several papers [5, 7–10, 13, 14], we started the study of classification problems of tight Euclidean t-designs on two concentric spheres. Considering the case $p = 2$ (i.e., the case on two concentric spheres) is not only it is the next first case beyond the spherical case, but also a very distinctive feature that the coherent configuration is associated to such tight Euclidean designs. (This does not necessarily hold, if $p \geq 3$.) So, in this paper, we mostly consider the case of $p = 2$.

We point out that the classification of tight Euclidean t-designs on two concentric spheres were copmpleted for odd values of t with $t \leq 9$ in our previous papers [5, 8, 13]. The main purpose of this paper is to develop the general theory for tight Euclidean t-designs on two concentric spheres. Our main theorem is Theorem 3.1. (Note that the classification problem for even t seems to be a bit more complicated compared with the case of odd t.) So, in the rest of this paper, we will treat mostly for odd t. In the consideration of tight spherical t-designs for odd t, by Bannai—Damerell II [12], the existence of such designs was reduced to show that Gegenbauer polynomial of degree $\left[\frac{t}{2}\right]$ must have all rational roots. Then the method of Newton-polygons settled the question, i.e., led the non-existence. In our case of tight Euclidean t-designs on two concentric spheres, the problem was reduced to show that all the square of the zeros of Gegenbauer polynomial of degree $\left[\frac{t}{2}\right]$ must be rational numbers, a considerably weaker condition compared with the spherical case. Nontheless, we can apply a similar technique, and we can prove our another main Theorem 3.2 that if $t \geq 13$ then there are only finitely many possible values of n for each t. Namely, there are only finitely many such tight Euclidean designs on two concentric spheres. Unfortunately, we could not get the complete nonexistence for each $t \geq 13$. Also, this technique did not allow us to get a similar conclusion for $t = 11$. Instead, we describe many strong restrictions in the case of $t = 11$, believing that these informations are very useful for the future study of such designs. We also point out that the method for odd t does not imply to get a similar results for even t. So, these problems will be left for future studies.

2. Definition and basic facts on the Euclidean t-designs

We use the following notation. Let $\mathcal{P}(\mathbb{R}^n)$ be the vector space over real number field $\mathbb{R}$ consists of all the polynomials in n variables $x_1, x_2, \ldots, x_n$ with real valued coefficients. For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $\boldsymbol{x} \cdot \boldsymbol{y} = \sum\limits_{i=1}^{n} x_i y_i$ and $||\boldsymbol{x}|| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$. For $f \in \mathcal{P}(\mathbb{R}^n)$, $\deg(f)$ denotes the degree of the polynomial f. Let $\mathrm{Harm}(\mathbb{R}^n)$ the subspace of $\mathcal{P}(\mathbb{R}^n)$ consists of all the harmonic polynomials. For each nonnegative integer l, let $\mathrm{Hom}_l(\mathbb{R}^n) = \langle f \in \mathcal{P}(\mathbb{R}^n) \mid f$ is homogeneous polynomial of degree $l \rangle$. The following are the notations for subspaces of $\mathcal{P}(\mathbb{R}^n)$ we use.

$$\mathrm{Harm}_l(\mathbb{R}^n) := \mathrm{Harm}(\mathbb{R}^n) \cap \mathrm{Hom}_l(\mathbb{R}^n), \qquad \mathcal{P}_e(\mathbb{R}^n) := \oplus_{l=0}^{e} \mathrm{Hom}_l(\mathbb{R}^n),$$

$$\mathcal{P}_e^*(\mathbb{R}^n) := \bigoplus_{l=0}^{[e/2]} \mathrm{Hom}_{e-2l}(\mathbb{R}^n), \; \mathcal{R}_{2(p-1)}(\mathbb{R}^n) := \langle ||\boldsymbol{x}||^{2i} \mid 0 \leq i \leq p-1 \rangle \subset \mathcal{P}_{2(p-1)}(\mathbb{R}^n)$$

For a subset $Y \subset \mathbb{R}^n$, $\mathcal{P}(Y) = \{f|_Y \mid f \in \mathcal{P}(\mathbb{R}^n)\}$. $\mathcal{P}_e(Y)$, $\mathrm{Hom}_l(Y)$, $\mathrm{Harm}_l(Y)$,..., etc., are defined in the same way. We denote the Gegenbauer polynomial of degree l, corresponding to the sphere S^{n-1}, by $Q_{l,n-1}$. We use the normalization so that $Q_{l,n-1}(1) = \dim(\mathrm{Harm}_l(\mathbb{R}^n))$ (see [2, 15] for explicit formula).

Let (X, w) be a weighted finite set in $\mathbb{R}^n$ whose weight satisfies $w(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in X$. Let $\{r_1, r_2, \ldots, r_p\}$ be the set $\{||\boldsymbol{x}|| \mid \boldsymbol{x} \in X\}$ of the length of the vectors in X. Let S_i, $1 \leq i \leq p$, be the sphere of radius r_i centered at the origin. We say that X is supported by p concentric spheres, or the union of p concentric spheres $S = S_1 \cup S_2 \cup \cdots \cup S_p$.

Definition 2.1 ([21]). *A weighted finite set (X, w) is a Euclidean t-design if*

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{S_i} f(\boldsymbol{x}) d\sigma_i(\boldsymbol{x}) = \sum_{\boldsymbol{x} \in X} w(\boldsymbol{x}) f(\boldsymbol{x})$$

holds for any $f \in \mathcal{P}_t(\mathbb{R}^n)$. *In above,* $w(X_i) = \sum\limits_{\boldsymbol{x} \in X_i} w(\boldsymbol{x})$, $\int\limits_{S_i} f(\boldsymbol{x}) d\sigma_i(\boldsymbol{x})$ *is the usual surface integral of the sphere* S_i *of radius* r_i, $|S_i|$ *is the surface area of* S_i.

THEOREM 2.1 ([9, 13, 16, 18, 19, 21], ETC). *Let* $X \subset \mathbb{R}^n$ *be a Euclidean* t*-design supported by a union* S *of* p *concentric spheres. Then the following hold.*

(1) *For* $t = 2e$,

$$|X| \geq \dim(\mathcal{P}_e(S)).$$

(2) *For* $t = 2e + 1$,

$$|X| \geq \begin{cases} 2\dim(\mathcal{P}_e^*(S)) - 1 & \textit{for } e \textit{ even and } \mathbf{0} \in X \\ 2\dim(\mathcal{P}_e^*(S)) & \textit{otherwise.} \end{cases}$$

DEFINITION 2.2 (TIGHTNESS OF DESIGNS). *If an equality holds in one of the inequalities given in Theorem* 2.1, *then* (X, w) *is a tight* t*-design on* p *concentric spheres in* $\mathbb{R}^n$. *Moreover if* $\dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$ *holds for* $t = 2e$, *or* $\dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n))$ *holds for* $t = 2e + 1$, *then* (X, w) *is a tight* t*-design of* $\mathbb{R}^n$.

Möller [19] proved that a tight $(2e+1)$-design (X, w) on p concentric spheres is antipodal and the weight function is center symmetric if e is odd or e is even and $\mathbf{0} \in X$. For the case e is even and $\mathbf{0} \notin X$ Theorem 2.3.6 in [9] implies if we assume $p \leq \frac{e}{2} + 1$, then X is antipodal and the weight function is center symmetric. Hence, Lemma 1.10 in [3] and Lemma 1.7 in [13] imply that weight function of a tight t-design on p concentric spheres is constant on each X_i for $t = 2e$; $t = 2e+1$ and e odd; $t = 2e + 1$, e even and $\mathbf{0} \in X$; $t = 2e + 1$, e even, $\mathbf{0} \notin X$, and $p \leq \frac{e}{2} + 1$.

For more information about the spherical designs and Euclidean designs please refer also [2, 3, 6], etc.

3. Main theorems and basic facts

Let (X, w) be a tight t-design on p concentric spheres. Then as we mentioned in §2, the weight function w is constant on each X_ν, $\nu = 1, \ldots, p$ ([3] and [13]). Let $w(\boldsymbol{x}) = w_\nu$ on X_ν $(1 \leq \nu \leq p)$. If t is odd, then X must be an antipodal set. Also any similarity transformation (including the normalization of the weight) of (X, w)

also gives a tight t-design on p concentric spheres ([3]). Please refer to [3] and [13] for more information of the basic facts on Euclidean t-designs.

Now we assume $p = 2$ and $\mathbf{0} \notin X$. Let $A(X_\lambda, X_\mu) = \left\{ \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{r_\lambda r_\mu} \mid \boldsymbol{x} \in X_\lambda, \boldsymbol{y} \in X_\mu, \boldsymbol{x} \neq \boldsymbol{y} \right\}$ for $\lambda, \mu = 1, 2$. It is known that $\frac{1}{r_1} X_1$ and $\frac{1}{r_2} X_2$ are spherical $(t-2)$-designs on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ ([13]). In the following we use the notation $A(X_\lambda) = A(X_\lambda, X_\lambda)$. Also the following inequalities are known [3, 13].

$$|A(X_\lambda)| = |A(X_\lambda, X_\lambda)| \leq \left[\frac{t+1}{2}\right], \lambda = 1, 2, \tag{3.1}$$

$$|A(X_1, X_2)| \leq \left[\frac{t}{2}\right], \tag{3.2}$$

where $[a]$ denotes the largest integer among those which are less than or equal to a. Let $s_\lambda = |A(X_\lambda)|$, $\lambda = 1,\ 2$. If $\frac{1}{r_\lambda} X_\lambda$ is not a tight spherical $(t-2)$-design, then X_λ must be a $\left[\frac{t+1}{2}\right]$-distance set, i.e., $s_\lambda = \left[\frac{t+1}{2}\right]$. Then X_λ has the structure of Q-polynomial scheme (see [15] for $t = 2e$ case and [4] for $t = 2e+1$ case.) The following are the main results of this paper.

