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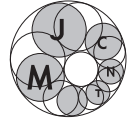
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# A strengthening of a theorem of Bourgain—Kontorovich II

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*Dedicated to the memory of professor N. M. Korobov.*

**Abstract:** Zaremba’s conjecture (1971) states that every positive integer number  $d$  can be represented as a denominator (continuant) of a finite continued fraction  $\frac{b}{d} = [d_1, d_2, \dots, d_k]$ , with all partial quotients  $d_1, d_2, \dots, d_k$  being bounded by an absolute constant  $A$ . Recently (in 2011) several new theorems concerning this conjecture were proved by Bourgain and Kontorovich. The easiest of them states that the set of numbers satisfying Zaremba’s conjecture with  $A = 50$  has positive proportion in  $\mathbb{N}$ . In 2013 we proved this result with  $A = 7$ . In this paper the same theorem is proved with  $A = 5$ .

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## 1. Historical background

Let  $\mathcal{A} \in \mathbb{N}$  be any finite alphabet ( $|\mathcal{A}| \geq 2$ ) and let  $\mathfrak{R}_{\mathcal{A}}$  be the set of rational numbers whose continued fraction expansion has all partial quotients from  $\mathcal{A}$  :

$$\mathfrak{R}_{\mathcal{A}} = \left\{ \frac{b}{d} = [d_1, d_2, \dots, d_k] \mid d_j \in \mathcal{A} \text{ for } j = 1, \dots, k \right\},$$

where

$$[d_1, \dots, d_k] = \frac{1}{d_1 + \frac{1}{\dots + \frac{1}{d_k}}} \tag{1.1}$$

is a finite continued fraction,  $d_1, \dots, d_k$  are partial quotients. Let  $\mathfrak{D}_A$  be the set of denominators of numbers in  $\mathfrak{R}_A$

$$\mathfrak{D}_A = \left\{ d \mid \exists b : (b, d) = 1, \frac{b}{d} \in \mathfrak{R}_A \right\},$$

and define  $\mathfrak{D}_A(N) = \left\{ d \in \mathfrak{D}_A \mid d \leq N \right\}$ .

CONJECTURE 1.1 (ZAREMBA'S CONJECTURE [12, p. 76], 1971). *For an alphabet  $\mathcal{A} = \{1, 2, \dots, A\}$  with sufficiently large  $A$  one has  $\mathfrak{D}_A = \mathbb{N}$ .*

That is, every  $d \geq 1$  can be represented as a denominator of a finite continued fraction  $\frac{b}{d}$  whose partial quotients are bounded by  $A$ . In fact Zaremba conjectured that  $A = 5$  is already large enough for his hypothesis being true. A detailed survey on results concerning Zaremba's conjecture can be found in [1], [11]. Let  $\mathfrak{C}_A$  be the set of infinite continued fractions whose partial quotients belong to  $\mathcal{A}$

$$\mathfrak{C}_A = \{ [d_1, \dots, d_j, \dots] : d_j \in \mathcal{A}, j = 1, \dots \}$$

and let  $\delta_A$  be the Hausdorff dimension of  $\mathfrak{C}_A$ . Then the Bourgain-Kontorovich's theorem [1, p. 13, Theorem 1.25] is as follows

THEOREM 1.1 ([1, p. 13, Theorem 1.25]). *For any alphabet  $\mathcal{A}$  with*

$$\delta_A > 1 - \frac{5}{312} = 0,983914\dots, \tag{1.2}$$

*the following inequality (positive proportion)*

$$\#\mathfrak{D}_A(N) \gg N \tag{1.3}$$

*holds.*

To be more precise, Bourgain-Kontorovich proved [1, p. 13, Theorem 1.29] much stronger result that under the condition (1.2) one has

$$\#\mathcal{D}_{\mathcal{A}}(N) = N + O\left(N^{1-c/\log \log N}\right) \quad (1.4)$$

with  $c = c(\mathcal{A}) > 0$ . This article deals only with the inequality (1.3). In view of the Hensly's [5] and Jenkinson's [7] results the alphabet

$$\{1, 2, \dots, A-1, A\} \quad (1.5)$$

with  $A = 50$  is likely to satisfy the condition (1.2). Using the method of Bourgain-Kontorovich [1] and Hensley's theorem (see [2, sec.2.2], [6]) we proved in [2] the following result

**THEOREM 1.2.** *For any alphabet  $\mathcal{A}$  such that*

$$\delta_{\mathcal{A}} > 1 - \frac{5}{\sqrt{369} + 23} = 0,8815\dots,$$

*the inequality (1.3) holds.*

The main result of the paper is the following theorem.

**THEOREM 1.3.** *For any alphabet  $\mathcal{A}$  such that*

$$\delta_{\mathcal{A}} > 1 - \frac{1}{6} = 0,8333\dots, \quad (1.6)$$

*the inequality (1.3) holds.*

**Remark 1.1.** It is proved in [7] that  $\delta_5 = 0,8368\dots$ ,  $\delta_7 = 0,8889\dots$ . So, the alphabet  $\{1, 2, \dots, 7\}$  satisfies the condition of Theorem 1.2 and the alphabet  $\{1, 2, \dots, 5\}$  satisfies the condition of Theorem 1.3.

**Remark 1.2.** It should be mentioned that the proof of Theorems 1.1 and 1.2 is essentially based on the result [1, p. 46, Lemma 7.1.] proved with the use of the spectral theory of automorphic forms. At the same time the proof of Theorem 1.3 is rather elementary and is based only on the theory of continued fractions.

**Remark 1.3.** It is worth noting that the proof of the Theorem 1.3 repeats significantly the proof of the Theorem 1.2 in [2]. So we will heavily refer to statements and

constructions in [2]. The complete proof of Theorem 1.3 can be found in our preprint [3].

## 2. Notation

Throughout  $\epsilon_0 = \epsilon_0(\mathcal{A}) \in (0, 1/2500)$ . For two functions  $f(x)$  and  $g(x)$  the Vinogradov notation  $f(x) \ll g(x)$  means that there exists a constant  $C$ , depending on  $A$  and  $\epsilon_0$ , such that  $|f(x)| \leq Cg(x)$ . The notation  $f(x) = O(g(x))$  means the same. The notation  $f(x) \asymp g(x)$  means that  $f(x) \ll g(x) \ll f(x)$ . Also a traditional notation  $e(x) = \exp(2\pi ix)$  is used. The cardinality of a finite set  $S$  is denoted either  $|S|$  or  $\#S$ .  $[\alpha]$  and  $\|\alpha\|$  denote the integral part of  $\alpha$  and the distance from  $\alpha$  to the nearest integer:  $[\alpha] = \max \{z \in \mathbb{Z} \mid z \leq \alpha\}$ ,  $\|\alpha\| = \min \{|z - \alpha| \mid z \in \mathbb{Z}\}$ .

## 3. The main properties of the ensemble $\Omega_N$

Let  $\Gamma_{\mathcal{A}} \subseteq SL(2, \mathbb{Z})$  be a semigroup with the unit element  $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix} = \begin{pmatrix} 1 & a_j \\ a_i & a_i a_j + 1 \end{pmatrix},$$

where  $a_i, a_j \in \mathcal{A}$ . Let  $V_{\mathcal{A}}(k)$  be the set of words with the length  $k$

$$V_{\mathcal{A}}(k) = \left\{ (d_1, d_2, \dots, d_k) \mid d_j \in \mathcal{A}, j = 1, \dots, k \right\},$$

and  $V_{\mathcal{A}^2} = \bigcup_{k \geq 1} V_{\mathcal{A}}(2k)$  be the set of all finite words with even length. It is well known that the map  $\mathcal{B} : V_{\mathcal{A}^2} \rightarrow \Gamma_{\mathcal{A}}$ ,

$$\mathcal{B}(d_1, d_2, \dots, d_k) = \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & d_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & d_2 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ 1 & d_k \end{pmatrix} \tag{3.1}$$

is a bijection and that

$$\frac{a}{c} = [d_1, \dots, d_{k-1}], \quad \frac{b}{d} = [d_1, \dots, d_k].$$

We use the following norm for matrices  $\gamma \in \Gamma_{\mathcal{A}}$

$$\|\gamma\| = \max\{|a|, |b|, |c|, |d|\} = d = \langle d_1, d_2, \dots, d_k \rangle, \quad (3.2)$$

where  $\langle d_1, d_2, \dots, d_k \rangle$  is the denominator (it is also called a continuant) of the continued fraction (1.1). We will use the following well known result [4]

$$\langle D \rangle \langle B \rangle \leq \langle D, B \rangle \leq 2 \langle D \rangle \langle B \rangle, \quad (3.3)$$

where  $B = \{b_1, b_2, \dots, b_n\}$ ,  $D = \{d_1, d_2, \dots, d_k\}$  and

$$D, B = \{d_1, d_2, \dots, d_k, b_1, b_2, \dots, b_n\}.$$

For all sufficiently large  $N$  we set

$$J = J(N) = \left\lceil \frac{\log \log N - 4 \log(10A) + 2 \log \epsilon_0}{-\log(1 - \epsilon_0)} \right\rceil. \quad (3.4)$$

The finite sequence

$$\{N_{-J-1}, N_{-J}, \dots, N_{-1}, N_0, N_1, \dots, N_{J+1}\},$$

where  $N_{J+1} = N$  and

$$N_j = \begin{cases} N^{\frac{1}{2-\epsilon_0}(1-\epsilon_0)^{1-j}}, & \text{if } -1 - J \leq j \leq 1; \\ N^{1 - \frac{1}{2-\epsilon_0}(1-\epsilon_0)^j}, & \text{if } 0 \leq j \leq J. \end{cases} \quad (3.5)$$

was defined in [2, sec.2.5]. We need the following statement [2].

LEMMA 3.1 ([2, Lemma 2.7]). *For any  $M$  such that*

$$N_{1-J} \leq M \leq N_J, \quad (3.6)$$

*there exist indices  $j$  and  $h$  for which the following inequalities hold*

$$2 \leq j \leq 2J, \quad 1 \leq h \leq 2J - 1, \quad h = 2J - j + 1, \quad (3.7)$$

$$N_{j-J}^{1-\epsilon_0} \leq M \leq N_{j-J}, \tag{3.8}$$

$$\left(\frac{N}{N_{h-J}}\right)^{1-\epsilon_0} \leq M \leq \frac{N}{N_{h-J}}. \tag{3.9}$$

The special set (ensemble)  $\Omega_N \subseteq \{\gamma \in \Gamma_{\mathcal{A}} \mid \|\gamma\| \leq N\}$  constructed in [2, sec.2.6] plays a key role in the proof of Theorem 1.2. It has the following form

$$\Omega_N = \Xi_1(M_1)\Xi_2(M_2) \dots \Xi_{2J}(M_{2J})\Xi_{2J+1}(M_{2J+1}),$$

where  $M_j \asymp N_{j-J}/N_{j-1-J}$  and the sets  $\Xi_1(M_j)$  were constructed in [2, sec. 2.4]. Let us formulate one of the properties of the set  $\Omega_N$ .

COROLLARY 3.1 ([2, Corollary 2.3]). *For any  $M$  satisfying the inequality*

$$\exp\left(\frac{10^5 A^4}{\epsilon_0^2}\right) \leq M \leq N \exp\left(-\frac{10^5 A^4}{\epsilon_0^2}\right), \tag{3.10}$$

*there exist indices  $j$  and  $h$  in the intervals (3.7), such that for any collection of matrices*

$$\xi_1 \in \Xi_1, \xi_2 \in \Xi_2, \dots, \xi_{2J+1} \in \Xi_{2J+1}, \tag{3.11}$$

*the inequalities*

$$0,99\|\xi_1\xi_2 \dots \xi_j\|^{1-\epsilon_0} \leq M \leq 70A^2\|\xi_1\xi_2 \dots \xi_j\|, \tag{3.12}$$

$$\frac{1}{73A^2}\|\xi_{h+1}\xi_{h+2} \dots \xi_{2J+1}\|^{1-\epsilon_0} \leq M \leq 150A^2\|\xi_{h+1}\xi_{h+2} \dots \xi_{2J+1}\| \tag{3.13}$$

*hold.*

For  $M$ , satisfying the inequality (3.10), we denote by  $\widehat{j}$  and  $\widehat{h}$  the numbers  $j$  and  $h$  from Lemma 3.1 corresponding to  $M$ . For brevity we will write that numbers  $\widehat{j}$  and  $\widehat{h}$  correspond to  $M$ . In the following theorem  $\widehat{j}^{(1)}$  corresponds to  $M^{(1)}$ , and  $\widehat{h}^{(3)}$  corresponds to  $M^{(3)}$ .