THEOREM 3.1. *Assume* $n \geq 3$. *Let* $e = \left[\frac{t}{2}\right]$, *then the following* (1), (2), *and* (3) *hold.*

1. $|A(X_1, X_2)| = e$. *Moreover if* $t = 2e+1$, *then* $A(X_1, X_2)$ *coincides with the set of zeros of the Gegenbauer polynomial* $Q_e(x)$ *of degree* e.
2. *If* $t = 2e$, *then* $s_\lambda = e-1$ *or* $s_\lambda = e$. *If* $t = 2e+1$, *then* $s_\lambda = e$ *or* $s_\lambda = e+1$, *for* $\lambda = 1, 2$. *Moreover if* $e \geq 5$, *then every element of* $A(X_\lambda)$ *is a rational number for* $\lambda = 1, 2$.
3. *If either* $A(X_1)$ *or* $A(X_2)$ *consists of rational numbers, then* γ^2 *is a rational number for any* $\gamma \in A(X_1, X_2)$.

THEOREM 3.2. *If* $t = 2e+1$ *and* $t \geq 13$, *then there are only finitely many tight* t*-designs on* 2 *concentric spheres up to similar transformations.*

This means that for any fixed odd integer $t \geq 13$, if n is large enough, then there is no tight t-designs on two concentric spheres in $\mathbb{R}^n$. Theorem 3.2 is implied by the following lemma.

LEMMA 3.1. *Let* $n \geq 3$ *and* $e \geq 6$. *Let* $Q_{e,n-1}(x)$ *be the Gegenbauer polynomial of degree* e. *Assume that* $\gamma^2 \in \mathbb{Q}$ *holds for any* γ *satisfying* $Q_{e,n-1}(\gamma) = 0$. *Then there exists a constant* C *depending only on* e *such that* $n \leq C$ *holds.*

Proof of Theorem 3.2. Let (X, w) be a tight $(2e+1)$-designs on 2 concentric spheres with $2e+1 \geq 13$. Then $e \geq 6$ and Theorem 3.1 implies that $\gamma^2 \in \mathbb{Q}$ for any $\gamma \in A(X_1, X_2)$. Therefore Lemma 3.1 implies Theorem 3.2. $\square$

In §4, we give the proof for Theorem 3.1.
In §5, we give the proof for Lemma 3.1.
In §6, we give more precise formulas to determine the values of the inner products in $A(X_1)$, $A(X_2)$, and $A(X_1, X_2)$.
In §7, we discuss the case $t = 11$.

4. Proof of Theorem 3.1

Proof of Theorem 3.1 (1). It is known that $|A(X_1, X_2)| \leq \left[\frac{t}{2}\right] = e$ (see [3, 13]). Let $\boldsymbol{a} \in X_2$. We may assume $\boldsymbol{a} = (r_2, 0, \ldots, 0)$. Let $\{\gamma_1, \ldots, \gamma_s\} = \left\{\frac{\boldsymbol{x} \cdot \boldsymbol{a}}{r_1 r_2} \mid \boldsymbol{x} \in X_1\right\} \subset A(X_1, X_2)$ and $n_j = \left|\{\boldsymbol{x} \in X_1 \mid \boldsymbol{x} \cdot \boldsymbol{a} = \gamma_j\}\right|$ for $j = 1, \ldots s$. Let $\boldsymbol{x} = r_1(x_1, \ldots, x_n)$ for $\boldsymbol{x} \in X_1$. Since $\left(\frac{\boldsymbol{x} \cdot \boldsymbol{a}}{r_1 r_2}\right)^i = x_1^i$ is a polynomial of x_1 with degree i, and $\frac{1}{r_1} X_1$ is a spherical $(t-2)$-design on S^{n-1}, we have

$$\frac{1}{|S^{n-1}|} \int\limits_{\boldsymbol{x} \in S^{n-1}} x_1^i d\sigma(\boldsymbol{x}) = \frac{1}{|X_1|} \sum_{\boldsymbol{x} \in X_1} \left(\frac{\boldsymbol{x} \cdot \boldsymbol{a}}{r_1 r_2}\right)^i = \frac{1}{|X_1|} \sum_{j=1}^{s} n_j \gamma_j^i \tag{4.1}$$

for any nonnegative integer $i \leq t-2$. On the other hand we have

$$\frac{1}{|S^{n-1}|} \int\limits_{\boldsymbol{x} \in S^{n-1}} x_1^i d\sigma(\boldsymbol{x}) = \frac{|S^{n-2}|}{|S^{n-1}|} \int_{-1}^{1} x^i (1-x^2)^{\frac{n-3}{2}} dx \tag{4.2}$$

for $i = 0, 1, \ldots, t-2$. Therefore we have the following cubature formula on $[-1, 1]$ of degree $t-2$.

$$\int_{-1}^{1} x^i (1-x^2)^{\frac{n-3}{2}} dx = \sum_{j=1}^{s} \lambda_j \gamma_j^i, \tag{4.3}$$

where $\lambda_j = \frac{|S^{n-1}|}{|S^{n-2}||X_1|} n_j$, $1 \leq j \leq s$. Hence

$$\int_{-1}^{1} f(x)(1-x^2)^{\frac{n-3}{2}} dx = \sum_{j=1}^{s} \lambda_j f(\gamma_j), \tag{4.4}$$

holds for any polynomial $f(x)$ of degree at most $t-2$.

Note that $Q_{l,n-1}(x)$ is the orthogonal polynomial with respect to the weight function $(1-x^2)^{\frac{n-3}{2}}$. Let $F(x) = \prod_{j=1}^{s}(x-\gamma_j)$. Let $F(x) = \sum_{l=0}^{s} c_l Q_{l,n-1}(x)$ be the Gegenbauer expansion of $F(x)$. If $s \leq \left[\frac{t}{2}\right] - 1 = e-1$, then $\deg(F(x)Q_{l,n-1}(x)) \leq s + l \leq t-2$ for any $0 \leq l \leq s$. Hence we have

$$\int_{-1}^{1} F(x)Q_{l,n-1}(x)\left(1-x^2\right)^{\frac{n-3}{2}} dx = \sum_{j=1}^{s} \lambda_j F(\gamma_j) Q_{l,n-1}(\gamma_j) = 0. \tag{4.5}$$

On the other hand, for any $l \leq s$ we have

$$\int_{-1}^{1} F(x)Q_{l,n-1}(x)\left(1-x^2\right)^{\frac{n-3}{2}} dx = c_l \int_{-1}^{1} (Q_{l,n-1}(x))^2\left(1-x^2\right)^{\frac{n-3}{2}} dx. \tag{4.6}$$

(4.5) and (4.6) imply $c_l = 0$ for any $l \leq s$. Since $\deg(F(x)) = s$, this is a contradiction and we must have $s = \left[\frac{t}{2}\right] = e$ and $|A(X_1, X_2)| = e$. Next assume $t = 2e+1$. Then a similar argument as given above implies $\deg(F(x)) = e$ and $F(x) = c_e Q_{e,n-1}(x) + c_{e-1}Q_{e-1,n-1}(x)$, $c_e \neq 0$. Then $\deg(F(x)Q_{e-1,n-1}(x)) = 2e-1 = t-2$ holds. Hence we must have

$$\int_{-1}^{1} F(x)Q_{e-1,n-1}(x)\left(1-x^2\right)^{\frac{n-3}{2}} dx = c_{e-1} \int_{-1}^{1} (Q_{e-1,n-1}(x))^2\left(1-x^2\right)^{\frac{n-3}{2}} dx = 0. \tag{4.7}$$

This implies $c_{e-1} = 0$ and $F(x)$ must be a nonzero scalar multiple of $Q_{e,n-1}(x)$. This completes the proof of Theorem 3.1 (1). □

PROOF OF THEOREM 3.1 (2). As we mentioned in §2, X_λ ($\lambda = 1,\ 2$) is a spherical $(t-2)$-design, $\left[\frac{t+1}{2}\right] - 1 \leq s_\lambda \leq \left[\frac{t+1}{2}\right]$ and X_λ has the structure of a Q-polynomial association scheme of class s_λ (see [4, 15]). If $t = 2e$, then $s_\lambda = e - 1$ or e and X_λ is antipodal. If $t = 2e + 1$, then $s_\lambda = e$ or $e + 1$, and X_λ is antipodal. Let $\left\{E_0^{(\lambda)}, E_1^{(\lambda)}, \dots, E_{s_\lambda}^{(\lambda)}\right\}$ be the basis of primitive idempotents corresponds to the Q-polynomial structure of X_λ. Then for $\boldsymbol{x}, \boldsymbol{y} \in X_\lambda$, we have

$$|X_\lambda| E_1^{(\lambda)}(\boldsymbol{x}, \boldsymbol{y}) = n \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{r_\lambda^2}, \tag{4.8}$$

where $\dfrac{\boldsymbol{x} \cdot \boldsymbol{y}}{r_\lambda^2} \in A(X_\lambda)$. Let $m_i^{(\lambda)}$ $(0 \le i \le s_\lambda)$ be the rank of $E_i^{(\lambda)}$ for $\lambda = 1, 2$. Therefore, if we prove that every entry of $E_1^{(\lambda)}$ is a rational number, then we have Theorem 3.1 (2). So we assume that there exist irrational numbers among the entries of $E_1^{(\lambda)}$. Then $E_1^{(\lambda)}$ is a matrix over a field K which is a finite extension over $\mathbb{Q}$. Since X_λ is a Q-polynomial scheme, all the primitive idempotents $E_i^{(\lambda)}$, $0 \le i \le s_\lambda$ are matrices over K. We consider the Bose—Mesner algebra over K. Denote it by $\mathfrak{M}_\lambda^K$. Let $G = G(K/\mathbb{Q})$ be the Galois group. Then Bose—Mesner algebra over K is invariant as a set under the action of G because the adjacency matrices are integral. Let σ be any element in G. Then $\left\{\sigma(E_0^{(\lambda)}) = E_0^{(\lambda)}, \sigma(E_1^{(\lambda)}), \dots, \sigma(E_{s_\lambda}^{(\lambda)})\right\}$ is a basis of primitive idempotents of $\mathfrak{M}_\lambda^K$ which is unique as a set. Hence we must have

$$\{\sigma(E_0^{(\lambda)}) = E_0^{(\lambda)}, \sigma(E_1^{(\lambda)}), \dots, \sigma(E_{s_\lambda}^{(\lambda)})\} = \{E_0^{(\lambda)}, E_1^{(\lambda)}, \dots, E_{s_\lambda}^{(\lambda)}\} \tag{4.9}$$

and $\left\{E_0^{(\lambda)}, \sigma(E_1^{(\lambda)}), \dots, \sigma(E_{s_\lambda}^{(\lambda)})\right\}$ gives a Q-polynomial structure on X_λ. It is known that $m_i^{(\lambda)} = Q_{n-1,i}(1) = \binom{n-1+i}{i} - \binom{n-1+i-2}{i-2}$ for

$$\begin{cases} 0 \le i \le s_\lambda, & \text{if } t = 2e \text{ and } s_\lambda = e - 1, \\ 0 \le i \le s_\lambda - 1, & \text{if } t = 2e \text{ and } s_\lambda = e, \\ 0 \le i \le s_\lambda - 2, & \text{if } t = 2e + 1 \text{ and } s_\lambda = e, e + 1. \end{cases} \tag{4.10}$$