LEMMA 3.2 ([2, Lemma 2.15]). *Let the inequality*

$$M^{(1)}M^{(3)} < N^{1-\epsilon_0}. \tag{3.14}$$

hold for some  $M^{(1)}$  and  $M^{(3)}$  in the interval (3.10). Then, firstly, we have  $\widehat{j}^{(1)} < \widehat{h}^{(3)}$ . Secondly, the inequality (3.12) holds for  $M = M^{(1)}$ ,  $j = \widehat{j}^{(1)}$  and the inequality (3.13) holds for  $M = M^{(3)}$ ,  $h = \widehat{h}^{(3)}$ .

For integers  $j$  and  $h$  we define the set  $\Omega(j, h) = \Xi_{j+1}\Xi_{j+2} \dots \Xi_h$ . We put  $\Omega_1(M) = \Omega(0, \widehat{j})$ ,  $\Omega_2(M) = \Omega(\widehat{h}, 2J+1)$ . It was proved in [2, Theorem 2.6] that

$$M^{2\delta-\epsilon_0} \leq |\Omega_1(M)| \ll M^{2\delta+4\epsilon_0}, \quad M^{2\delta-\epsilon_0} \leq |\Omega_2(M)|. \quad (3.15)$$

In [2, Corollary 2.1], we also proved that

$$N^{2\delta-\epsilon_0} \leq |\Omega_N| \ll N^{2\delta}. \quad (3.16)$$

**THEOREM 3.1** ([2, Theorem 2.3]). *For any  $M = M^{(1)}$  satisfying the inequality (3.10), the ensemble  $\Omega_N$  can be represented in the following form*

$$\Omega_N = \Omega^{(1)}\Omega^{(3)}, \quad (3.17)$$

where

$$\Omega^{(1)} = \Omega_1(M^{(1)}) = \Xi_1\Xi_2 \dots \Xi_{\widehat{j}}, \quad \Omega^{(3)} = \Xi_{\widehat{j}+1}\Xi_{j+2} \dots \Xi_{2J+1} \quad (3.18)$$

and for any  $\gamma_1 \in \Omega^{(1)}$ ,  $\gamma_3 \in \Omega^{(3)}$  the following inequalities hold

$$\frac{M^{(1)}}{70A^2} \leq \|\gamma_1\| \leq 1, 03(M^{(1)})^{1+2\epsilon_0}, \quad (3.19)$$

$$\frac{N}{140A^2H_1(M^{(1)})} \leq \|\gamma_3\| \leq \frac{73A^2N}{M^{(1)}}, \quad \text{where } H_1(M^{(1)}) = 1, 03(M^{(1)})^{1+2\epsilon_0}, \quad (3.20)$$

The following theorem is similar to [2, Theorem 2.4 and 2.5].

**THEOREM 3.2.** *Let  $M^{(4)} \geq (M^{(1)})^{2\epsilon_0}$  and let the inequality (3.10) hold for  $M = M^{(1)}$  and  $M = M^{(1)}M^{(4)}$ . Let  $M^{(1)}M^{(4)}$  and  $M^{(6)}$  satisfy the hypotheses of Lemma 3.2. Then the ensemble  $\Omega_N$  can be represented in the following form*

$$\Omega_N = \Omega^{(1)}\Omega^{(3)} = \Omega^{(1)}\Omega^{(4)}\Omega^{(5)}\Omega^{(6)}.$$

In doing so one has (3.18) and

$$\begin{aligned} \Omega^{(1)}\Omega^{(4)} &= \Omega_1(M^{(1)}M^{(4)}), \quad \Omega^{(4)} = \Xi_{\widehat{j}_1+1}\Xi_{\widehat{j}_1+2}\cdots\Xi_{\widehat{j}_4}, \\ \Omega^{(5)} &= \Xi_{\widehat{j}_4+1}\Xi_{\widehat{j}_4+2}\cdots\Xi_{\widehat{h}_6}, \quad \Omega^{(6)} = \Omega_2(M^{(6)}) = \Xi_{\widehat{h}_6+1}\Xi_{\widehat{h}_6+2}\cdots\Xi_{2J+1}, \end{aligned}$$

where  $\widehat{j}_1$  corresponds to  $M^{(1)}$ ,  $\widehat{j}_4$  corresponds to  $M^{(1)}M^{(4)}$ , and  $\widehat{h}_6$  to  $M^{(6)}$ . Moreover for any  $\gamma_i \in \Omega^{(i)}$ ,  $i = 1, \dots, 6$  one has (3.19), (3.20), and

$$\frac{M^{(1)}M^{(4)}}{70A^2} \leq \|\gamma_1\gamma_4\| \leq 1, 03(M^{(1)}M^{(4)})^{1+2\epsilon_0}, \quad (3.21)$$

$$\frac{M^{(4)}}{150A^2(M^{(1)})^{2\epsilon_0}} \leq \|\gamma_4\| \leq 73A^2(M^{(4)})^{1+2\epsilon_0}(M^{(1)})^{2\epsilon_0}, \quad (3.22)$$

$$\frac{M^{(6)}}{150A^2} \leq \|\gamma_6\| \leq 80A^{2,1}(M^{(6)})^{1+2\epsilon_0}, \quad (3.23)$$

$$\frac{N}{25000A^5(M^{(1)}M^{(6)})^{1+2\epsilon_0}} \leq \|\gamma_4\gamma_5\| \leq 11000A^4 \frac{N}{M^{(1)}M^{(6)}}. \quad (3.24)$$

PROOF. To ensure that the partition of the ensemble is well-defined, it is enough to verify that  $\widehat{j}_1 < \widehat{j}_4 < \widehat{h}_6$ . The second inequality follows from Lemma 3.2 ( $\widehat{j}_4 = j^{(1)}$ ,  $\widehat{h}_6 = h^{(3)}$ ). To prove  $\widehat{j}_1 < \widehat{j}_4$  it is sufficient to show that the inequality  $N_{\widehat{j}_1-J} \leq M^{(1)}M^{(4)}$  holds for any  $M^{(1)}$  in the interval  $N_{\widehat{j}_1-1-J} \leq M^{(1)} \leq N_{\widehat{j}_1-J}$ . It follows from the conditions of the theorem that

$$M^{(4)} \geq (M^{(1)})^{2\epsilon_0} \geq (M^{(1)})^{\frac{1}{1-\epsilon_0}-1} \geq \frac{N_{\widehat{j}_1-J}}{M^{(1)}}, \quad (3.25)$$

where the last inequality holds because of  $M^{(1)} \geq N_{\widehat{j}_1-J}^{1-\epsilon_0}$  (see (3.8)). Thus we have proved that the partition of the ensemble is well-defined. The bound (3.21) can be proved in the same way as (3.19), and (3.22) follows from (3.19), (3.21), and (3.3). It also follows from Lemma 3.2 that the inequality (3.13) holds for  $\gamma_6 \in \Omega^{(6)}$ . Using this inequality we obtain the estimate (3.23). Next, in the same way as [2, (2.80)] we obtain

$$\frac{N}{280A^2} \leq \|\gamma_1\|\|\gamma_4\gamma_5\|\|\gamma_6\| \leq 1, 01N. \quad (3.26)$$

Substituting (3.19) and (3.23) into (3.26) we have (3.24). This completes the proof of the theorem.  $\square$

#### 4. General estimates of exponential sums over the ensemble

It is common knowledge (for example, see [2, sec.2.3]) that to prove Theorem 1.3 it is sufficient to obtain the following estimate

$$\int_0^1 |S_N(\theta)|^2 d\theta \ll \frac{|\Omega_N|^2}{N}, \text{ where } S_N(\theta) := \sum_{\gamma \in \Omega_N} e(\theta \|\gamma\|), \quad (4.1)$$

and  $N$  is a sufficiently large integer. In view of (3.2) we have

$$S_N(\theta) = \sum_{\gamma \in \Omega_N} e(\theta \|\gamma\|) = \sum_{\gamma \in \Omega_N} e((0, 1)\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} \theta). \quad (4.2)$$

Let  $Z$  be a finite subset of  $[0, 1]$ . Using Vinogradov's method, in [2, sec.3.1] we reduced the problem of estimating the sum

$$\sigma_{N,Z} = \sum_{\theta \in Z} |S_N(\theta)|,$$

to the evaluation of cardinality of some sets. To do this we used the partition  $\Omega^{(1)}\Omega^{(2)}\Omega^{(3)}$  of the ensemble  $\Omega_N$  and rewrote  $S_N(\theta)$  as a triple sum. After that the sum over  $\Omega^{(1)}$  was expanded to the sum over all integer vectors  $\|\gamma_1\| \leq H_1$  and the Cauchy—Schwarz inequality was applied. Thereafter, standard bounds of the absolute value of linear exponential sums were used. It was done in [2, Lemmas 3.1 and 3.2]. Now we present a slight simplification of the method. Using Theorem 3.1 or 3.2 we obtain the following partition of the ensemble  $\Omega_N$

$$\Omega_N = \Omega^{(1)}\Omega^{(3)}. \quad (4.3)$$

So in [2, sec.3.1], in particular, in [2, Lemma 3.1 and 3.2], we can put  $\mu = 3$  and think of  $\tilde{\Omega}^{(2)}$  as the set consisting only the unit matrix.

Let

$$N_{1/2} = \frac{N^{1/2}}{10A}, \quad N_{1/2}^* = \frac{N}{N_{1/2}}. \quad (4.4)$$

It follows from the Dirichlet's theorem that for any  $\theta \in [0, 1]$  there exist  $a, q \in \mathbb{N} \cup \{0\}$  and  $\beta \in \mathbb{R}$  such that

$$\theta = \frac{a}{q} + \beta, \quad (a, q) = 1, \quad 0 \leq a \leq q \leq N_{1/2}, \quad \beta = \frac{K}{N}, \quad |K| \leq \frac{N_{1/2}^*}{q}, \quad (4.5)$$

with  $a = 0$  and  $a = q$  being possible if only  $q = 1$ . Let

$$P_{Q_1, Q}^{(\beta)} = \left\{ \theta = \frac{a}{q} + \beta \mid (a, q) = 1, \quad 0 \leq a \leq q, \quad Q_1 \leq q \leq Q \right\}. \quad (4.6)$$

We will always write numbers  $\theta^{(1)}, \theta^{(2)} \in P_{Q_1, Q}^{(\beta)}$  in the following way

$$\theta^{(1)} = \frac{a^{(1)}}{q^{(1)}} + \beta, \quad \theta^{(2)} = \frac{a^{(2)}}{q^{(2)}} + \beta. \quad (4.7)$$

For  $n \in \{1, 3\}$  we denote

$$\tilde{\Omega}^{(n)} = \begin{cases} (0, 1)\Omega^{(1)} = \left\{ (0, 1)g_1 \mid g_1 \in \Omega^{(1)} \right\}, & \text{if } n = 1, \\ \Omega^{(3)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left\{ g_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid g_3 \in \Omega^{(3)} \right\}, & \text{if } n = 3. \end{cases}, \quad (4.8)$$

Let

$$\mathfrak{N} = \left\{ (g_3^{(1)}, g_3^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(3)} \times \tilde{\Omega}^{(3)} \times Z^2 \mid (4.10) \text{ and } (4.11) \text{ hold} \right\}, \quad (4.9)$$

where

$$\left\| g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \leq \frac{74A^2\overline{K}}{M^{(1)}}, \quad (4.10)$$

$$\left| g_3^{(1)} - g_3^{(2)} \right|_{1,2} \leq \min \left\{ \frac{73A^2N}{M^{(1)}}, \frac{73A^2N}{M^{(1)}\overline{K}} + \frac{N}{\overline{K}} \left\| g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \right\}, \quad (4.11)$$

and  $\overline{K} = \max\{1, |K|\}$ .