Since $n \ge 3$, we have

$$\begin{cases} m_0^{(\lambda)} < m_1^{(\lambda)} < \cdots < m_{s_\lambda}^{(\lambda)}, & \text{if } t = 2e \text{ and } s_\lambda = e - 1, \\ m_0^{(\lambda)} < m_1^{(\lambda)} < \cdots < m_{s_\lambda - 1}^{(\lambda)}, & \text{if } t = 2e \text{ or } t = 2e + 1 \text{ and } s_\lambda = e, \\ m_0^{(\lambda)} < m_1^{(\lambda)} < \cdots < m_{s_\lambda - 2}^{(\lambda)}, & \text{if } t = 2e + 1 \text{ and } s_\lambda = e + 1. \end{cases} \tag{4.11}$$

More details of the formula for $m_i^{(\lambda)}$ please refer to [2, 4, 15]. If $t = 2e$ and $s_\lambda = e - 1$ (this means X_λ is a spherical tight $(2e-2)$-design), then $m_1^{(\lambda)} \neq m_i^{(\lambda)}$ for any $i \neq 1$. Hence $\sigma(E_1^{(\lambda)}) = E_1^{(\lambda)}$ for any $\sigma \in G$. Therefore $E_1^{(\lambda)}$ is a matrix over $\mathbb{Q}$ (this is more or less a known fact). For other cases we need more arguments. We introduce the following theorem proved in [24].

THEOREM 4.1 (SUZUKI [24]). *Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq s})$ be a Q-polynomial association scheme with $m_1 > 2$. Then the following hold.*

(1) *$\mathfrak{X}$ has at most 2 distinct orderings of the primitive idempotents which give Q-polynomial structures.*

(2) *If $E_0, E_1, \ldots, E_s$ is an ordering of the primitive idempotents of $\mathfrak{X}$ which gives a Q-polynomial structure of $\mathfrak{X}$. Then possible second ordering must be one of the followings:*

  I. *If $s > 5$, then*

    (a) $E_0, E_2, E_4, E_6, \ldots, E_5, E_3, E_1$,
    (b) $E_0, E_s, E_1, E_{s-1}, E_2, E_{s-2}, E_3, E_{s-3}, \ldots$,
    (c) $E_0, E_s, E_2, E_{s-2}, E_4, E_{s-4}, \ldots, E_{s-5}, E_5, E_{s-3}, E_3, E_{s-1}, E_1$,
    (d) $E_0, E_{s-1}, E_2, E_{s-3}, E_4, E_{s-5}, \ldots, E_5, E_{s-4}, E_3, E_{s-2}, E_1, E_s$.

  II. *If $s = 5$, then*

    (e) $E_0, E_5, E_3, E_2, E_4, E_1$.

Case $t = 2e$ and $s_\lambda = e$: Then $m_0^{(\lambda)} < m_1^{(\lambda)} < \cdots < m_{e-1}^{(\lambda)} = m_{s_\lambda - 1}^{(\lambda)}$ holds. Hence if $e \geq 5$, then none of the second possible ordering is impossible and we must have $\sigma(E_1^{(\lambda)}) = E_1^{(\lambda)}$.

Case $t = 2e + 1$ and $s_\lambda = e$: Then $m_0^{(\lambda)} < m_1^{(\lambda)} < \cdots < m_{e-1}^{(\lambda)} = m_{s_\lambda - 1}^{(\lambda)}$ holds. Hence if $e \geq 5$, then none of the second ordering is impossible and we must have $\sigma(E_1^{(\lambda)}) = E_1^{(\lambda)}$.

Case $t = 2e + 1$ and $s_\lambda = e + 1$: Then $m_0^{(\lambda)} < m_1^{(\lambda)} < \cdots < m_{e-1}^{(\lambda)} = m_{s_\lambda - 2}^{(\lambda)}$. Hence if $e \geq 6$, then none of the second ordering is impossible. If $e = 5$, then $s_\lambda = 6$, and $E_0^{(\lambda)}, E_5^{(\lambda)}, E_2^{(\lambda)}, E_3^{(\lambda)}, E_4^{(\lambda)}, E_1^{(\lambda)}, E_6^{(\lambda)}$ is the unique possible second ordering. In this case we must have $m_5^{(\lambda)} = m_1^{(\lambda)}$. Then Theorem 1.2 (3) in [4] implies $\dfrac{|X_\lambda|}{2} - \dbinom{n+2}{3} = n$. Since X_λ is a spherical 9-design, $|X_\lambda| \geq \dfrac{1}{12} n(n+1)(n+2)(n+3)$. This is a contradiction. Hence we must have $\sigma(E_1^{(\lambda)}) = E_1^{(\lambda)}$. This completes the proof of Theorem 3.1 (2). □

PROOF OF THEOREM 3.1 (3). Let $\{\varphi_{l,i} \mid i = 1, 2, \ldots, h_l\}$ be an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^n)$ with respect to the inner product $\langle f, g\rangle = \frac{1}{|S^{n-1}|}\int\limits_{S^{n-1}} f(\boldsymbol{x})g(\boldsymbol{x})d\sigma(\boldsymbol{x})$. Here $h_l = \dim(\mathrm{Harm}_l(\mathbb{R}^n))$. Let $H_l^{(\lambda)}$ be the matrix whose rows are indexed by X, columns are indexed by $\{\varphi_{l,i} \mid i = 1, 2, \ldots, h_l\}$ and $(\boldsymbol{x}, \varphi_{l,i})$-entry is given by $H_l^{(\lambda)}(\boldsymbol{x}, i) = \varphi_{l,i}\big(\frac{\boldsymbol{x}}{r_\lambda}\big)$ for $\boldsymbol{x} \in X_\lambda$ and $H_l^{(\lambda)}(\boldsymbol{x}, i) = 0$ for $\boldsymbol{x} \notin X_\lambda$. Let I_λ $(\lambda = 1, 2)$ be the matrix indexed by $X \times X$ defined by $I_\lambda(\boldsymbol{x}, \boldsymbol{y}) = \delta_{\boldsymbol{x},\boldsymbol{y}}$ for $\boldsymbol{x}, \boldsymbol{y} \in X_\lambda$, $I_\lambda(\boldsymbol{x}, \boldsymbol{y}) = 0$ other wise. Let

$$E_l^{(\lambda,\mu)} = \frac{1}{\sqrt{|X_\lambda||X_\mu|}} H_l^{(\lambda)}\ {}^tH_l^{(\mu)}$$

for $l = 0, 1, \ldots$. Since X_λ and X_μ are spherical $(t-2)$-designs ${}^tH_l^{(\lambda)}H_k^{(\lambda)} = \delta_{l.k}|X_\lambda|I_\lambda$, ${}^tH_l^{(\mu)}H_k^{(\mu)} = \delta_{l.k}|X_\mu|I_\mu$ hold for any nonnegative integers k and l satisfying $0 \le k + l \le t - 2$ ([2, 15]). In [7], it is proved that $X = X_1 \cup X_2$ has a structure of coherent configuration. Let $A(X_\lambda, X_\mu) = \left\{\alpha_{\lambda,\mu}^{(1)}, \ldots, \alpha_{\lambda,\mu}^{(s_{\lambda,\mu})}\right\}$ and $\alpha_{\lambda,\lambda}^{(0)} = 1$, for $\lambda = 1,\ 2$. Let $D_{\lambda,\mu}^{(u)}$ be the adjacency matrix of the coherent configuration defined on $X \times X$ whose $(\boldsymbol{x}, \boldsymbol{y})$-entry is defined by

$$D_{\lambda,\mu}^{(u)} = \begin{cases} 1 \text{ if } (\boldsymbol{x}, \boldsymbol{y}) \in X_\lambda \times X_\mu \text{ and } \dfrac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_\lambda r_\mu} = \alpha_{\lambda,\mu}^{(u)}, \\ 0 \text{ otherwise.} \end{cases} \tag{4.12}$$

Hence if $(\boldsymbol{x}, \boldsymbol{y}) \in X_\lambda \times X_\mu$, then

$$E_l^{(\lambda,\mu)}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\sqrt{|X_\lambda||X_\mu|}} \sum_{i=1}^{h_l} \varphi_{l,i}\Big(\frac{\boldsymbol{x}}{r_\lambda}\Big)\varphi_{l,i}\Big(\frac{\boldsymbol{y}}{r_\mu}\Big) = \frac{1}{\sqrt{|X_\lambda||X_\mu|}} Q_{l,n-1}\Big(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_\lambda r_\mu}\Big) \tag{4.13}$$

and we have

$$E_l^{(\lambda,\mu)} = \frac{1}{\sqrt{|X_\lambda||X_\mu|}} \sum_{u=1-\delta_{\lambda,\mu}}^{s_{\lambda,\mu}} Q_{l,n-1}\left(\alpha_{\lambda,\mu}^{(u)}\right) D_{\lambda,\mu}^{(u)}.$$

Now we consider the case $\lambda \neq \mu$. Theorem 3.1 (1) implies $|A(X_1, X_2)| = e$. Therefore if $l \le e - 1$, then there exists $\gamma \in A(X_1, X_2)$ satisfying $Q_{l,n-1}(\gamma) \neq 0$.