LEMMA 4.1 ([2, Lemma 3.3]). *For  $Z \subseteq P_{Q_1, Q}^{(\beta)}$  and for  $M^{(1)}$  in (3.10) such that  $M^{(1)} > 146A^2\overline{K}$ , the following inequality holds*

$$\sigma_{N, Z} \ll H_1 |\Omega^{(1)}|^{1/2} |\mathfrak{N}|^{1/2}. \quad (4.12)$$

We denote

$$\mathfrak{M} = \left\{ (g_3^{(1)}, g_3^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(3)} \times \tilde{\Omega}^{(3)} \times Z^2 \mid (4.14) \text{ and } (4.15) \text{ hold} \right\}, \quad (4.13)$$

where

$$|g_3^{(1)} - g_3^{(2)}|_{1,2} \leq \frac{73A^2N}{M^{(1)}\bar{K}}, \quad (4.14)$$

$$\left\| g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} = 0. \quad (4.15)$$

Now we transform the equation (4.15). Let  $\mathbf{q} = [q^{(1)}, q^{(2)}]$ , then (4.15) can be written as

$$\left( g_3^{(1)} \frac{a^{(1)}q^{(2)}}{(q^{(1)}, q^{(2)})} - g_3^{(2)} \frac{a^{(2)}q^{(1)}}{(q^{(1)}, q^{(2)})} \right)_{1,2} \equiv 0 \pmod{\mathbf{q}}. \quad (4.16)$$

By setting  $g_3^{(1)} = (x_1, x_2)^t$ ,  $g_3^{(2)} = (y_1, y_2)^t$  in (4.16) we obtain the congruence

$$x_1 \frac{a^{(1)}q^{(2)}}{(q^{(1)}, q^{(2)})} \equiv y_1 \frac{a^{(2)}q^{(1)}}{(q^{(1)}, q^{(2)})} \pmod{\mathbf{q}} \quad (4.17)$$

and, the same for  $x_2, y_2$ . But  $\left( a^{(1)}, \frac{q^{(1)}}{(q^{(1)}, q^{(2)})} \right) \leq (a^{(1)}, q^{(1)}) = 1$  and, therefore,

$$x_1 \equiv 0 \pmod{\frac{q^{(1)}}{(q^{(1)}, q^{(2)})}}, \quad x_2 \equiv 0 \pmod{\frac{q^{(1)}}{(q^{(1)}, q^{(2)})}}$$

and, the same for  $y_1, y_2$ . At the same time  $(x_1, x_2) = (y_1, y_2) = 1$  as the components of the vectors  $g_3^{(1)}, g_3^{(2)}$ , thus  $q^{(1)} = (q^{(1)}, q^{(2)}) = q^{(2)} = \mathbf{q}$ . So

$$\mathfrak{M} = \left\{ (g_3^{(1)}, g_3^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(3)} \times \tilde{\Omega}^{(3)} \times Z^2 \mid (4.14) \text{ and } (4.19) \text{ hold} \right\}, \quad (4.18)$$

where

$$(a^{(1)}g_3^{(1)} - a^{(2)}g_3^{(2)})_{1,2} \equiv 0 \pmod{\mathbf{q}} \quad \text{and} \quad \mathbf{q} = q^{(1)} = q^{(2)}. \quad (4.19)$$

LEMMA 4.2 ([2, Lemma 3.4]). *Let the hypotheses of one of the Theorem 3.1 or 3.2, on which the partition of  $\Omega_N$  in the form (4.3) is based, hold. Let  $M^{(1)}$  be such that for any  $\theta^{(1)}, \theta^{(2)} \in Z$  the following inequality holds*

$$\max\{2, [q^{(1)}, q^{(2)}]\} < \frac{M^{(1)}}{74A^2\overline{K}}. \quad (4.20)$$

*Then the following bound holds*

$$\sigma_{N,Z} \ll H_1 |\Omega^{(1)}|^{1/2} |\mathfrak{M}|^{1/2}. \quad (4.21)$$

Thus we reduced the problem of estimating  $\sigma_{N,Z}$  to the evaluation the cardinality of one of the sets  $\mathfrak{N}$ ,  $\mathfrak{M}$ . Let us state one more lemma of a general nature. A similar statement was used by S. V. Konyagin in [8, sec.17].

LEMMA 4.3 ([2, Lemma 3.5]). *Let  $W$  be a finite subset of the interval  $[0, 1]$  and let  $|W| > 1$ . Let  $f : W \rightarrow \mathbb{R}_+$  be a function such that, for any subset  $Z \subseteq W$  the following bound holds*

$$\sum_{\theta \in Z} f(\theta) \leq C_1 |Z|^{1/2},$$

*where  $C_1$  is a non-negative constant not depending on the set  $Z$ . Then the following estimate holds*

$$\sum_{\theta \in W} f^2(\theta) \ll C_1^2 \log |W| \quad (4.22)$$

*with the absolute constant in Vinogradov symbol.*

## 5. Consequences of general estimates of exponential sum

Let  $Q_1$ ,  $Q$ , and  $\beta$  be given. For any  $q$  in  $Q_1 \leq q \leq Q$  we define by any means the number  $a_q$ , such that  $(a_q, q) = 1$ ,  $0 \leq a_q \leq q$ . Let us denote

$$Z^* = \left\{ \theta = \frac{a_q}{q} + \beta \mid Q_1 \leq q \leq Q \right\}, \quad (5.1)$$

then  $|Z^*| \leq Q$ . The following trivial bound holds

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \leq Q \sum_{Q_1 \leq q \leq Q} \max_{1 \leq a \leq q, (a, q) = 1} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 = Q \sum_{\theta \in Z^*} |S_N(\theta)|^2, \quad (5.2)$$

where as  $a_q$  we have chosen numerators for which the maximum is achieved. We write  $\gamma = 1 - \delta$  and

$$Q_0 = \max \left\{ \exp \left( \frac{10^5 A^4}{\epsilon_0^2} \right), \exp(\epsilon_0^{-5}) \right\}.$$

LEMMA 5.1. *If  $\bar{K}^{5/2} Q^3 \leq N^{1-2\epsilon_0}$ ,  $\bar{K}Q \geq Q_0$ , then for any  $Z \subseteq Z^*$  then the following bound holds*

$$\sigma_{N, Z} \ll \frac{|\Omega_N| |Z|}{|Z|^{1/2} \bar{K} Q_1} \left( \bar{K}^{5/2} Q^3 \right)^\gamma \left( \bar{K}^{15} Q^{21} \right)^{\epsilon_0}. \quad (5.3)$$

PROOF. We put

$$M^{(1)} = 76A^2 \bar{K} Q^2, \quad M^{(4)} = \bar{K}^{1/2} (M^{(1)})^{2\epsilon_0}, \quad M^{(6)} = \bar{K} Q. \quad (5.4)$$

It follows from the statement of the lemma that all conditions of the Theorem 3.2 and Lemma 4.2 hold. Let us prove that

$$|\mathfrak{M}| \ll |Z| |\Omega^{(3)}| \left( (M^{(4)})^{4\epsilon_0} (M^{(1)})^{10\epsilon_0} \right) |\Omega^{(5)}| \left( \frac{(M^{(6)})^{1+2\epsilon_0}}{Q_1} + 1 \right) \left( \frac{(M^{(1)} M^{(6)})^{1+2\epsilon_0}}{M^{(1)} \bar{K} Q_1} + 1 \right), \quad (5.5)$$

where  $\mathfrak{M}$  is defined in (4.18). Let us fix  $\mathbf{q}$ , for which there are  $|Z|$  choices, this gives the first factor in (5.5). Then it follows from the conditions on the set  $Z$  that  $a^{(1)} = a^{(2)}$  and the congruence (4.19) can be simplified to

$$\left( g_3^{(1)} - g_3^{(2)} \right)_{1,2} \equiv 0 \pmod{\mathbf{q}}. \quad (5.6)$$

We fix the vector  $g_3^{(2)}$ , for which there are  $|\Omega^{(3)}|$  choices. This gives the second factor in (5.5). Using Theorem 3.2 we can write the second vector in the following form  $g_3^{(2)} = \gamma_4 \gamma_5 \gamma_6 (0, 1)^t$ , where  $\gamma_i \in \Omega^{(i)}$ ,  $i = 4, 5, 6$ . Let  $g_3^{(1)} = (x_1, x_2)^t$ ,  $g_3^{(2)} = (y_1, y_2)^t$ . It follows from (4.14) that

$$\left| \frac{x_1}{x_2} - \frac{y_1}{y_2} \right| \ll \frac{(M^{(1)})^{2\epsilon_0}}{\bar{K}} \quad (5.7)$$

Let  $\gamma_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  then

$$\left| \frac{y_1}{y_2} - \frac{b}{d} \right| \ll \frac{1}{d^2} \tag{5.8}$$

It follows from (3.22) that

$$\frac{1}{d^2} = \frac{1}{\|\gamma_4\|^2} \ll \frac{(M^{(1)})^{4\epsilon_0}}{(M^{(4)})^2}. \tag{5.9}$$

Using (5.7), (5.8), and (5.9), we obtain  $\left| \frac{x_1}{x_2} - \frac{b}{d} \right| \ll \frac{(M^{(1)})^{2\epsilon_0}}{\overline{K}} + \frac{(M^{(1)})^{4\epsilon_0}}{(M^{(4)})^2}$ . It follows from (3.22) that for any two different matrices  $\gamma_4^{(1)}$  and  $\gamma_4^{(2)}$  in  $\Omega^{(4)}$ , we have

$$\left| \frac{b_1}{d_1} - \frac{b_2}{d_2} \right| \geq \frac{1}{d_1 d_2} \gg \frac{1}{(M^{(1)})^{4\epsilon_0} (M^{(4)})^{2+4\epsilon_0}}$$

Hence, the number of different fractions  $\frac{b}{d}$ , that is the number of different matrices  $\gamma_4$  is less than

$$\begin{aligned} & (M^{(4)})^{2+4\epsilon_0} (M^{(1)})^{4\epsilon_0} \left( \frac{(M^{(1)})^{2\epsilon_0}}{\overline{K}} + \frac{(M^{(1)})^{4\epsilon_0}}{(M^{(4)})^2} \right) = \\ & = (M^{(4)})^{4\epsilon_0} (M^{(1)})^{8\epsilon_0} \left( 1 + \frac{(M^{(4)})^2}{(M^{(1)})^{2\epsilon_0} \overline{K}} \right) \ll (M^{(4)})^{4\epsilon_0} (M^{(1)})^{10\epsilon_0} \end{aligned}$$

this is the third factor in (5.5). We fix the matrix  $\gamma_5$  for which there are  $|\Omega^{(5)}|$  choices. This gives the fourth factor in (5.5). Let  $\gamma_6(0, 1)^t = (x, y)^t$ , then (4.14) and (5.6) can be written in the following form

$$|x_1 - (ax + by)| \leq \frac{73A^2N}{M^{(1)}\overline{K}}, \quad |x_2 - (cx + dy)| \leq \frac{73A^2N}{M^{(1)}\overline{K}}, \tag{5.10}$$

$$x_1 \equiv (ax + by) \pmod{\mathfrak{q}}, \quad x_2 \equiv (cx + dy) \pmod{\mathfrak{q}}, \tag{5.11}$$

where  $g_3^{(1)} = (x_1, x_2)^t$ , and  $\gamma_4\gamma_5 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  are fixed. As  $\det(\gamma_4\gamma_5) = 1$ , we obtain from (5.11) that

$$x \equiv (dx_1 - bx_2) \pmod{\mathfrak{q}}, \quad y \equiv (ax_2 - cx_1) \pmod{\mathfrak{q}}, \quad (5.12)$$

Since  $x \leq \|\gamma_6\|$  and applying (3.23) and (5.12) we obtain that for  $x$  there are less than  $\left(\frac{(M^{(6)})^{1+2\epsilon_0}}{\mathfrak{q}} + 1\right)$  choices. This gives the fifth factor in (5.5). It follows from (5.10) and (5.12) that for  $y$  there are less than  $\left(\frac{73A^2N}{M^{(1)}\overline{K}\mathfrak{q}d} + 1\right)$  choices. Using (3.24), we obtain that it is less than

$$\frac{73A^2N}{M^{(1)}\overline{K}Q_1} \frac{25000A^5(M^{(1)}M^{(6)})^{1+2\epsilon_0}}{N} + 1 \ll \frac{(M^{(1)}M^{(6)})^{1+2\epsilon_0}}{M^{(1)}\overline{K}Q_1} + 1.$$

This gives the sixth factor in (5.5) Thus the bound (5.5) is proved. Substituting (5.5) into (4.21) and applying (5.4), we obtain

$$\sigma_{N,Z} \ll |Z|^{1/2} |\Omega^{(1)}|^{1/2} |\Omega^{(3)}|^{1/2} |\Omega^{(5)}|^{1/2} (M^{(1)})^{1+7\epsilon_0} (M^{(4)})^{2\epsilon_0} \frac{(M^{(1)})^{1/2+\epsilon_0} (M^{(6)})^{1+2\epsilon_0}}{(M^{(1)})^{1/2} \overline{K}^{1/2} Q_1},$$

Since

$$|\Omega^{(1)}|^{1/2} |\Omega^{(3)}|^{1/2} |\Omega^{(5)}|^{1/2} = |\Omega^{(1)}|^{1/2} |\Omega^{(4)}|^{1/2} |\Omega^{(5)}| |\Omega^{(6)}|^{1/2} = \frac{|\Omega_N|}{|\Omega^{(1)}|^{1/2} |\Omega^{(4)}|^{1/2} |\Omega^{(6)}|^{1/2}}, \quad (5.13)$$

and applying (3.15) we obtain

$$\sigma_{N,Z} \ll |Z|^{1/2} |\Omega_N| \frac{(M^{(1)})^{1+8\epsilon_0} (M^{(4)})^{2\epsilon_0} (M^{(6)})^{1+2\epsilon_0}}{(M^{(1)}M^{(4)}M^{(6)})^{\delta-\epsilon_0/2} \overline{K}^{1/2} Q_1}. \quad (5.14)$$

Substituting  $M^{(i)}$  from (5.4) into (5.14), one has

$$\sigma_{N,Z} \ll |Z|^{1/2} |\Omega_N| \overline{K}^{5\gamma/2-1+15\epsilon_0} Q^{3\gamma-1+21\epsilon_0} \frac{Q}{Q_1}.$$

This completes the proof of the lemma.  $\square$

LEMMA 5.2. *If  $\bar{K}^{5/2} Q^3 \leq N^{1-2\epsilon_0}$ ,  $\bar{K}Q \geq Q_0$ , then the following estimate holds*

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \ll \frac{|\Omega_N|^2 Q}{\bar{K}^2 Q_1^2} (\bar{K}^5 Q^6)^\gamma (\bar{K}^{30} Q^{42})^{\epsilon_0}. \quad (5.15)$$

PROOF. Applying Lemma 4.3 with  $W = Z^*$ , we obtain from (5.3) that

$$\sum_{\theta \in Z^*} |S_N(\theta)|^2 \ll |\Omega_N|^2 \bar{K}^{5\gamma-2+30\epsilon_0} Q^{6\gamma-2+42\epsilon_0} \frac{Q^2}{Q_1^2}.$$

Using the trivial bound (5.2), we obtain the desired estimate (5.15). The lemma is proved.  $\square$

LEMMA 5.3 ([2, Lemma 3.9]). *If  $\bar{K}q \geq Q_0$  then the following bound holds*

$$|S_N(\theta)| \ll \frac{|\Omega_N|}{\bar{K}q} N^\gamma (\bar{K}^2 q^2 N)^{\epsilon_0}. \quad (5.16)$$

LEMMA 5.4. *Let the following inequalities hold*

$$Q_0 \leq Q^{1-\epsilon_0} \leq Q_1 \leq Q \leq \frac{N^{1/2}}{15A}, \quad \bar{K} \leq 15A \frac{N^{1/2}}{Q_1}.$$

*Then for any  $Z \subseteq P_{Q_1, Q}^{(\beta)}$  the following bound holds*

$$\sum_{\theta \in Z} |S_N(\theta)| \ll |Z|^{1/2} |\Omega_N| \left( \frac{N^{\gamma+4\epsilon_0}}{(\bar{K}Q_1)^{1/2}} + N^{3\gamma/2-1/2+5\epsilon_0} \frac{(\bar{K}Q)^{1/2-\gamma/2}}{\bar{K}^{1/2}} \right). \quad (5.17)$$

PROOF. We use the partition of the ensemble  $\Omega_N$  given by Theorem 3.1. We put  $Z \subseteq P_{Q_1, Q}^{(\beta)}$  and

$$M^{(1)} = 150A^2 N^{1/2} (\bar{K}Q)^{1/2}, \quad U = \frac{73A^2 N}{M^{(1)}}. \quad (5.18)$$

Then  $M^{(1)} \geq 150A^2 \bar{K}$  and conditions of Lemma 4.1 hold. Let

$$g_3^{(1)} = (x_1, x_2)^t, \quad g_3^{(2)} = (y_1, y_2)^t, \quad \mathcal{Y} = x_1 y_2 - y_1 x_2. \quad (5.19)$$

It was proved in [2, Lemma 3.10] that if (4.10) and (4.11) hold then  $\mathcal{Y} \equiv 0 \pmod{\mathbf{q}}$ , where  $\mathbf{q} = [q^{(1)}, q^{(2)}]$ . The set  $\mathfrak{N}$  can be represented as a union of the sets  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ . For the first set  $\mathcal{Y} = 0$ , for the second one  $\mathcal{Y} \neq 0$ . To prove the estimate (5.17) we use the following bounds

$$|\mathfrak{M}_1| \ll |Z| |\Omega^{(3)}|, \quad (5.20)$$

$$|\mathfrak{M}_2| \ll |Z| \frac{U^2}{\overline{K} Q_1} \left( |\Omega^{(3)}| N^{2\epsilon_0} + Q^{1+\epsilon_0} \right). \quad (5.21)$$

The proof of (5.20) is given in [2, Lemma 3.12]. The bound (5.21) will be proved in Lemma 5.5. Hence

$$|\mathfrak{M}|^{1/2} \ll |Z|^{1/2} \left( \frac{N^{1+\epsilon_0}}{M^{(1)}} \frac{|\Omega^{(3)}|^{1/2}}{\overline{K}^{1/2} Q_1^{1/2}} + \frac{NQ^\epsilon}{M^{(1)} \overline{K}^{1/2}} + |\Omega^{(3)}|^{1/2} \right). \quad (5.22)$$

Substituting (5.22) into (4.12) one has

$$\sigma_{N,Z} \ll (M^{(1)})^{1+2\epsilon_0} |\Omega^{(1)}|^{1/2} |Z|^{1/2} \left( \frac{N^{1+\epsilon_0}}{M^{(1)}} \frac{|\Omega^{(3)}|^{1/2}}{\overline{K}^{1/2} Q_1^{1/2}} + \frac{NQ^{\epsilon_0}}{M^{(1)} \overline{K}^{1/2}} + |\Omega^{(3)}|^{1/2} \right). \quad (5.23)$$

Using the bounds (3.15) and (3.16), we have

$$|\Omega^{(1)}|^{1/2} |\Omega^{(3)}|^{1/2} = |\Omega_N| \frac{1}{|\Omega_N|^{1/2}} \leq |\Omega_N| \frac{1}{N^{\delta-\epsilon_0/2}}. \quad (5.24)$$

$$|\Omega^{(1)}|^{1/2} = |\Omega_N| \frac{|\Omega^{(1)}|^{1/2}}{|\Omega_N|} \leq |\Omega_N| \frac{(M^{(1)})^{\delta+2\epsilon_0}}{N^{2\delta-\epsilon_0}}. \quad (5.25)$$

Substituting (5.24) and (5.25) into (5.23), we obtain

$$\sigma_{N,Z} \ll |\Omega_N| |Z|^{1/2} \left( \frac{N^{1-\delta+2\epsilon_0} (M^{(1)})^{2\epsilon_0}}{\overline{K}^{1/2} Q_1^{1/2}} + N^{1-2\delta+\epsilon_0} (M^{(1)})^{\delta+4\epsilon_0} \frac{Q^{\epsilon_0}}{\overline{K}^{1/2}} + \frac{(M^{(1)})^{1+2\epsilon_0}}{N^{\delta-\epsilon_0/2}} \right).$$

Substituting  $M^{(1)}$  from (5.18) one has

$$\sigma_{N,Z} \ll |Z|^{1/2} |\Omega_N| \left( \frac{N^{\gamma+4\epsilon_0}}{(\overline{K} Q_1)^{1/2}} + N^{3\gamma/2-1/2+5\epsilon_0} \frac{(\overline{K} Q)^{1/2-\gamma/2}}{\overline{K}^{1/2}} + N^{\gamma-1/2+3\epsilon_0} (\overline{K} Q)^{1/2} \right).$$

Under the hypotheses of the lemma the last summand is less than the first one and we obtain (5.17). This completes the proof of the lemma.  $\square$

LEMMA 5.5. *Under the hypotheses of Lemma 5.4 one has*

$$|\mathfrak{M}_2| \ll |Z| \frac{U^2}{\overline{K}Q_1} \left( |\Omega^{(3)}| N^{2\epsilon_0} + Q^{1+\epsilon_0} \right). \quad (5.26)$$

PROOF. In the same way as in [2, Lemma 3.13] we obtain

$$q^{(1)} = q^{(2)} = \mathfrak{q}, \quad (g_3^{(1)} a^{(1)} - g_3^{(2)} a^{(2)})_{1,2} \equiv 0 \pmod{\mathfrak{q}}. \quad (5.27)$$

Then, using (4.11), one has

$$|g_3^{(1)} - g_3^{(2)}|_{1,2} \leq \frac{U}{\overline{K}}. \quad (5.28)$$

It was proved in [2, Lemma 3.13] that if  $g_3^{(1)}$ ,  $g_3^{(2)}$ , and  $a^{(1)}$  are fixed then  $a^{(2)}$  is determined uniquely by (5.27). Let  $g_3^{(1)} = (x_1, x_2)^t$  and  $g_3^{(2)} = (y_1, y_2)^t$ . Since  $\mathfrak{q}|\mathcal{Y}$  one has  $x_1 y_2 \equiv x_2 y_1 \pmod{\mathfrak{q}}$  and so

$$|\mathfrak{M}_2| \leq |Z| \sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{\substack{g_3^{(2)} \in \Omega^{(3)} \\ |g_3^{(1)} - g_3^{(2)}|_{1,2} \leq U/\overline{K}}} \mathbf{1}_{\{x_1 y_2 \equiv x_2 y_1 \pmod{\mathfrak{q}}\}} \quad (5.29)$$

The change of variables  $z_1 = x_1 - y_1$  and  $z_2 = x_2 - y_2$  leads to

$$|\mathfrak{M}_2| \leq |Z| \sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{|z_{1,2}| \leq U/\overline{K}} \mathbf{1}_{\{x_1 z_2 \equiv x_2 z_1 \pmod{\mathfrak{q}}\}} \quad (5.30)$$

We consider three cases.