This implies $E_l^{(\lambda,\mu)} \neq 0$ for $0 \leq l \leq e-1$ and $\lambda \neq \mu$. Let $0 \leq l, k \leq e-1$. Then $l+k \leq 2e-2 \leq t-2$. Hence we have

$$E_l^{(\lambda,\lambda)} E_k^{(\lambda,\mu)} = \frac{1}{|X_\lambda|} \frac{1}{\sqrt{|X_\lambda||X_\mu|}} H_l^{(\lambda)t} H_l^{(\lambda)} H_k^{(\lambda)t} H_k^{(\mu)} = \delta_{l,k} E_l^{(\lambda,\mu)}.$$

Similarly we have

$$E_l^{(\lambda,\mu)} E_k^{(\mu,\mu)} = \delta_{l,k} E_l^{(\lambda,\mu)}, \quad E_l^{(\mu,\lambda)} E_k^{(\lambda,\mu)} = \delta_{l,k} E_l^{(\mu,\mu)},$$

for $0 \leq l, k \leq e-1$. Let $\sum_{l=0}^{e-1} c_l E_l^{(\lambda,\mu)} = 0$. Then for any nonnegative integer $k \leq e-1$ we have $E_k^{(\lambda,\lambda)} \sum_{l=0}^{e-1} c_l E_l^{(\lambda,\mu)} = c_k E_k^{(\lambda,\mu)} = 0$. Since $E_k^{\lambda,\mu} \neq 0$, we must have $c_k = 0$ for any $k = 0, 1, \ldots, e-1$. Hence $E_0^{(\lambda,\mu)}, E_1^{(\lambda,\mu)}, \ldots, E_{e-1}^{(\lambda,\mu)}$ is linearly independent and we must have $\left\langle E_0^{(\lambda,\mu)}, E_1^{(\lambda,\mu)}, \ldots, E_{e-1}^{(\lambda,\mu)} \right\rangle = \left\langle D_{\lambda,\mu}^{(1)}, D_{\lambda,\mu}^{(2)}, \ldots, D_{\lambda,\mu}^{(e)} \right\rangle$. The vector space $\left\langle E_0^{(\lambda,\mu)}, E_1^{(\lambda,\mu)}, \ldots, E_{e-1}^{(\lambda,\mu)} \right\rangle$ is closed under the Hadamard product. Since Gegenbauer polynomial $Q_{l,n-1}(x)$ is a linear sum of the $(Q_{1,n-1}(x))^k, k = 0, 1, \ldots, l$, $E_l^{(\lambda,\mu)}$ is a polynomial of $E_1^{(\lambda,\mu)}$ with respect to the Hadamard product. Hence we can consider $\left\langle E_0^{(\lambda,\mu)}, E_1^{(\lambda,\mu)}, \ldots, E_{e-1}^{(\lambda,\mu)} \right\rangle$ as a vector space over the field $K = \mathbb{Q}\big(\alpha_{1,2}^{(1)}, \ldots, \alpha_{1,2}^{(e)}\big)$. This vector space is invariant as a set under the action of the Galois group G of the extention $K/\mathbb{Q}$. Here we note that, by definition, $D_{\lambda,\mu}^{(l)} = {}^t D_{\mu,\lambda}^{(l)}$ $(1 \leq l \leq e)$ and $E_{\lambda,\mu}^{(l)} = {}^t E_{\mu,\lambda}^{(l)}$ $(0 \leq l \leq e-1)$ hold. Let σ be any element in the Galois group G. Let $\sigma\big(E_l^{(\lambda,\mu)}\big) = \sum_{k=0}^{e-1} c_k E_k^{(\lambda,\mu)}$, $c_k \in K$. Assume that every element of $A(X_\lambda)$ is a rational number. Then we must have $\sigma(E_l^{(\lambda,\lambda)}) = E_l^{(\lambda,\lambda)}$. Then we have

$$\sum_{k=0}^{e-1} c_k E_k^{(\lambda,\mu)} = \sigma(E_l^{(\lambda,\mu)}) = \sigma(E_l^{(\lambda,\lambda)} E_l^{(\lambda,\mu)}) = E_l^{(\lambda,\lambda)} \sigma(E_l^{(\lambda,\mu)}) = c_l E_l^{(\lambda,\mu)}.$$

Since $\big\{E_0^{(\lambda,\mu)}, E_1^{(\lambda,\mu)}, \ldots, E_{e-1}^{(\lambda,\mu)}\big\}$ is linearly independent we must have $c_k = 0$ for $k \neq l$. Hence $\sigma\big(E_l^{(\lambda,\mu)}\big) = c_l E_l^{(\lambda,\mu)}$ with some $c_l \in K$. Since $E_l^{(\lambda,\mu)} = {}^t E_l^{(\mu,\lambda)}$, we must have $\sigma(E_l^{(\mu,\lambda)}) = c_l E_l^{(\mu,\lambda)}$. Then

$$E_l^{(\lambda,\lambda)} = \sigma\big(E_l^{(\lambda,\lambda)}\big) = \sigma\big(E_l^{(\lambda,\mu)} E_l^{(\mu,\lambda)}\big) = \sigma\big(E_l^{(\lambda,\mu)}\big)\sigma\big(E_l^{(\mu,\lambda)}\big) = c_l^2 E_l^{(\lambda,\mu)} E_l^{(\mu,\lambda)} = c_l^2 E_l^{(\lambda,\lambda)}.$$

Hence we must have $c_l^2 = 1$. In particular

$$\sigma\left(E_1^{(\lambda,\mu)}\right) = \frac{1}{\sqrt{|X_1||X_2|}} \sum_{u=1}^{e} Q_{1,n-1}\left(\sigma(\alpha_{\lambda,\mu}^{(u)})\right) D_{\lambda,\mu}^{(u)}$$

and

$$c_1 E_1^{(\lambda,\mu)} = \frac{1}{\sqrt{|X_1||X_2|}} \sum_{u=1}^{e} Q_{1,n-1}\left(c_1 \alpha_{\lambda,\mu}^{(u)}\right) D_{\lambda,\mu}^{(u)}.$$

Hence $\sigma\left(\alpha_{\lambda,\mu}^{(u)}\right) = c_1 \alpha_{\lambda,\mu}^{(u)}$ holds. This implies $\sigma\left(\left(\alpha_{\lambda,\mu}^{(u)}\right)^2\right) = \left(\sigma\left(\alpha_{\lambda,\mu}^{(u)}\right)\right)^2 = c_1^2\left(\alpha_{\lambda,\mu}^{(u)}\right)^2 = \left(\alpha_{\lambda,\mu}^{(u)}\right)^2$ for any $\sigma \in G$ and $u = 1, \ldots, e$. Hence, $\left(\alpha_{\lambda,\mu}^{(u)}\right)^2$, $1 \leq u \leq e$, are all rational numbers. This completes the proof of Theorem 3.1 (3). ☐

5. Proof of Lemma 3.1

Note that in [12], it was shown that if $n \geq 3$, $e \geq 6$, and $\gamma \in \mathbb{Q}$ hold for any γ satisfying $Q_{e,n-1}(\gamma) = 0$, then we get contradiction. We prove Lemma 3.1 using the method similar to the one used in [12]. Let $\{\gamma_1, \ldots, \gamma_e\}$ be the set of all the zeros of $Q_{e,n-1}(x)$. Note that $Q_{e,n-1}(x) = 0$ has only simple roots. Let $\{\gamma_1, \ldots, \gamma_e\}$ be the set of all the zeros of $Q_{e,n-1}(x)$. Then $-\gamma_i \in \{\gamma_1, \ldots, \gamma_e\}$ for $i = 1, 2, \ldots, e$, and if e is even, $\gamma_i \neq 0$ $(1 \leq i \leq e)$. On the other hand, if e is odd, then $Q_{e,n-1}(0) = 0$ holds. Let $m = \left[\frac{e}{2}\right]$ and $\{\gamma_1, \ldots, \gamma_{2m}\}$ be the set of all the nonzero zeros of $Q_{e,n-1}(x)$. We define a polynomial $S_m(x)$ by

$$S_m(x) = \prod_{i=1}^{2m} \left(x - \frac{1}{\gamma_i}\right). \tag{5.1}$$

Then the following expression is given in [12].

$$S_m(x) = x^{2m} + \sum_{i=1}^{m} (-1)^i u_i x^{2(m-i)}, \tag{5.2}$$

where if e is even,

$$u_i = \binom{m}{i} \frac{(h + 2(i-1))(h + 2(i-2)) \cdot \ldots \cdot (h+2)h}{(2i+\varepsilon)(2i+\varepsilon-2) \cdot \ldots \cdot 3 \cdot 1},$$

(5.3)

where $\varepsilon = (-1)^{e-1}$ and $h = n + 2m + 1 + \varepsilon$. Let $F(x)$ be the polyomial of degree m defined by

$$F(x) = x^m + \sum_{i=1}^{m} (-1)^i u_i x^{m-i}, \tag{5.4}$$

where u_i $(1 \leq i \leq m)$ are given by (5.3). Then $F(x)$ has m zeros $\{\gamma_j^{-2}, 1 \leq j \leq m\}$. By assumption, $\{\gamma_j^{-2}, 1 \leq j \leq m\}$ are all rational numbers. For any fixed γ_j^{-2} $(1 \leq j \leq m)$, let $\gamma_j^{-2} = p_j/q_j$, where p_j and q_j are integers relatively prime to each other. Then the following holds.

$$\left(\frac{p_j}{q_j}\right)^m + \sum_{i=1}^{m} (-1)^i u_i \left(\frac{p_j}{q_j}\right)^{m-i} = 0 \quad (1 \leq j \leq m). \tag{5.5}$$

Multiply this equation by q_j^m, then we obtain

$$p_j^m + \sum_{i=1}^{m} (-1)^i u_i p_j^{m-i} q_j^i = 0 \quad (1 \leq j \leq m). \tag{5.6}$$

Among the integers in the denominators of $u_1, u_2 \ldots, u_m$, $(2m + \varepsilon)!! = (2m + \varepsilon)(2m + \varepsilon - 2) \cdot \ldots \cdot 3 \cdot 1$ is the largest one. So we multiply (5.6) by $(2m + \varepsilon)!!$. Then we obtain

$$\begin{aligned} &(2m + \varepsilon)!!\, p_j^m + \sum_{i=1}^{m-1} (-1)^i (2m + \varepsilon)!!\, u_i p_j^{m-i} q_j^i + \\ &+ (h + 2(m-1))(h + 2(m-2)) \cdot \ldots \cdot (h + 2) h q_j^m = 0 \quad (1 \leq j \leq m). \end{aligned} \tag{5.7}$$