1. Let  $z_1 > 0$  and  $z_2 > 0$ . We fix the vector  $g_3^{(1)} \in \Omega^{(3)}$ , then  $x_1 z_2 - x_2 z_1 = j\mathfrak{q}$ . Let us estimate the number of  $j$ 's which contribute. We have

$$x_1 - x_2 \frac{U}{\overline{K}} \leq j\mathfrak{q} \leq x_1 \frac{U}{\overline{K}} - x_2$$

and, hence,  $\#j \leq \frac{U^2}{\mathfrak{q}\overline{K}} + 1$ . For a fixed  $j$  the solution of the congruence is given by the formulae

$$z_1 = z_{1,0} + nx_1, \quad z_2 = z_{2,0} + nx_2.$$

In view of  $x_2 \gg \frac{U}{(M^{(1)})^{2\epsilon_0}}$  and  $|z_{1,2}| \leq \frac{U}{\overline{K}}$  we have  $\#n \ll \frac{(M^{(1)})^{2\epsilon_0}}{\overline{K}} + 1$ . Thus

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{0 < z_{1,2} \leq T/\overline{K}} \mathbf{1}_{\{x_1 z_2 \equiv x_2 z_1 \pmod{\mathfrak{q}}\}} \ll |\Omega^{(3)}| \left( \frac{U^2}{\overline{K}Q_1} + 1 \right) \left( \frac{(M^{(1)})^{2\epsilon_0}}{\overline{K}} + 1 \right). \quad (5.31)$$

It follows from the conditions of Lemma 5.4 that  $\frac{U^2}{\overline{K}Q_1} \gg Q^{-\epsilon_0}$ , so one has

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{0 < z_{1,2} \leq U/\overline{K}} \mathbf{1}_{\{x_1 z_2 \equiv x_2 z_1 \pmod{\mathfrak{q}}\}} \ll |\Omega^{(3)}| \frac{U^2}{\overline{K}Q_1} N^{2\epsilon_0}. \quad (5.32)$$

2. Let  $z_1 > 0$  and  $z_2 < 0$ . In the same way as in the previous case we obtain

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{0 < -z_2, z_1 \leq U/\overline{K}} \mathbf{1}_{\{x_1 z_2 \equiv x_2 z_1 \pmod{\mathfrak{q}}\}} \ll |\Omega^{(3)}| \frac{U^2}{\overline{K}Q_1} N^{2\epsilon_0}. \quad (5.33)$$

3. Let  $z_1 = 0$ . One has

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{|z_2| \leq U/\overline{K}} \mathbf{1}_{\{x_1 z_2 \equiv 0 \pmod{\mathfrak{q}}\}} \leq \sum_{g_3^{(1)} \in \Omega^{(3)}} \left( \frac{U}{q\overline{K}}(x_1, q) + 1 \right) \quad (5.34)$$

$$\leq |\Omega^{(3)}| + \frac{U^2}{q\overline{K}} \sum_{x_1 \leq U} (x_1, q) \quad (5.35)$$

Next,

$$\sum_{x_1 \leq U} (x_1, q) \leq \sum_{d|q} d \left( \frac{U}{d} + 1 \right) \ll_{\epsilon} Uq^{\epsilon} + q^{1+\epsilon}$$

and so

$$\sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{|z_2| \leq U/\overline{K}} \mathbf{1}_{\{x_1 z_2 \equiv 0 \pmod{\mathfrak{q}}\}} \ll_{\epsilon} |\Omega^{(3)}| + \frac{U^2}{q\overline{K}} \left( Uq^{\epsilon} + q^{1+\epsilon} \right). \quad (5.36)$$

Using (5.32), (5.33), and (5.36) and putting  $\epsilon = \epsilon_0$ , we obtain

$$|\mathfrak{M}_2| \ll |Z| \frac{U^2}{\overline{K}Q_1} \left( UQ_1^{\epsilon_0} + Q^{1+\epsilon_0} + |\Omega^{(3)}|N^{2\epsilon_0} \right) \leq |Z| \frac{U^2}{\overline{K}Q_1} \left( Q^{1+\epsilon_0} + |\Omega^{(3)}|N^{2\epsilon_0} \right). \quad (5.37)$$

The lemma is proved. □

**COROLLARY 5.1.** *Under the hypotheses of Lemma 5.4, one has*

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \ll |\Omega_N|^2 Q^{\epsilon_0} \left( \frac{N^{2\gamma+8\epsilon_0}}{\overline{K}Q_1} + N^{3\gamma-1+10\epsilon_0} \frac{(\overline{K}Q)^{1-\gamma}}{\overline{K}} \right), \quad (5.38)$$

**PROOF.** Using Lemma 5.4 and Lemma 4.3 with  $W = P_{Q_1, Q}^{(\beta)}$ ,  $f(\theta) = |S_N(\theta)|$  we obtain (5.38). This completes the proof of the corollary. □

## 6. Replacing integrals by sums

The purpose of the following arguments is a slight modification of the results of Sec. 4. It follows from the statement of Lemma 7.1 that we need to know how to estimate the following expression

$$\frac{1}{N} \sum_{0 \leq a \leq q \leq X}^* \int_{|K| \leq Y} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK, \quad (6.1)$$

where  $Y$  may depend on  $q$ . The following arguments are similar to [9, Lemma 26, p. 145]. Let us take a sufficiently large number  $T$ , then

$$\int_{|K| \leq Y} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \leq \sum_{|l| \leq TY} \int_{l/T}^{(l+1)/T} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK. \quad (6.2)$$

Hence  $K = \frac{l}{T} + \lambda$ ,  $\theta = \frac{a}{q} + \frac{K}{N} = \frac{a}{q} + \frac{l}{TN} + \frac{\lambda}{N}$ , and

$$\left| S_N(\theta) - S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right| \leq \lambda |\Omega_N| \implies |S_N(\theta)|^2 \leq \left| S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right|^2 + \lambda^2 |\Omega_N|^2. \quad (6.3)$$

So, one has

$$\begin{aligned} \frac{1}{N} \sum_{0 \leq a \leq q \leq X}^* \int_{|K| \leq Y} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK &\leq \\ &\leq \frac{1}{N} \sum_{0 \leq a \leq q \leq X}^* \sum_{|l| \leq TY} \left( \frac{1}{T} \left| S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right|^2 + \frac{|\Omega_N|^2}{T^3} \right). \end{aligned} \quad (6.4)$$

Choosing  $T$  sufficiently large, we obtain that the investigation of the expression of the form (6.1) reduces to the investigation of the quantity

$$\frac{1}{TN} \sum_{0 \leq a \leq q \leq X}^* \sum_{|l| \leq TY} \left| S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right|^2. \quad (6.5)$$

Our next purpose is to modify Lemma 4.1 and Lemma 4.2. Let

$$P_{Q_1, Q}^{\lambda_1, \lambda} = \left\{ \theta = \frac{a}{q} + \frac{l}{TN} \mid (a, q) = 1, 0 \leq a \leq q, Q_1 \leq q \leq Q, \lambda_1 \leq |l| \leq \lambda \right\}, \quad (6.6)$$

and  $\Lambda_1 = \frac{\lambda_1}{T}$  and  $\Lambda = \frac{\lambda}{T}$ . It should be mentioned that  $\Lambda$  is similar to  $\bar{K}$  and that  $|P_{Q_1, Q}^{\lambda_1, \lambda}| \ll \lambda Q^2 = \Lambda T Q^2$ . Let  $Z \subseteq P_{Q_1, Q}^{\lambda_1, \lambda}$  we denote

$$\mathfrak{N}^* = \left\{ (g_3^{(1)}, g_3^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(3)} \times \tilde{\Omega}^{(3)} \times Z^2 \mid (6.8) \text{ and } (6.9) \text{ hold} \right\}, \quad (6.7)$$

where

$$\left\| g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \leq \frac{74A^2\Lambda}{M^{(1)}}, \quad (6.8)$$

$$\left| g_3^{(1)} \frac{l_1}{TN} - g_3^{(2)} \frac{l_2}{TN} \right|_{1,2} \leq \min \left\{ \frac{73A^2\Lambda}{M^{(1)}}, \frac{73A^2}{M^{(1)}} + \left\| g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \right\}, \quad (6.9)$$

$$\mathfrak{M}^* = \left\{ (g_3^{(1)}, g_3^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(3)} \times \tilde{\Omega}^{(3)} \times Z^2 \mid (6.11) \text{ and } (6.12) \text{ hold} \right\}, \quad (6.10)$$

where

$$\left| g_3^{(1)} \frac{l_1}{TN} - g_3^{(2)} \frac{l_2}{TN} \right|_{1,2} \leq \frac{1}{M^{(1)}}, \quad (6.11)$$

$$\left\| g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} = 0. \quad (6.12)$$

The following lemma can be proved in the same manner as Lemma 4.1.

LEMMA 6.1. *For  $Z \subseteq P_{Q_1, Q}^{\lambda_1, \lambda}$  and for  $M^{(1)}$  in (3.10) such that  $M^{(1)} > 146A^2\Lambda$ , the following inequality holds*

$$\sigma_{N,Z} \ll H_1 |\Omega^{(1)}|^{1/2} |\mathfrak{Y}^*|^{1/2}. \quad (6.13)$$

The following lemma can be proved in the same manner as Lemma 4.2.

LEMMA 6.2. *Let  $Z \subseteq P_{Q_1, Q}^{\lambda_1, \lambda}$  and let ensemble  $\Omega_N$  satisfy the hypotheses of Lemma 4.2. Let  $M^{(1)}$  be such that for any  $\theta^{(1)}, \theta^{(2)} \in Z$  the following inequality holds*

$$M^{(1)} > 76A^2 \left[ q^{(1)}, q^{(2)} \right] \Lambda. \quad (6.14)$$

Then the following bound holds

$$\sigma_{N,Z} \ll H_1 |\Omega^{(1)}|^{1/2} |\mathfrak{M}^*|^{1/2}. \quad (6.15)$$

For any  $q$  in  $Q_1 \leq q \leq Q$  we define by any means the number  $a_q$ , such that  $(a_q, q) = 1$ ,  $0 \leq a_q \leq q$ . Let us denote

$$Z^* = \left\{ \theta = \frac{a_q}{q} + \frac{l}{TN} \mid Q_1 \leq q \leq Q, \lambda_1 \leq |l| \leq \lambda \right\}, \quad (6.16)$$

then  $|Z^*| \leq \Lambda T Q$ . The following statements are similar to Lemma 5.1 and Lemma 5.2.

LEMMA 6.3. *If  $\Lambda^{3/2} Q^3 \leq N^{1-2\epsilon_0}$ ,  $\Lambda \geq Q^{10\epsilon_0}$ , and  $\Lambda Q \geq Q_0$ , then for any  $Z \subseteq Z^*$  then the following bound holds*

$$\sigma_{N,Z} \ll |Z|^{1/2} T^{1/2} |\Omega_N| \frac{(\Lambda^{3/2} Q^3)^\gamma}{\Lambda^{1/2} Q} (\Lambda^{15} Q^{22})^{\epsilon_0} \frac{Q \Lambda^{\delta/2}}{Q_1 \Lambda_1^{\delta/2}}. \quad (6.17)$$

PROOF. We put

$$M^{(1)} = 76A^2 \Lambda Q^2, \quad M^{(4)} = \Lambda_1^{1/2} (M^{(1)})^{\epsilon_0}, \quad M^{(6)} = Q. \quad (6.18)$$

It follows from the statement of the lemma that all conditions of the Theorem 3.2 and Lemma 6.2 hold. We prove that

$$|\mathfrak{M}^*| \ll |Z| T(M^{(1)})^{2\epsilon_0} |\Omega^{(3)}| \left( (M^{(4)})^{4\epsilon_0} (M^{(1)})^{10\epsilon_0} \right) |\Omega^{(5)}| \left( \frac{(M^{(6)})^{1+2\epsilon_0}}{Q_1} + 1 \right)^2. \quad (6.19)$$

In the same manner as in Lemma 5.1 one can prove that  $q^{(1)} = (q^{(1)}, q^{(2)}) = q^{(2)} = \mathbf{q}$ . We fix  $\theta_1$  (thus  $q_1 = \mathbf{q}$  and  $l_1$  are fixed) for which there are  $|Z|$  choices, this gives the first factor in (6.19). Then it follows from the conditions on the set  $Z$  that  $a^{(1)} = a^{(2)}$  and the congruence (6.12) can be simplified to (5.6). Using the first inequality in (6.11) and the lower bound (3.20) we obtain that there are less than  $T(M^{(1)})^{2\epsilon_0}$  choices for  $l_2$  (if  $g_3^{(1)}, g_3^{(2)}$ , and  $l_1$  have been already fixed). This gives the second factor in (6.19). We fix the vector  $g_3^{(1)}$  for which there are  $|\Omega^{(3)}|$  choices. This gives the third factor in (6.19). Using Theorem 3.2, we can write the second vector in the following form  $g_3^{(2)} = \gamma_4 \gamma_5 \gamma_6 (0, 1)^t$ , where  $\gamma_i \in \Omega^{(i)}$ ,  $i = 4, 5, 6$ . Let  $g_3^{(1)} = (x_1, x_2)^t$  and  $g_3^{(2)} = (y_1, y_2)^t$ . It follows from (6.11) that

$$\left| \frac{x_1}{x_2} - \frac{y_1}{y_2} \right| \ll \frac{(M^{(1)})^{2\epsilon_0}}{\Lambda_1}. \quad (6.20)$$

In the same manner as in Lemma 5.1 we obtain that the number of matrices  $\gamma_4$  is less than  $(M^{(4)})^{4\epsilon_0} (M^{(1)})^{10\epsilon_0}$ . This is the fourth factor in (6.19). We fix the matrix  $\gamma_5$  for which there are  $|\Omega^{(5)}|$  choices. This gives the fifth factor in (6.19). Let  $\gamma_6(0, 1)^t = (x, y)^t$ , then (5.6) can be written in the form

$$x_1 \equiv (ax + by) \pmod{\mathbf{q}}, \quad x_2 \equiv (cx + dy) \pmod{\mathbf{q}},$$

where  $g_3^{(1)} = (x_1, x_2)^t$ , and  $\gamma_4 \gamma_5 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  have been already fixed. Using (3.23),

we obtain that for  $x$  and for  $y$  there are less than  $\left( \frac{(M^{(6)})^{1+2\epsilon_0}}{\mathbf{q}} + 1 \right)$  choices.

This gives the sixth factor in (6.19). Thus the estimate (6.19) is proved. Substituting (6.19) into (6.15) and applying (6.18), we obtain

$$\sigma_{N,Z} \ll |Z|^{1/2} T^{1/2} |\Omega^{(1)}|^{1/2} |\Omega^{(3)}|^{1/2} |\Omega^{(5)}|^{1/2} (M^{(1)})^{1+7\epsilon_0} (M^{(4)})^{2\epsilon_0} \frac{(M^{(6)})^{1+2\epsilon_0}}{Q_1}.$$

Using (5.13) and (3.15), one has

$$\sigma_{N,Z} \ll |Z|^{1/2} T^{1/2} |\Omega_N| \frac{(M^{(1)})^{1+7\epsilon_0} (M^{(6)})^{1+2\epsilon_0} (M^{(4)})^{2\epsilon_0}}{(M^{(1)} M^{(4)} M^{(6)})^{\delta-\epsilon_0/2} Q_1} \tag{6.21}$$

Substituting  $M^{(i)}$  from (6.18) into (6.21), we have

$$\sigma_{N,Z} \ll |Z|^{1/2} T^{1/2} |\Omega_N| \Lambda^{3\gamma/2-1/2+15\epsilon_0} Q^{3\gamma-1+22\epsilon_0} \frac{Q \Lambda^\delta}{Q_1 \Lambda_1^\delta}.$$

The lemma is proved. □

LEMMA 6.4. *If  $\Lambda^{3/2} Q^3 \leq N^{1-2\epsilon_0}$ ,  $\Lambda \geq Q^{10\epsilon_0}$ , and  $\Lambda Q \geq Q_0$ , then the following estimate holds*

$$\sum_{\theta \in P_{Q_1, Q}^{\lambda_1, \lambda}} |S_N(\theta)|^2 \ll |\Omega_N|^2 T \frac{(\Lambda^3 Q^6)^\gamma}{\Lambda Q} (\Lambda^{30} Q^{44})^{\epsilon_0} \frac{Q^2 \Lambda^\delta}{Q_1 \Lambda_1^\delta}. \tag{6.22}$$

PROOF. Applying Lemma 4.3 with  $W = Z^*$ , we obtain from (6.17) that

$$\sum_{\theta \in Z^*} |S_N(\theta)|^2 \ll |\Omega_N|^2 T \Lambda^{3\gamma-1+30\epsilon_0} Q^{6\gamma-2+44\epsilon_0} \frac{Q^2 \Lambda^\delta}{Q_1 \Lambda_1^\delta}.$$

Using the trivial bound (5.2), we obtain the desired estimate (6.22). The lemma is proved. □

The following statement is similar to Lemma 5.4.

LEMMA 6.5. *Let the following inequalities hold  $Q_0 \leq Q \leq N^{1/2-10\epsilon_0}$ ,  $\Lambda Q \leq N^{1/2+\epsilon_0}$ . Then for any  $Z \subseteq P_{Q_1, Q}^{\lambda_1, \lambda}$  the following bound holds*

$$\sum_{\theta \in Z} |S_N(\theta)| \ll |\Omega_N| |Z|^{1/2} T^{1/2} \left( \frac{N^{\gamma+2\epsilon_0} (\Lambda Q)^{2\epsilon_0}}{(\Lambda_1 Q_1)^{1/2}} + N^{\gamma-1/2+2\epsilon_0} (\Lambda Q)^{1/2+2\epsilon_0} + \right)$$

$$+ N^{3\gamma/2-1/4+4\epsilon_0} \frac{(\Lambda Q)^{1/4-\gamma/2+3\epsilon_0} Q^{\epsilon_0}}{Q_1^{1/2}} + N^{3\gamma/2-3/4+4\epsilon_0} (\Lambda Q)^{3/4-\gamma/2+3\epsilon_0} Q^{1/2+\epsilon_0} \Big). \quad (6.23)$$

PROOF. We use the partition of the ensemble  $\Omega_N$  given by Theorem 3.1. We put  $Z \subseteq P_{Q_1, Q}^{\lambda_1, \lambda}$  and

$$M^{(1)} = 150A^2 N^{1/2} (\Lambda Q)^{1/2}, \quad U = \frac{73A^2 N}{M^{(1)}}. \quad (6.24)$$

Then  $M^{(1)} \geq 150A^2 \Lambda$  and conditions of Lemma 6.1 hold. Therefore, inequality (6.13) holds. Let

$$g_3^{(1)} = (x_1, x_2)^t, \quad g_3^{(2)} = (y_1, y_2)^t, \quad \mathcal{Y} = x_1 y_2 - y_1 x_2.$$

In the same way as in [2, Lemma 3.10] we obtain that  $\mathcal{Y} \equiv 0 \pmod{\mathfrak{q}}$ , where  $\mathfrak{q} = [q^{(1)}, q^{(2)}]$ . The set  $\mathfrak{N}^*$  can be represented as a union of the sets  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ . For the first set  $\mathcal{Y} = 0$ , for the second one  $\mathcal{Y} \neq 0$ . To prove the estimate (6.23) we use the following bounds

$$|\mathfrak{M}_1| \ll |Z| T (M^{(1)})^{2\epsilon_0} |\Omega^{(3)}|, \quad (6.25)$$

$$|\mathfrak{M}_2| \ll_\epsilon |Z| T (M^{(1)})^{2\epsilon_0} \left( \frac{|\Omega^{(3)}| U^2}{\Lambda_1 Q_1} + \frac{U^3 Q^\epsilon}{Q_1} + U Q^{1+\epsilon} \right), \quad (6.26)$$

which will be proved below in Lemma 6.7 and 6.8 respectively. Hence,

$$|\mathfrak{N}^*|^{1/2} \ll |Z|^{1/2} T^{1/2} (M^{(1)})^{\epsilon_0} \left( \frac{|\Omega^{(3)}|^{1/2} U}{(\Lambda_1 Q_1)^{1/2}} + |\Omega^{(3)}|^{1/2} + \frac{U^{3/2} Q^{\epsilon_0}}{Q_1^{1/2}} + U^{1/2} Q^{1/2+\epsilon_0} \right). \quad (6.27)$$

Substituting (6.27) into (6.13), we obtain

$$\begin{aligned} \sigma_{N, Z} &\ll |Z|^{1/2} T^{1/2} (M^{(1)})^{1+3\epsilon_0} |\Omega^{(1)}|^{1/2} \times \\ &\times \left( \frac{|\Omega^{(3)}|^{1/2} U}{(\Lambda_1 Q_1)^{1/2}} + |\Omega^{(3)}|^{1/2} + \frac{U^{3/2} Q^{\epsilon_0}}{Q_1^{1/2}} + U^{1/2} Q^{1/2+\epsilon_0} \right). \end{aligned}$$

Using the bounds (5.24) and (5.25) and the definition of  $M^{(1)}$  and  $M^{(3)}$ , we have

$$\begin{aligned} \sigma_{N,Z} \ll & |\Omega_N| |Z|^{1/2} T^{1/2} \left( \frac{N^{\gamma+2\epsilon_0} (\Lambda Q)^{2\epsilon_0}}{(\Lambda_1 Q_1)^{1/2}} + N^{\gamma-1/2+2\epsilon_0} (\Lambda Q)^{1/2+2\epsilon_0} + \right. \\ & \left. + N^{3\gamma/2-1/4+4\epsilon_0} \frac{(\Lambda Q)^{1/4-\gamma/2+3\epsilon_0} Q^{\epsilon_0}}{Q_1^{1/2}} + N^{3\gamma/2-3/4+4\epsilon_0} (\Lambda Q)^{3/4-\gamma/2+3\epsilon_0} Q^{1/2+\epsilon_0} \right). \end{aligned}$$

The lemma is proved. □

To prove Lemma 6.7 we use the following statement.

LEMMA 6.6 ([2, Lemma 3.11]). *Let*

$$1 \leq a_i \leq q_i \leq Q, (a_i, q_i) = 1, i = 1, 2; \quad \frac{a_1}{q_1} \neq \frac{a_2}{q_2},$$

*then for  $R > \max \left\{ \frac{Q^2}{Y_1}, 4Y, 2Q \right\}$  there are no coprime numbers  $y_1$  and  $y_2$ , satisfying the following conditions*

$$\left\| y_i \left( \frac{a_1}{q_1} - \frac{a_2}{q_2} \right) \right\| < \frac{1}{R}, \quad 0 < Y_1 \leq y_i \leq Y, \quad i = 1, 2. \tag{6.28}$$

LEMMA 6.7. *Under the hypotheses of Lemma 6.5 one has*

$$|\mathfrak{M}_1| \ll |Z| T (M^{(1)})^{2\epsilon_0} |\Omega^{(3)}|. \tag{6.29}$$

PROOF. It follows from the condition  $\mathcal{Y} = 0$  that  $\frac{x_1}{x_2} = \frac{y_1}{y_2}$ . Since

$$g_3^{(1)} = (x_1, x_2)^t, \quad g_3^{(2)} = (y_1, y_2)^t,$$

we have  $(x_1, x_2) = 1$  and  $(y_1, y_2) = 1$  and hence  $x_1 = y_1$  and  $x_2 = y_2$ . Then it follows from (6.8) that

$$\left\| y_1 \left( \frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} \right) \right\| < \frac{1}{R} \tag{6.30}$$

and similarly, one has  $\left\| y_2 \left( \frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} \right) \right\| < \frac{1}{R}$ . It follows from (6.9) that

$$\left| y_{1,2} \frac{l_1 - l_2}{TN} \right| \leq \frac{73A^2}{M^{(1)}} + \left\| y_{1,2} \left( \frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} \right) \right\|.$$

Thus we obtain

$$|\mathfrak{M}_1| \leq \sum_{g_3^{(2)} \in \tilde{\Omega}^{(3)}} \sum_{\theta^{(1)} \in Z} \sum_{\theta^{(2)} \in Z} \mathbf{1} \left\{ \left\| y_{1,2} \left( \frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} \right) \right\| < \frac{1}{R} \right\} \mathbf{1} \left\{ \left| y_{1,2} \frac{l_1 - l_2}{T} \right| \leq U + N \left| y_{1,2} \left( \frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} \right) \right| \right\}. \quad (6.31)$$