Since $(2m + \varepsilon)!!\, u_i$, $i = 1, \ldots, m - 1$, are all integers and q_j and p_j are relatively prime to each other, q_j must divide $(2m + \varepsilon)!!$ for any $j = 1, 2, \ldots, m$. Let $l.c.m.(q)$ be the least common multiple of $q_1, \ldots, q_m$. Then there exist $y_1, y_2, \ldots, y_m \in \mathbb{Z}$ satisfying $\dfrac{p_j}{q_j} = \dfrac{y_j}{l.c.m.(q)}$. Then by factoring out q_j^m, (5.7) is transformed into

$$(2m+\varepsilon)!!\; y_j^m + \sum_{i=1}^{m-1} (-1)^i (2m+\varepsilon)!!\; u_i y_j^{m-i} (l.c.m.(q))^i + (h+2(m-1))\times$$

$$\times\ (h+2(m-2))\cdot\ldots\cdot(h+2)h(l.c.m.(q))^m = 0 \quad (1 \le j \le m). \tag{5.8}$$

Thus, $y_1, \ldots, y_m$ must be the zeros of the following polynomial $f(x)$ of degree m

$$f(x) = a_0 x^m + \sum_{i=1}^{m} a_i x^{m-i} \tag{5.9}$$

where

$$a_0 = (2m+\varepsilon)!!, \tag{5.10}$$

$$a_i = (-1)^i (l.c.m.(q))^i \binom{m}{i} (2m+\varepsilon)(2m-3)\cdot\ldots\cdot(2i+\varepsilon+2)\times$$

$$\times\ (h+2(i-1))(h+2(i-2))\cdot\ldots\cdot(h+2)h \qquad (1 \le i \le m-1), \tag{5.11}$$

$$a_m = (l.c.m.(q))^m (h+2(m-1))(h+2(m-2))\cdot\ldots\cdot(h+2)h. \tag{5.12}$$

Note that since $l.c.m.(q)$ must divide $(2m+\varepsilon)!!$, every prime number satisfying $p > 2m+\varepsilon$ is relatively prime to $l.c.m.(q)$.

Next we use the method so called Newton polygon method ([17]). Let $F(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x + b_m$ be a polynomial of integral coefficients with $b_m b_0 \neq 0$. Let p be a prime number, and let $c_i(p)$ be the largest integer that $p^{c_i(p)}$ divides b_i. (We put $c_i(p) = \infty$ if $b_i = 0$.) Let plot the points $(0,0)$, $(1, c_0(p))$, $(2, c_1(p)), \ldots, (m+1, c_m(p))$ in the x-y plane $\mathbb{R}^2$. Take the lower side of the convex hull of these points (which is called the Newton-polygon). Then, in order that all the zeros of $F(x) = 0$ are integers, all the slopes (of the Newton-polygon) must be integers. (If the coefficients are rational numbers, then we consider factorization of the polynomial as p-adic numbers.) It is said that Newton had a similar figure, but this method was discovered later (Dumas [17]).

Now we apply this to our polynomial $f(x)$ defined by (5.9). Let p be any prime satisfying $p > 2m+\varepsilon$ and $p | h+2(m-2)$. Then $p \nmid h+2(m-i)$ for $i = 1, 3, 4, \ldots, m$. Hence $p \nmid a_i$, for $i = 0, 1, \ldots, m-2$ and we must have $c_0(p) = c_1(p) = \ldots = c_{m-3}(p) = c_{m-2}(p) = 0$. Moreover both a_{m-1} and a_m are of the form $(h+2(m-2)) \times$ (an integer prime to p). Hence we must have $c_{m-1}(p) = c_m(p) > 0$.

This implies that the line segment from $(m-2, c_{m-2}(p)) = (m-2, 0)$ to $(m, c_m(p))$ is an edge of the Newton polygon. Therefore the slope must be an integer. Hence $c_m(p)$ must be an even integer. Consider the factorization of $h + 2(m-2)$ into prime factors and let $h + 2(m-2) = Ay^2$ where A is a product of distinct prime numbers. Then every prime factor of A must be less than or equal to $2m + \varepsilon$. Hence A is a bounded umber for each $m = \left[\frac{e}{2}\right]$. Next we consider the prime factor of $h + 2(m-3)$. Let p' be a prime number satisfying $p' > 2m + \varepsilon$ and $p' | h + 2(m-3)$. Then p' is prime to $h + 2(m-j)$ for any $j \neq 3$. Hence we must have $c_{m-2}(p') = c_{m-1}(p') = c_m(p')$ and $c_0(p') = c_1(p') = \ldots = c_{m-3}(p') = 0$. Hence the line segment from $(m-3, c_{m-3}(p')) = (m-3, 0)$ to $(m, c_m(p'))$ is an edge of the Newton polygon. Hence $c_m(p')$ must be a multiple of 3. Consider the factorization of $h + 2(m-3)$ into distinct primes, and let $h + 2(m-3) = Bz^3$, where every prime factor p of B satisfies $p^3 \nmid B$. Then p must be less than or equal to $2m + \varepsilon$. Hence B is at most a product of squares of prime numbers less than or equal to $2m + \varepsilon$. Hence B is a bounded number each $m = \left[\frac{e}{2}\right]$. Thus we have an equation

$$Ay^2 - Bz^3 = 2.$$

It is known in the theory of diophantine equations that there are only finitely many integer solutions of this equation (see Theorem 3 or Theorem 4 in [20, Chapter 28]. See also [1, 23]). Since choices for each m, we have finitely many choices for A and B. Hence there are finitely many solutions for y and z for each m. Hence there are finitely many choices for $h = n + 2m + 1 + (-1)^{e-1}$ for a fixed e. This implies Lemma 3.1.

6. Formulas

Let (X, w) be supported by p concentric spheres. Notations are the same as in Sec. 2. Then $\dim(\mathcal{R}_{2(p-1)}(X)) = p$ holds. For each l, we define an inner product $\langle -, \ -\rangle_l$ on $\mathcal{P}_{2(p-1)}(X)$ by $\langle f, g\rangle_l = \sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})||\boldsymbol{x}||^{2l} f(\boldsymbol{x})g(\boldsymbol{x})$. Then $\langle -, \ -\rangle_l$ is a positive definite for each l. For each l, we define polynomials $\{g_{l,j} \mid 0 \leq j \leq p-1\} \subset \mathcal{R}_{2(p-1)}(\mathbb{R}^n)$ so that $\{g_{l,j}|_X \mid 0 \leq j \leq p-1\}$ is an orthnomal basis of $\mathcal{R}_{2(p-1)}(X)$ with respect $\langle -, \ -\rangle_l$. We define so that $g_{l,j}(\boldsymbol{x})$ is a polynomial of degree $2j$ and a linear combination of $\left\{||\boldsymbol{x}||^{2i} \mid 0 \leq i \leq j\right\}$. We abuse the notation and we identify $g_{l,j}(\boldsymbol{x}) = g_{l,j}(r_\nu)$ for $\boldsymbol{x} \in X_\nu$ $(1 \leq \nu \leq p)$. First we prove the following proposition.

PROPOSITION 6.1. *Let (X, w) be a positive weighted finite subset in $\mathbb{R}^n$ supported by p concentric spheres. Assume $\mathbf{0} \notin X$ and the weight function is constant on each X_i $(1 \le i \le p)$. Then the following holds.*

$$\sum_{j=0}^{p-1} g_{l,j}(r_\nu) g_{l,j}(r_\mu) = \delta_{\nu,\mu} \frac{1}{|X_\nu| w_\nu r_\nu^{2l}}.$$

PROOF. Let M_l be the $p \times p$ matrix whose (ν, j) entry is defined by $\sqrt{|X_\nu| w_\nu} r_\nu^l g_{l,j}(r_\nu)$ for $1 \le \nu \le p$ and $0 \le j \le p-1$. Then

$$\begin{aligned}({}^t M_l M_l)(j_1, j_2) &= \sum_{\nu=1}^{p} M_{\nu,j_1} M_{\nu,j_2} = \sum_{\nu=1}^{p} |X_\nu| w_\nu r_\nu^{2l} g_{l,j_1}(r_\nu) g_{l,j_2}(r_\nu) = \\ &= \sum_{\nu=1}^{p} \sum_{\boldsymbol{x} \in X_\nu} w(\boldsymbol{x}) \|\boldsymbol{x}\|^{2l} g_{l,j_1}(r_\nu) g_{l,j_2}(r_\nu) = \sum_{\boldsymbol{x} \in X} w(\boldsymbol{x}) \|\boldsymbol{x}\|^{2l} g_{l,j_1}(\boldsymbol{x}) g_{l,j_2}(\boldsymbol{x}) = \\ &= \delta_{j_1, j_2} \end{aligned} \tag{6.1}$$

Hence M_l is invertible and $M_l^{-1} = {}^t M_l$. Hence we have $M_l \, {}^t M_l = I$.

$$(M_l \, {}^t M_l)(\nu, \mu) = r_\nu^l r_\mu^l \sqrt{|X_\nu||X_\mu| w_\nu w_\mu} \sum_{j=0}^{p-1} g_{l,j}(r_\nu) g_{l,j}(r_\mu) = \delta_{\nu,\mu} \tag{6.2}$$

Hence we must have

$$\sum_{j=0}^{p-1} g_{l,j}(r_\nu) g_{l,j}(r_\mu) = \delta_{\nu,\mu} \frac{1}{|X_\nu| w_\nu r_\nu^{2l}}.$$

□

Remark. By definition $g_{l,0}(\boldsymbol{x}) \equiv \dfrac{1}{\sqrt{\sum_{\boldsymbol{x} \in X} w(\boldsymbol{x}) \|\boldsymbol{x}\|^{2l}}}$. More information on $g_{l,j}(\boldsymbol{x})$, please refer to [3, 13].

Now let $X = X_1 \cup X_2$ be a tight t-design on two concentric spheres. Assume $0 \notin X$. In the following we consider the case $n \ge 3$ and $t = 4, 6, 8$ and $t \ge 10$. (Classification of tight t-designs on two concentric spheres are done for the following cases: For $n = 2$ see [9]. For $t = 2, 3$, see [10]. $n \geq 3$ and $t = 5$ see [13]. $n \geq 3$ and $t = 7$ see [5]. $n \geq 3$ and $t = 9$ see [8]. For $t = 4$ and 6, interesting examples are given in [11, 14])

If $t = 2e$, then it is known that X_1 and X_2 are spherical $2(e-1)$-designs and $e - 1 \leq s_\nu \leq e$ (see [13]). If $s_\nu = e - 1$, then it is known that $A(X_\nu)$ coincide the set of zeros of the polynomial $\sum_{i=0}^{e-1} Q_{i,n-1}(x)$ ([15]). If $t = 2e + 1$, then it is known that X_1 and X_2 are spherical $(2e-1)$-designs and $e \leq s_\nu \leq e + 1$ (see [13]). If $s_\nu = e$, then $A(X_\nu)$ coincide with the set of zeros of the polynomial $(x + 1) \sum_{i=0}^{[\frac{e-1}{2}]} Q_{e-1-2i,n-1}(x)$ ([15]). In the following we use the following notation for the inner products between the points in X. $\alpha_0^{(\nu)} = 1$, $\alpha_i^{(\nu)} = \alpha_{\nu,\nu}^{(i)} \in A(X_\nu)$ and $\gamma_i = \alpha_{1,2}^{(i)} \in A(X_1, X_2)$.

Proposition 6.2. *We assume $n \geq 3$. Let $X = X_1 \cup X_2$ be a tight t-design on 2 concentric spheres in $\mathbb{R}^n$. Assume $0 \notin X$. Then the following* (1) *and* (2) *hold.*

(1) *If $t = 2e$, then the followings hold.*

a) *$A(X_1, X_2)$ coincide with the set of zeros of the polynomial*

$$Q_{e,n-1}(x) + \frac{|X_1|w_1r_1^{2e} + |X_2|w_2r_2^{2e}}{r_1r_2(|X_1|w_1r_1^{2(e-1)} + |X_2|w_2r_2^{2(e-1)})} Q_{e-1,n-1}(x).$$

b) *If $s_1 = s_2 = e$, then the following conditions hold.*

$$\frac{|X_\nu|w_\nu r_\nu^{2e}}{|X_1|w_1r_1^{2e} + |X_2|w_2r_2^{2e}} Q_{e,n-1}(\alpha^{(\nu)}) + \frac{|X_\nu|w_\nu r_\nu^{2e-2}}{|X_1|w_1r_1^{2e-2} + |X_2|w_2r_2^{2e-2}} Q_{e-1,n-1}(\alpha^{(\nu)}) + \sum_{l=0}^{e-2} Q_{l,n-1}(\alpha^{(\nu)}) = 0 \tag{6.3}$$

for any $\alpha^{(\nu)} \in A(X_\nu)$, $\nu = 1, 2$.

$$\frac{|X_\nu|w_\nu r_\nu^{2e}}{|X_1|w_1r_1^{2e}+|X_2|w_2r_2^{2e}} Q_{e,n-1}(1) + \frac{|X_\nu|w_\nu r_\nu^{2e-2}}{|X_1|w_1r_1^{2e-2}+|X_2|w_2r_2^{2e-2}} Q_{e-1,n-1}(1) + + \sum_{l=0}^{e-2} Q_{l,n-1}(1) = |X_\nu|, \quad \nu = 1, 2. \tag{6.4}$$

(2) *If* $t = 2e + 1$ *and* $s_1 = s_2 = e + 1$, *then the following conditions hold.*

$$\left(\frac{|X_\nu|(n-2)!\ e!}{2(n-2+2e)(n-3+e)!} - \frac{(n-2+e)(n-1+e)}{(n-2+2e)(n-1)}\right) Q_{e,n-1}(\alpha^{(\nu)}) \\ +C_{e,n-1}(\alpha^{(\nu)}) = 0, \tag{6.5}$$

for any $\alpha^{(\nu)} \in A(X_\nu)$ *satisfying* $\alpha^{(\nu)} \neq -1$, $\nu = 1, 2$. *Here* $C_{e,n-1}(x) = \sum_{j=0}^{[e/2]} Q_{e-2j,n-1}(x)$.

$$\frac{w_1}{w_2}\left(\frac{r_1}{r_2}\right)^{2e} = \frac{|X_2|}{|X_1|}\left(\frac{|X_1| - 2\binom{n-3+e}{e-2}}{|X_2| - 2\binom{n-3+e}{e-2}}\right). \tag{6.6}$$

Note that $|X_1| + |X_2| = |X| = 2\left(\binom{n+e-1}{e} + \binom{n+e-3}{e-2}\right)$ *by assumption.*

Proof. Equations (2.3) and (2.4) in the proof of Lemma 1.10 in [3] imply that

$$\sum_{l=0}^{e} a_l^{(\nu)} Q_{l,n-1}(1) = \frac{1}{w_\nu}, \quad \nu = 1, 2$$

$$\sum_{l=0}^{e} a_l^{(\nu)} Q_{l,n-1}(\alpha^{(\nu)}) = 0, \quad \text{for any } \alpha^{(\nu)} \in A(X_\nu),\ \nu = 1, 2$$

and

$$\sum_{l=0}^{e} c_l Q_{l,n-1}(\gamma) = 0, \quad \text{for any } \gamma \in A(X_1, X_2)$$

where $a_l^{(\nu)} = r_\nu^{2l} g_{l,0}(r_\nu)^2 = \dfrac{r_\nu^{2l}}{|X_1|w_1r_1^{2l} + |X_2|w_2r_2^{2l}}$ for $l = e-1,\ e$ and $a_l^{(\nu)} = r_\nu^{2l}(g_{l,0}(r_\nu)^2 + g_{l,1}(r_\nu)^2)$ for $l = 0, 1, \dots, e-2$. Also $c_l = (r_1r_2)^l g_{l,0}(r_1)g_{l,0}(r_2) = \dfrac{(r_1r_2)^l}{|X_1|w_1r_1^{2l} + |X_2|w_2r_2^{2l}}$ for $l = e-1$, e and $c_l = (r_1r_2)^l(g_{l,0}(r_1)g_{l,0}(r_2) + g_{l,1}(r_1)g_{l,1}(r_2))$ for $l = 0, 1, \dots, e-2$. Then Proposition 6.1 implies (1).

(2) Let $X_\nu = X_\nu^* \cup (-X_\nu^*)$, $X_\nu^* \cap (-X_\nu^*) = \varnothing$ for $\nu = 1, 2$, where $-X_\nu^* = \{-\boldsymbol{x} \mid \boldsymbol{x} \in X_\nu^*\}$. Then we must have $A(X_\nu) = \{-1\} \cup A(X_\nu^*)$ and by assumption $|A(X_\nu^*)| = e$. Equations (3.1) and (3.2) in the proof of Lemma 1.7 in [13] implies

the following equations.

$$\sum_{l=0}^{[e/2]} c^*_{e-2l} Q_{e-2l,n-1}(1) = \frac{1}{w_\nu}, \tag{6.7}$$

and

$$\sum_{l=0}^{[e/2]} c^*_{e-2l} Q_{e-2l,n-1}\Big(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_\nu^2}\Big) = 0 \tag{6.8}$$

for any $\boldsymbol{x},\boldsymbol{y} \in X^*_\nu$, where $c^*_{e-2l} = r_\nu^{2(e-2l)} \sum_{j=0}^{\min\{p-1,l\}} g^*_{e-2l,j}(r_\nu)^2$ $(0 \le l \le [\frac{e}{2}])$ for $\nu = 1,2$. (Here we denote the orthonormal basis of $\mathcal{R}_{2(p-1)}(X^*)$ by $\{g^*_{l,j}(\boldsymbol{x}) \mid 0 \le j \le p-1\}$.) We apply Proposition 6.1 for (X^*, w). Then we obtain the following equations.

$$\frac{|X_\nu| w_\nu r_\nu^{2e}}{|X_1|w_1 r_1^{2e} + |X_2| w_2 r_2^{2e}} Q_{e,n-1}(\alpha^{(\nu)}) + \sum_{l=1}^{[e/2]} Q_{e-2l,n-1}(\alpha^{(\nu)}) = 0 \tag{6.9}$$

for any $\alpha^{(\nu)} \in A(X_\nu)$ satisfying $\alpha^{(\nu)} \neq -1$.

$$\frac{|X_\nu| r_\nu^{2e} w_\nu}{|X_1|w_1 r_1^{2e} + |X_2| w_2 r_2^{2e}} Q_{e,n-1}(1) + \sum_{l=1}^{[e/2]} Q_{e-2l,n-1}(1) = \frac{1}{2}|X_\nu|. \tag{6.10}$$

Then $Q_{e,n-1}(1) = \binom{n-1+e}{e} - \binom{n-1+e-2}{e-2}$ and $\sum_{l=0}^{[e/2]} Q_{e-2l,n-1}(1) = \binom{n-1+e}{e}$ imply (2). □

Remark. In Proposition 6.2, $t = 2e$: Let $A(X_\nu) = \{\alpha_1^{(\nu)}, \alpha_2^{(\nu)}, \ldots, \alpha_e^{(\nu)}\}$, $\alpha_0^{(\nu)} = 1$ and $b_l^{(\nu)} = \dfrac{|X_\nu| w_\nu r_\nu^{2l}}{|X_1| w_1 r_1^{2l} + |X_2| w_2 r_2^{2l}}$ for $l = e-1, e$ and $\nu = 1,2$. Let M_ν be the square matrix of size $e+1$ whose (i,j)-entry is given by $M_\nu(i,j) = Q_i(\alpha_j^{(\nu)})$. Since $\alpha_0^{(\nu)}, \alpha_1^{(\nu)}, \ldots, \alpha_e^{(\nu)}$ are distinct rational numbers. M_ν is a invertible matrix. Then (6.3) and (6.4) imply

$$\Big(1, 1, \ldots, 1, b_{e-1}^{(\nu)}, b_e^{(\nu)}\Big)\, M_\nu = (|X_\nu|, 0, 0, \ldots, 0).$$

Hence, $b_{e-1}^{(\nu)}$ and $b_e^{(\nu)}$ are determined uniquely by $\{\alpha_1^{(\nu)}, \alpha_2^{(\nu)}, \ldots, \alpha_e^{(\nu)}\}$. Hence, the ratios r_1/r_2 and w_1/w_2 are uniquely determined by n, $|X_1|$, $|X_2|$, and $\{\alpha_1^{(\nu)}, \alpha_2^{(\nu)}, \ldots, \alpha_e^{(\nu)}\}$, $(\nu = 1, 2)$. On the other hand, the tight $(2e+1)$-designs on two concentric spheres are flexible (see equation (6.6)).