Let us prove that all conditions of Lemma 6.6 hold. It follows from Theorem 3.1 that

$$Y_1 \leq y_i \leq Y, \text{ where } Y_1 = \frac{N}{150A^2(M^{(1)})^{1+2\epsilon_0}}, \quad Y = \frac{73A^2N}{M^{(1)}}.$$

Under the hypotheses of Lemma 6.5 one has  $R > \max \left\{ \frac{Q^2}{Y_1}, 4Y, 2Q \right\}$ , and so we can apply Lemma 6.6. We obtain that in (6.31) only summands corresponding to  $\frac{a^{(1)}}{q^{(1)}} = \frac{a^{(2)}}{q^{(2)}}$  are nonzero. We fix  $\theta^{(1)}$  for which there are  $Z$  choices. Thus  $\frac{a^{(1)}}{q^{(1)}} = \frac{a^{(2)}}{q^{(2)}}$  and  $l_1$  are fixed. Using the inequality  $\left| y_{1,2} \frac{l_1 - l_2}{T} \right| \leq U$  and the lower bound from (3.20) we obtain that for  $l_2$  there are less than  $T(M^{(1)})^{2\epsilon_0}$  choices. Hence  $|\mathfrak{M}_1| \ll |Z|T(M^{(1)})^{2\epsilon_0}|\Omega^{(3)}|$ . The lemma is proved.  $\square$

LEMMA 6.8. *Under the hypotheses of Lemma 6.5 one has*

$$|\mathfrak{M}_2| \ll_\epsilon |Z|T(M^{(1)})^{2\epsilon_0} \left( \frac{|\Omega^{(3)}|U^2}{\Lambda_1 Q_1} + \frac{U^3 Q^\epsilon}{Q_1} + UQ^{1+\epsilon} \right). \quad (6.32)$$

PROOF. In the same way as in Lemma 5.5 we obtain (5.27). Then it follows from (6.9) that

$$\left| g_3^{(1)} \frac{l_1}{TN} - g_3^{(2)} \frac{l_2}{TN} \right|_{1,2} \leq \frac{73A^2N}{M^{(1)}}. \quad (6.33)$$

We fix  $\theta_1$  (that is  $q_1 = \mathbf{q}$  and  $l_1$  are fixed) for which there are  $|Z|$  choices. Using the first inequality (6.33) and the lower bound (3.20) we obtain that  $(g_3^{(1)}, g_3^{(2)})$  and  $l_1$  are fixed) there are less than  $T(M^{(1)})^{2\epsilon_0}$  choices for  $l_2$ . As in Lemma 5.5 we obtain that  $a^{(2)}$  is determined uniquely by (5.27). Hence,

$$|\mathfrak{M}_2| \leq |Z|T(M^{(1)})^{2\epsilon_0} \sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{g_3^{(2)} \in \Omega^{(3)}} \mathbf{1}_{\{x_1 y_2 \equiv x_2 y_1 \pmod{\mathbf{q}}\}} \quad (6.34)$$

$$\left| \frac{l_1}{x_2 T N^{-y_2}} - \frac{l_2}{T N} \right| \leq \frac{73A^2}{M^{(1)}}$$

To simplify we put  $U = \frac{73A^2 N}{M^{(1)}}$ . Let

$$\delta_1 = (x_1, q), \delta_2 = (x_2, q), x_3 = \frac{x_1}{\delta_1}, x_4 = \frac{x_2}{\delta_2}, y_3 = \frac{y_1}{\delta_1}, y_4 = \frac{y_2}{\delta_2}, p = \frac{\mathbf{q}}{\delta_1 \delta_2},$$

then one has

$$|\mathfrak{M}_2| \leq |Z|T(M^{(1)})^{2\epsilon_0} \sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{y_3 \leq \frac{U}{\delta_1}} \sum_{y_4 \leq \frac{U}{\delta_2}} \mathbf{1}_{\{x_3 y_4 \equiv x_4 y_3 \pmod{p}\}} \cdot \quad (6.35)$$

$$\left| x_4 \frac{l_1}{T} - y_4 \frac{l_2}{T} \right| \leq \frac{U}{\delta_2}$$

Since  $(x_3, p) = (x_4, p) = 1$ , the congruence (6.35) can be written in the form  $y_4 \equiv cy_3 \pmod{p}$ , where  $c \equiv x_3^{-1} x_4 \pmod{p}$ . Then using the function  $\delta_p(a)$

$$\delta_p(a) = \frac{1}{p} \sum_{x=1}^p e\left(\frac{ax}{p}\right) = \begin{cases} 1, & \text{if } a \equiv 0 \pmod{p}; \\ 0, & \text{else,} \end{cases}$$

we obtain

$$|\mathfrak{M}_2| \leq |Z|T(M^{(1)})^{2\epsilon_0} \sum_{g_3^{(1)} \in \Omega^{(3)}} \frac{1}{p} \sum_{k=1}^p \sum_{y_3 \leq \frac{U}{\delta_1}} \sum_{y_4 \leq \frac{U}{\delta_2}} e\left(k \frac{x_3 y_4 - x_4 y_3}{p}\right).$$

$$\left| x_4 \frac{l_1}{T} - y_4 \frac{l_2}{T} \right| \leq \frac{U}{\delta_2}$$

In the same way as in [9, p.18], we obtain

$$\begin{aligned} \frac{1}{p} \sum_{k=1}^p \sum_{y_3 \leq \frac{U}{\delta_1}} \sum_{\substack{y_4 \leq \frac{U}{\delta_2} \\ \left| x_4 \frac{l_1}{T} - y_4 \frac{l_2}{T} \right| \leq \frac{U}{\delta_2}}} e \left( k \frac{x_3 y_4 - x_4 y_3}{p} \right) &\leq \\ &\leq \frac{1}{p} \left( \frac{U}{\delta_1} + 1 \right) \left( \frac{U}{\delta_2 \Lambda_1} + 1 \right) + O \left( s \left( \frac{c}{p} \right) \log^2 p \right), \end{aligned} \quad (6.36)$$

where  $s(\alpha) = \sum_{1 \leq i \leq s} a_i$  is the sum of partial quotients of the number  $\alpha = [a_1, \dots, a_s]$ .

Substituting (6.36) into (6.35), we have

$$\begin{aligned} |\mathfrak{M}_2| \leq |Z|T(M^{(1)})^{2\epsilon_0} &\left( \sum_{g_3^{(1)} \in \Omega^{(3)}} \frac{1}{q} \left( \frac{U^2}{\Lambda_1} + U\delta_2 + \frac{U\delta_1}{\Lambda_1} + \delta_1\delta_2 \right) + \right. \\ &\left. + \log^2 Q \sum_{x_1 \leq U} \sum_{x_2 \leq U} s \left( \frac{x_3^{-1} x_2}{q/\delta_1} \right) \right). \end{aligned} \quad (6.37)$$

Using the following result of Knuth and Yao [10],

$$\sum_{a \leq b} s(a/b) \ll b \log^2 b,$$

we have

$$\sum_{x_1 \leq U} \sum_{x_2 \leq U} s \left( \frac{x_3^{-1} x_2}{q/\delta_1} \right) \leq \sum_{x_1 \leq U} \left( \frac{U}{q/\delta_1} + 1 \right) \frac{q}{\delta_1} \log^2 q \leq (U^2 + Uq) \log^2 q. \quad (6.38)$$

One has

$$\sum_{x_1 \leq U} \delta_1 \ll (U + q) q^{\epsilon_0}. \quad (6.39)$$

Substituting (6.38) and (6.39) into (6.37) we obtain

$$|\mathfrak{M}_2| \ll_{\epsilon} |Z|T(M^{(1)})^{2\epsilon_0} \left( \frac{|\Omega^{(3)}|U^2}{\Lambda_1 Q_1} + \frac{U^3 Q^{\epsilon}}{Q_1} + UQ^{1+\epsilon} \right).$$

The lemma is proved. □

Applying Lemmas 6.5 and 4.3 with  $W = P_{Q_1, Q}^{\lambda_1, \lambda}$ , we obtain

LEMMA 6.9. *Let  $N^{\epsilon_0} \leq Q \leq N^{1/2-10\epsilon_0}$ ,  $\Lambda Q \leq N^{1/2+\epsilon_0}$ , then then the following bound holds*

$$\begin{aligned} \sum_{\theta \in P_{Q_1, Q}^{\lambda_1, \lambda}} |S_N(\theta)|^2 &\ll |\Omega_N|^2 T \left( \frac{N^{2\gamma+4\epsilon_0} (\Lambda Q)^{4\epsilon_0}}{\Lambda_1 Q_1} + N^{2\gamma-1+4\epsilon_0} (\Lambda Q)^{1+4\epsilon_0} + \right. \\ &\left. + N^{3\gamma-1/2+8\epsilon_0} \frac{(\Lambda Q)^{1/2-\gamma+6\epsilon_0} Q^{2\epsilon_0}}{Q_1} + N^{3\gamma-3/2+8\epsilon_0} (\Lambda Q)^{3/2-\gamma+6\epsilon_0} Q^{1+2\epsilon_0} \right). \end{aligned} \quad (6.40)$$

### 7. Estimates for integrals of $|S_N(\theta)|^2$

We recall that the sequence  $\{N_j\}_{j=-J-1}^{J+1}$  was defined in (3.5) and the quantities  $N_{1/2}$  and  $N_{1/2}^*$  were defined in (4.4). Note that  $N_0 < N_{1/2} < N_1$ . Using the Dirichlet’s theorem in the form (4.5) we obtain the following statement

LEMMA 7.1 ([2, Lemma 5.1]). *The following inequality holds*

$$\int_0^1 |S_N(\theta)|^2 d\theta \leq \frac{1}{N} \sum_{0 \leq a \leq q \leq N_{1/2}}^* \int_{\substack{|K| \leq \frac{N_{1/2}^*}{q}}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK, \quad (7.1)$$

where  $\sum^*$  means that the sum is taken over  $a$  and  $q$  being coprime for  $q \geq 1$ , and  $a = 0, 1$  for  $q = 1$ .

Recall that

$$Q_0 = \max \left\{ \exp \left( \frac{10^5 A^4}{\epsilon_0^2} \right), \exp(\epsilon_0^{-5}) \right\}, \quad \gamma = 1 - \delta.$$

LEMMA 7.2 ([2, Lemma 5.2]). *The following inequality holds*

$$\int_0^1 |S_N(\theta)|^2 d\theta \leq 2Q_0^2 \frac{|\Omega_N|^2}{N} + \frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq N_{1/2} \\ q > Q_0}}^* \int_{\substack{|K| \leq \frac{N_{1/2}^*}{q}}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK +$$

$$+ \frac{1}{N} \sum_{0 \leq a \leq q \leq Q_0}^* \int_{\substack{Q_0 \\ q} \leq |K| \leq \frac{N_{1/2}^*}{q}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK. \quad (7.2)$$

First we estimate the second integral in the right side of (7.2). It is convenient to use the following notation

$$\xi_1 = N^{2\gamma+10\epsilon_0}. \quad (7.3)$$

LEMMA 7.3. For  $\gamma < \frac{1}{5} - 7\epsilon_0$  and  $\epsilon_0 \in (0, 1/2500)$  the following inequality holds

$$\frac{1}{N} \sum_{0 \leq a \leq q \leq Q_0}^* \int_{\substack{Q_0 \\ q} \leq |K| \leq \xi_1} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (7.4)$$

PROOF. We denote by  $I$  the integral in the left side of (7.4). Applying Lemma 5.2 we obtain

$$I \ll |\Omega_N|^2 Q_0^{6\gamma+1+42\epsilon_0} \int_{\substack{Q_0 \\ q} \leq |K| \leq \xi_1} \overline{K}^{5\gamma-2+30\epsilon_0} dK \ll |\Omega_N|^2, \quad (7.5)$$

by the choice of the parameter  $\gamma$ . Summing (7.5) over  $0 \leq a \leq q \leq Q_0$ , we obtain inequality (7.4). The lemma is proved.  $\square$