7. Tight 11-design on 2 concentric spheres

Let (X, w) be a tight 11-design on two concentric spheres. Since there is no tight spherical 9-designs, we have the following conditions:

$$|X| = 2\left(\binom{n+4}{5} + \binom{n+2}{3}\right) = \frac{1}{60}n(n+1)(n+2)(n^2+7n+32),$$

$$\frac{1}{60}n(n+2)(n+1)(n^2+2n+17) > |X_2| \geq \frac{1}{120}n(n+1)(n+2)(n^2+7n+32)$$

$$\geq |X_1| > \frac{1}{12}n(n+1)(n+2)(n+3).$$

Since X, X_1, and X_2 are antipodal, $|A(X_\nu)| = 6$ and $|A(X_1, X_2)| = 5$, $A(X_1, X_2)$, and $A(X_\nu)$ are expressed in the following way:

$$A(X_1, X_2) = \{\gamma_1, \gamma_2, \ldots, \gamma_5\},\ \gamma_1 = -\gamma_2,\ \gamma_3 = -\gamma_4,\ \gamma_5 = 0,$$

$$A(X_\nu) = \{\alpha_1^{(\nu)}, \ldots, \alpha_6^{(\nu)}\},\ \alpha_1^{(\nu)} = -1,\ \alpha_6^{(\nu)} = 0,\ \alpha_3^{(\nu)} = -\alpha_2^{(\nu)},\ \alpha_5^{(\nu)} = -\alpha_4^{(\nu)}.$$

Here $A(X_\nu)$ coincides with the set of zeros of the following polynomial:

$$x(x-1)\times \qquad (7.1)$$

$$\times\left(x^4 - \frac{n(n+9)(n+2)(n+1) - 30N_\nu}{((n+6)(n(n+2)(n+1) - 3N_\nu)}x^2 + \frac{3(n(n+4)(n+2)(n+1) - 15N_\nu)}{(n+6)(n+4)(n(n+2)(n+1) - 3N_\nu)}\right)$$

and $A(X_1, X_2)$ coincides with the set of zeros of the Gegenbauer polynomial $Q_{5,n-1}(x)$ given by

$$Q_{5,n-1}(x) = \frac{1}{120}n(n+8)(n+2)x((n+6)(n+4)x^4 - 10(n+4)x^2 + 15) \qquad (7.2)$$

Then we have

$$\gamma_1 = \frac{\sqrt{(n+6)(n+4)\left(5n+20+\sqrt{10(n+4)(n+1)}\right)}}{(n+6)(n+4)},$$

$$\gamma_3 = \frac{\sqrt{(n+6)(n+4)\left(5n+20-\sqrt{10(n+4)(n+1)}\right)}}{(n+6)(n+4)}. \tag{7.3}$$

$$\alpha_2^{(\nu)} = \frac{1}{2(n+6)(n+4)(3|X_\nu|-n(n+2)(n+1))}\Big\{2(n+6)(n+4)(3|X_\nu|-n(n+2)(n+1))\times$$
$$\times\Big(30(n+4)|X_\nu|-n(n+9)(n+4)(n+2)(n+1)+\sqrt{(n+4)(n+1)}\times$$
$$\times\sqrt{360|X_\nu|^2-24n(n+9)(n+2)(n+1)|X_\nu|+n^2(n+4)(n+1)(n+3)^2(n+2)^2}\Big)\Big\}^{\frac{1}{2}}, \tag{7.4}$$

$$\alpha_4^{(\nu)} = \frac{1}{2(n+6)(n+4)(3|X_\nu|-n(n+2)(n+1))}\Big\{2(n+6)(n+4)(3|X_\nu|-n(n+2)(n+1))\times$$
$$\times\Big(30(n+4)|X_\nu|-n(n+9)(n+4)(n+2)(n+1)-\sqrt{(n+4)(n+1)}\times$$
$$\times\sqrt{360|X_\nu|^2-24n(n+9)(n+2)(n+1)|X_\nu|+n^2(n+4)(n+1)(n+3)^2(n+2)^2}\Big)\Big\}^{\frac{1}{2}}. \tag{7.5}$$

Since $\gamma_1^2, \gamma_2^2, \dots, \gamma_5^2$ are rational numbers and $\sqrt{10(n+1)(n+4)}$ must be an integer. On the other hand it is shown in [7] that $X = X_1 \cup X_2$ has the structure of an coherent configuration. Let $\alpha_0^{(\mu)} = 1$ for $\nu = 1, 2$. For any fixed $\boldsymbol{x}$, $\boldsymbol{y} \in X_\nu$, let $p_{\alpha_i^\nu,\alpha_j^{(\nu)}}^{\alpha_k^{(\nu)}} = \left|\left\{\boldsymbol{z} \in X_\nu \mid \frac{\boldsymbol{x}\cdot\boldsymbol{z}}{r_\nu^2} = \alpha_i^{(\nu)},\ \frac{\boldsymbol{z}\cdot\boldsymbol{y}}{r_\nu^2} = \alpha_j^{(\nu)}\right\}\right|$ and $p_{\gamma_i,\gamma_j}^{\alpha_k^{(\nu)}} = \left|\left\{\boldsymbol{z} \in X_\mu \mid \mu \neq \nu,\ \frac{\boldsymbol{x}\cdot\boldsymbol{z}}{r_1 r_2} = \gamma_i,\ \frac{\boldsymbol{z}\cdot\boldsymbol{y}}{r_1 r_2} = \gamma_j\right\}\right|$, etc. Then using the formula given in Proposition 3.2 in [7], we have the following expressions for the intersection numbers. Actually, we can determine all the intersection numbers but we list some of them here. We also use the notation $N_\nu = |X_\nu|$, $\nu = 1, 2$.

$$p_{\gamma_1,\gamma_1}^{\alpha_0^{(\nu)}} = p_{\gamma_2,\gamma_2}^{\alpha_0^{(\nu)}} = \frac{|X_\nu|((7n+2)(n+4)(n+1)-(2n+7)(n-2)\sqrt{10(n+4)(n+1)})}{60n(n+1)(n+2)},$$

$$p_{\gamma_3,\gamma_3}^{\alpha_0^{(\nu)}} = p_{\gamma_4,\gamma_4}^{\alpha_0^{(\nu)}} = \frac{|X_\nu|((7n+2)(n+4)(n+1)+(2n+7)(n-2)\sqrt{10(n+4)(n+1)})}{60n(n+1)(n+2))},$$

$$p_{\gamma_5,\gamma_5}^{\alpha_0^{(\nu)}} = p_{\gamma_5,\gamma_5}^{\alpha_1^{(\nu)}} = N_\nu - N_\nu\frac{(n+2)(\gamma_1^2+\gamma_3^2)-3}{n(n+2)\gamma_1^2\gamma_3^2} = N_\nu - \frac{N_\nu(n+4)(7n+2)}{15n(n+2)} \tag{7.6}$$

$$p^{\alpha_0^{(\nu)}}_{\alpha_1^{(\nu)},\alpha_1^{(\nu)}} = 1,$$

$$p^{\alpha_0^{(\nu)}}_{\alpha_2^{(\nu)},\alpha_2^{(\nu)}} = \frac{3N_\nu - 2n^2 - 4n - (n+2)(N_\nu - 2n)(\alpha_4^{(\nu)})^2}{2n(n+2)(\alpha_2^{(\nu)})^2\left((\alpha_2^{(\nu)})^2 - (\alpha_4^{(\nu)})^2\right)},$$

$$p^{\alpha_0^{(\nu)}}_{\alpha_4^{(\nu)},\alpha_4^{(\nu)}} = \frac{(n+2)(N_\nu - 2n)(\alpha_2^{(\nu)})^2 - 3N_\nu + 2n^2 + 4n}{2n(n+2)(\alpha_4^{(\nu)})^2\left((\alpha_2^{(\nu)})^2 - (\alpha_4^{(\nu)})^2\right)} =$$

$$= \frac{1}{12n(n+2)(n+1)(-15N_\nu + n(n+4)(n+2)(n+1))} \times$$

$$\times \frac{1}{n^2(n+4)(n+1)(n+3)^2(n+2)^2 - 24nN_\nu(n+9)(n+2)(n+1) + 360N_\nu^2} \times$$

$$\times \Bigg\{ \bigg((n+4)(n+1)(-3N_\nu^2(7n+2) + nN_\nu(n+2)(n^3 + 9n^2 + 2n + 24) -$$

$$-6n^2(n+1)(n+2)^2 \bigg) \bigg(n^8 + 15n^7 + 91n^6 + 144n^2 + 285n^5 + 484n^4 + 420n^3 -$$

$$-24nN_\nu(n+9)(n+2)(n+1) + 360N_\nu^2 \bigg) -$$

$$-\bigg(6n^3(n+4)(n+3)(n+1)^2(n+2)^3 +$$

$$+N_\nu n^2(n+1)(n^5 + 16n^4 + 77n^3 + 2n^2 - 528n - 648)(n+2)^2 -$$

$$-3N_\nu^2 n(n+2)(n+1)(11n^3 + 111n^2 + 34n - 696) +$$

$$+180N_\nu^3(2n+7)(n-2) \bigg) \sqrt{(n+4)(n+1)} \times$$

$$\times \sqrt{n^2(n+4)(n+1)(n+3)^2(n+2)^2 - 24nN_\nu(n+9)(n+2)(n+1) + 360N_\nu^2},$$

$$p^{\alpha_0^{(\nu)}}_{\alpha_6^{(\nu)},\alpha_6^{(\nu)}} = \frac{3N_\nu - 2n^2 - 4n - (n+2)(N_\nu - 2n)\left((\alpha_4^{(\nu)})^2 + (\alpha_2^{(\nu)})^2\right)}{(n+2)n(\alpha_4^{(\nu)})^2(\alpha_2^{(\nu)})^2} + N_\nu - 2 =$$

$$= \frac{(n+4)(3(7n+2)N_\nu^2 - n(n+2)(n^3+9n^2+2n+24)N_\nu + 6n^2(n+1)(n+2)^2)}{3n(n+2)\left(n(n+1)(n+2)(n+4) - 15N_\nu\right)} + N_\nu - 2. \quad (7.7)$$

Equalities (7.6) and (7.7) imply that

$$\frac{N_\nu(n+4)(7n+2)}{15n(n+2)} \tag{7.8}$$

and

$$\frac{(n+4)(3(7n+2)N_\nu^2-n(n+2)(n^3+9n^2+2n+24)N_\nu+6n^2(n+1)(n+2)^2)}{3n(n+2)\Bigl(n(n+1)(n+2)(n+4)-15N_\nu\Bigr)} \tag{7.9}$$

are integers. Fraction (7.9) has the following expression:

$$-\frac{2(n+4)(n-1)^2(n+1)^3n(n+2)}{225(15N_\nu-(n+4)(n+1)n(n+2))}-\frac{2}{225}(n+4)(n-4)(n^2+14)-\frac{N_\nu(n+4)(7n+2)}{15n(n+2)}$$

Hence, by (7.8),

$$-\frac{2(n+4)(n-1)^2(n+1)^3n(n+2)}{225(15N_\nu-(n+4)(n+1)n(n+2))}-\frac{2}{225}(n+4)(n-4)(n^2+14) \tag{7.10}$$

is an integer. Since $\alpha_2^{(\nu)}$ $(\nu=1,2)$ is a rational number and $\sqrt{10(n+4)(n+1)}$ is an integer, (7.4) implies that

$$\sqrt{60^2N_\nu^2-240n(n+9)(n+2)(n+1)N_\nu+10n^2(n+4)(n+1)(n+3)^2(n+2)^2} \tag{7.11}$$

is an integer. Since If $n \leq 60000$, then $\sqrt{10(n+4)(n+1)}$ is an integer only for $n = 26, 124, 241, 1079, 4801, 9244, 41066$. For these cases we can show that there is no tight 11-design on two concentric spheres using the integrality conditions (7.8), (7.10), and (7.11). We expect the nonexistence of such designs. However it seems very difficult even to prove that there exists at most finitely many tight 11-design on two concentric spheres comparing to the case $t = 2e+1 \geq 13$. In the following we give some more useful integrality conditions. Since $\alpha_i^{(\nu)}$, $1 \leq i \leq 6, \nu = 1, 2$ are rational numbers (7.4) and (7.5) implies that $C_{\nu,+}$ and $C_{\nu,-}$ $(\nu = 1,2)$ defined by

$$\begin{aligned}C_{\nu,\pm} &= \Biggl\{2(n+6)(n+4)(3N_\nu-n(n+2)(n+1))\times\\ &\times\Biggl(30(n+4)N_\nu-n(n+9)(n+4)(n+2)(n+1)\pm\sqrt{(n+4)(n+1)}\times\\ &\times\sqrt{360N_\nu^2-24n(n+9)(n+2)(n+1)N_\nu+n^2(n+4)(n+1)(n+3)^2(n+2)^2}\Biggr)\Biggr\}^{\frac{1}{2}}\end{aligned} \tag{7.12}$$

are integers. Then we have

$$C_{\nu,+}^2C_{\nu,-}^2 = 48(n+6)^3(n+4)^3(15N_\nu-n(n+4)(n+2)(n+1))(3N_\nu-n(n+2)(n+1))^3$$

(7.13)

and $C^2_{\nu,+}C^2_{\nu,-}$ is the square of an integer. Hence,

$$3(n+6)(n+4)(15N_\nu - n(n+4)(n+2)(n+1))(3N_\nu - n(n+2)(n+1)) \tag{7.14}$$

must be the square of an integer for $\nu = 1, 2$.

Remark. A generalization of the Larman—Rogers—Seidel's theorem was obtained by Nozaki [22]. We can apply Nozaki's theorem for X_2. Then it follows that $K_2 = \dfrac{1-\left(\alpha_4^{(2)}\right)^2}{\left(\alpha_2^{(2)}\right)^2\left(\left(\alpha_2^{(2)}\right)^2 - \left(\alpha_4^{(2)}\right)^2\right)}$ and $K_4 = \dfrac{1-\left(\alpha_2^{(2)}\right)^2}{\left(\alpha_4^{(2)}\right)^2\left(\left(\alpha_4^{(2)}\right)^2 - \left(\alpha_2^{(2)}\right)^2\right)}$ must be integers. Then we have

$$K_2 + K_4 = -\frac{(n+4)n(n+2)(n-1)(n+1)^2}{15(15N_2 - n(n+4)(n+2)(n+1))} - \frac{1}{15}(n+4)(n-4), \tag{7.15}$$

$$\begin{aligned}
K_2K_4 = &-\frac{1}{600}(n+6)^2(n+4)(n-1) - \frac{1}{600}\frac{n(n+2)(n+1)(n+6)^2(n+4)(n-1)(2n+3)}{15N_2 - n(n+4)(n+2)(n+1)} + \\
&+\frac{1}{3000}\frac{n^3(n+2)^3(n+1)^2(n+6)^2(n+4)^2(n-1)^3(2n+7)}{15N_2 - n(n+4)(n+2)(n+1)} \times \\
&\times \frac{1}{360N_2^2 - 24n(n+9)(n+2)(n+1)N_2 + n^2(n+4)(n+1)(n+3)^2(n+2)^2} - \\
&-\frac{1}{3000}\frac{n^2(n+2)^2(n+1)(n+6)^2(n+4)^2(n-1)^2(3n+13)}{(360N_2^2 - 24n(n+9)(n+2)(n+1)N_2 + n^2(n+4)(n+1)(n+3)^2(n+2)^2)}.
\end{aligned} \tag{7.16}$$

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Bibliography

1. ***A. Baker,*** *Bounds for the solutions of the hyperelliptic equation*, Proc. Camb. Phil. So., **65** (1969), 439-444.
2. ***Ei. Bannai and Et. Bannai,*** *Algebraic Combinatorics on Spheres*, Springer, Tokyo, 1999 (in Japanese).

3. ***Ei. Bannai and Et. Bannai,*** *On Euclidean tight* 4*-designs*, J. Math. Soc. Japan, **58**:3 (2006), 775–804.
4. ***Ei. Bannai and Et. Bannai,*** *On antipodal spherical t-designs of degree s with $t \geq 2s - 3$*, J. Comb. Inf. Syst. Sci., **34** (2009), 33–50.
5. ***Ei. Bannai and Et. Bannai,*** *Spherical designs and Euclidean designs*, in: *Recent Developments in Algebra and Related Areas* (Beijing, 2007), 1–37, Adv. Lect. Math. **8,** Higher Education Press, Beijing; International Press, Boston, 2009.
6. ***Ei. Bannai and Et. Bannai,*** *A survey on spherical designs and algebraic combinatorics on spheres*, European J. Combin., **30** (2009), 1392–1425.
7. ***Ei. Bannai and Et. Bannai,*** *Euclidean designs and coherent configurations*, Contemporary Mathematics, **531** (2010), 59–93.
8. ***Ei. Bannai and Et. Bannai,*** *Tight* 9*-designs on two concentric spheres*, J. Math. Soc. Japan, **63** (2011), 1359–1376.
9. ***Ei. Bannai, Et. Bannai, M. Hirao, and M. Sawa,*** *Cubature formulas in numerical analysis and Euclidean tight designs*, (Special Issue in Honour of Prof. Michel Deza), European J. Combin., **31** (2010), 429–441.
10. ***Ei. Bannai, Et. Bannai, and D. Suprijanto,*** *On the strong non-rigidity of certain tight Euclidean designs*, European Journal of Combinatorics **28** (2007), 1662–1680.
11. ***Ei. Bannai, Et. Bannai, and J. Shigezumi,*** *A new Euclidean tight* 6*-design*, Ann. Comb., **16**:4 (2012), 651–659.
12. ***Ei. Bannai and R. M. Damerell,*** *Tight spherical designs II*, J. London Math. Soc. (2), **21** (1980), 13–30.
13. ***Et. Bannai,*** *On Antipodal Euclidean Tight* $(2e+1)$*-Designs*, J. Algebraic Combinatorics, **24**:4 (2006), 391–414.
14. ***Et. Bannai,*** *New examples of Euclidean tight* 4*-designs*, European J. Combin., **30** (2009), 655–667.
15. ***P. Delsarte, J. M. Goethals, and J. J. Seidel,*** *Spherical codes and designs*, Geom. Dedicata, **6**:3 (1977), 363–388.
16. ***P. Delsarte and J. J. Seidel,*** *Fisher type inequalities for Euclidean t-designs*, Linear Algebra Appl., **114-115** (1989), 213–230.
17. ***G. Dumas,*** *Sur quelques cas d'irreductibilite des polynomes a coefficients rationnel*, J. Math. Pures Appl., **2** (1906), 191–258.
18. ***H. M. Möller,*** *Kubaturformeln mit minimaler Knotenzahl*, Numer. Math. **25**:2 (1975/ 76), 185–200.
19. ***H. M. Möller,*** *Lower bounds for the number of nodes in cubature formulae*, Numerische Integration (Tagung, Math. Forschungsinst., Oberwolfach, 1978), 221–230, Internat. Ser. Numer. Math., **45,** Birkhäuser, Basel—Boston, 1979.

20. ***L. J. Mordell,*** *Diophantine Equations*, Academic Press, London, 1969.
21. ***A. Neumaier and J. J. Seidel,*** *Discrete measures for spherical designs, eutactic stars and lattices*, Nederl. Akad. Wetensch. Proc. Ser. A **91**=Indag. Math. **50**:3 (1988), 321–334.
22. ***H. Nozaki,*** *A generalization of the Larman—Rogers—Seidel's theorem*, Discrete Math., **311** (2011), 792–799.
23. ***C. L. Siegel,*** *The integer solutions of the equation* $y^2 = ax^n + bx^{n-1} + \ldots + k$, J. London Math Soc., **1** (1920), 66–68.
24. ***H. Suzuki,*** *Association schemes with multiple Q-polynomial structures*, J. Algebraic Combin., **7** (1998), 181–196.

EIICHI BANNAI

Department of Mathematics,
Shanghai Jiao Tong University
800 Dongchuan Road, Shanghai,
200240, China
bannai@sjtu.edu.cn

ETSUKO BANNAI

Misakigaoka 2-8-21, Itoshimashi, Fukuoka,
819-1136, Japan
et-ban@rc4.so-net.ne.jp