The integral

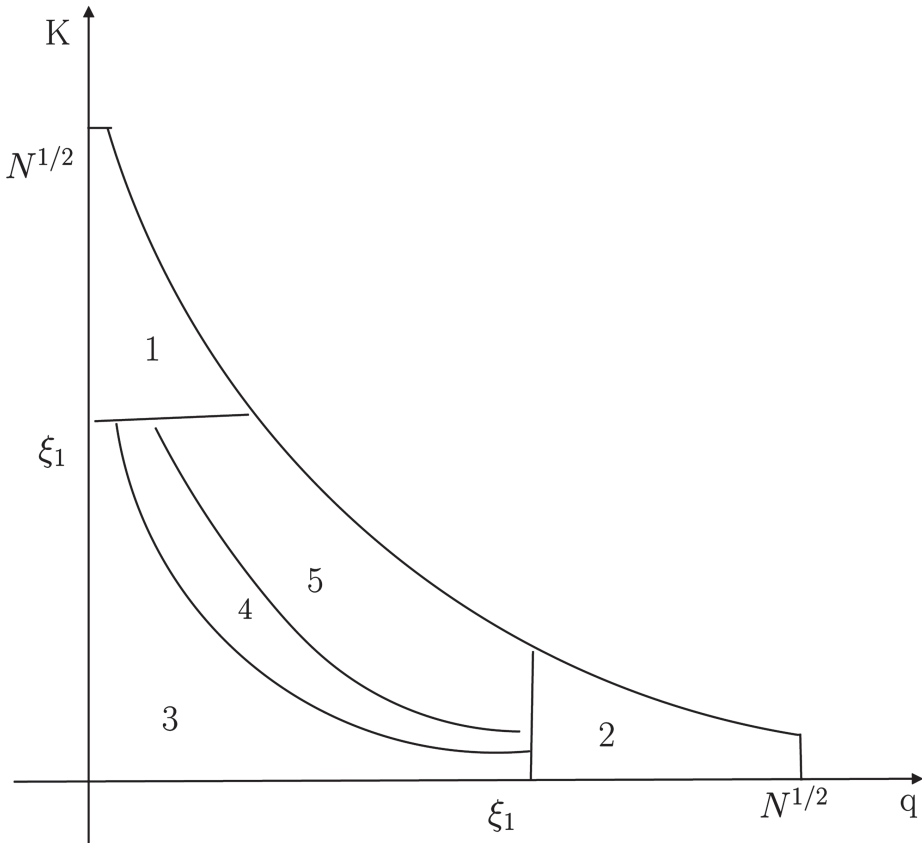
$$\frac{1}{N} \sum_{0 \leq a \leq q \leq Q_0}^* \int_{\substack{N_{1/2}^* \\ \xi_1} \leq |K| \leq \frac{N_{1/2}^*}{q}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK. \quad (7.6)$$

will be estimated later in Lemma 7.6.

It remains to estimate the first integral in the right side of (7.2), that is,

$$\frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq N_{1/2} \\ q > Q_0}}^* \int_{\substack{Q_0 \\ q} \leq |K| \leq \frac{N_{1/2}^*}{q}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK. \quad (7.7)$$

The following lemmas will be devoted to this. We partition the range of summation and integration over  $q, K$  into five subareas:



**Fig. 1.** Decomposition on subdomains

Lemma 7.6 corresponds to the domain 1, Lemma 7.7 corresponds to the domain 2, Lemma 7.8 corresponds to the domain 3, Lemma 7.9 corresponds to the domain 4, and Lemma 7.10 corresponds to the domain 5. Let

$$c_1 = c_1(N), c_2 = c_2(N), Q_0 \leq c_1 < c_2 \leq N^{1/2},$$

$$f_1 = f_1(N, q), f_2 = f_2(N, q), f_1 < f_2 \leq \frac{N^{1/2}}{q},$$

$$m_1 = \min\{f_1(N, N_j), f_1(N, N_{j+1})\}, m_2 = \max\{f_2(N, N_j), f_2(N, N_{j+1})\}.$$

LEMMA 7.4. *If the functions  $f_1(N, q)$  and  $f_2(N, q)$  are monotonic for  $q$ , then the following inequality holds*

$$\begin{aligned} & \sum_{\substack{c_1 \leq q \leq c_2 \\ 1 \leq a \leq q}}^* \int_{f_1 \leq |K| \leq f_2} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \leq \\ & \leq \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK. \end{aligned} \quad (7.8)$$

PROOF. The interval  $[c_1, c_2]$  can be covered by the intervals  $[N_j, N_{j+1}]$ , then

$$\sum_{c_1 \leq q \leq c_2} \leq \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \sum_{N_j \leq q \leq N_{j+1}}.$$

Interchanging the order of summation over  $q$  and integration over  $K$ , we obtain inequality (7.8). This completes the proof of the lemma.  $\square$

Using (6.4), we in the same way as in Lemma 7.4 obtain the following statement.

LEMMA 7.5. *If the functions  $f_1(N, q)$  and  $f_2(N, q)$  are monotonic for  $q$  and  $m_1 \geq Q_0$ , then the following inequality holds*

$$\begin{aligned} & \sum_{\substack{c_1 \leq q \leq c_2 \\ 1 \leq a \leq q}}^* \int_{f_1 \leq |K| \leq f_2} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \\ & \ll \frac{1}{T} \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \sum_{i: m_1^{1-\epsilon_0} \leq N_i \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \sum_{TN_i \leq |l| \leq TN_{i+1}} \left| S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right|^2. \end{aligned} \quad (7.9)$$

To simplify we denote

$$Q_1 = N_j, \quad Q = N_{j+1}, \quad \lambda_1 = TN_i, \quad \lambda = TN_{i+1}, \quad \Lambda_1 = N_i, \quad \Lambda = N_{i+1}.$$

Using the results of [2, Lemma 2.5] we obtain

$$\frac{Q}{Q_1} \leq Q^{\epsilon_0} \leq N^{\epsilon_0/2}, \quad c_1^{1-\epsilon_0} \leq Q_1 \leq c_2, \quad c_1 \leq Q \leq c_2^{1+2\epsilon_0},$$

$$\frac{\Lambda}{\Lambda_1} \leq \Lambda^{\epsilon_0} \leq N^{\epsilon_0/2}, \quad m_1^{1-\epsilon_0} \leq \Lambda_1 \leq m_2, \quad m_1 \leq \Lambda \leq m_2^{1+2\epsilon_0}$$

Further, we will use this bounds without reference to them.

LEMMA 7.6. *For  $\epsilon_0 \in (0, 1/2500)$  the following inequality holds*

$$\frac{1}{N} \sum_{1 \leq a \leq q \leq N_{1/2}}^* \int_{\xi_1 \leq |K| \leq \frac{N_{1/2}^*}{q}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \tag{7.10}$$

PROOF. We denote by  $I$  the integral in the left side of (7.10). Applying Lemma 5.3, we obtain

$$I \ll |\Omega_N|^2 \frac{N^{2\gamma+2\epsilon_0}}{q^{2-4\epsilon_0}} \int_{\xi_1 \leq |K| \leq \frac{N_{1/2}^*}{q}} \frac{dK}{K^{2-4\epsilon_0}} \ll |\Omega_N|^2 \frac{N^{2\gamma+2\epsilon_0}}{q^{2-4\epsilon_0} \xi_1^{1-4\epsilon_0}}. \tag{7.11}$$

Substituting (7.11) into the left side of (7.10), we have

$$\frac{1}{N} \sum_{1 \leq a \leq q \leq N_{1/2}}^* \int_{\xi_1 \leq |K| \leq \frac{N_{1/2}^*}{q}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N} \sum_{1 \leq q \leq N_{1/2}} \frac{N^{2\gamma+2\epsilon_0}}{q^{1-4\epsilon_0} \xi_1^{1-4\epsilon_0}}. \tag{7.12}$$

By the choice of the parameter  $\xi_1$ , we obtain

$$\sum_{1 \leq q \leq N_{1/2}} \frac{N^{2\gamma+2\epsilon_0}}{q^{1-4\epsilon_0} \xi_1^{1-4\epsilon_0}} \ll \frac{N^{2\gamma+4\epsilon_0}}{\xi_1^{1-4\epsilon_0}} \ll N^{-\epsilon_0/2} \ll 1. \tag{7.13}$$

Substituting (7.13) into (7.12), we obtain (7.10). This completes the proof of the lemma. □

LEMMA 7.7. *For  $\gamma \leq \frac{1}{5} - 4\epsilon_0$ ,  $\epsilon_0 \in (0, 1/2500)$  the following inequality holds*

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N_{1/2} \\ q > \xi_1}}^* \int_{|K| \leq \frac{N_{1/2}^*}{q}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \tag{7.14}$$

PROOF. We use Lemma 7.4 with

$$c_1 = \xi_1, c_2 = N_{1/2}, f_1 = 0, f_2 = \frac{N_{1/2}^*}{q}, m_1 = \frac{Q_0}{Q}, m_2 = \frac{N_{1/2}^*}{Q_1}.$$

We note that  $KQ \leq N_{1/2}^* \frac{Q}{Q_1} \leq N^{1/2+\epsilon_0/2}$ , thus, applying Corollary 5.1, we obtain

$$\sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 \ll |\Omega_N|^2 \left( \frac{N^{2\gamma+8.5\epsilon_0}}{\overline{K}Q_1} + N^{3\gamma-1+11\epsilon_0} \frac{(\overline{K}Q)^{1-\gamma}}{\overline{K}} \right). \quad (7.15)$$

Integrating over  $K$ , we have

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll |\Omega_N|^2 \left( \frac{N^{2\gamma+9\epsilon_0}}{Q_1} + N^{5/2\gamma-1/2+11\epsilon_0} \right).$$

Since  $Q_1 \geq \xi_1^{1-2\epsilon_0}$  and  $\gamma < \frac{1}{5} - 4\epsilon_0$ , we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll |\Omega_N|^2 N^{-0,1\epsilon_0}. \quad (7.16)$$

Since the number of summands in the sum over  $j$  is less than  $\log \log N$ , one has

$$\sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll |\Omega_N|^2.$$

The lemma is proved. □

LEMMA 7.8. For  $\gamma \leq \frac{1}{6} - 10\epsilon_0$ ,  $\epsilon_0 \in (0, 1/2500)$  the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{|K| \leq \frac{N^{2/5-2\epsilon_0}}{q^{6/5}}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (7.17)$$

PROOF. We use Lemma 7.4 with

$$c_1 = Q_0, c_2 = \xi_1, f_1 = 0, f_2 = \frac{N^{2/5-2\epsilon_0}}{q^{6/5}}, m_1 = \frac{Q_0}{Q}, m_2 = \frac{N^{2/5-2\epsilon_0}}{Q_1^{6/5}}.$$

Applying Lemma 5.2 one has

$$\sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 \ll |\Omega_N|^2 \overline{K}^{5\gamma-2+30\epsilon_0} Q^{6\gamma-1+42\epsilon_0} \frac{Q^2}{Q_1^2}. \tag{7.18}$$

Integrating over  $K$  we have

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll |\Omega_N|^2 Q^{6\gamma-1+42\epsilon_0} \frac{Q^2}{Q_1^2} \ll |\Omega_N|^2 Q^{6\gamma-1+44\epsilon_0}.$$

For the sum over  $j$  to be bounded by a constant, it is sufficient to have  $\gamma \leq \frac{1}{6} - 10\epsilon_0$ . The lemma is proved.  $\square$

LEMMA 7.9. For  $\gamma \leq \frac{1}{6} - 6\epsilon_0$ ,  $\epsilon_0 \in (0, 1/2500)$  the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > N^{(1-5\gamma)/3-9\epsilon_0}}}^* \int_{\substack{N^{2/5-2\epsilon_0} \\ q^{6/5}} \leq |K| \leq \frac{N^{2/3-3\epsilon_0}}{q^2}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \tag{7.19}$$

PROOF. We use Lemma 7.5 with  $c_1 = N^{(1-5\gamma)/3-9\epsilon_0}$ ,  $c_2 = \xi_1$ ,  $f_1 = \frac{N^{2/5-2\epsilon_0}}{q^{6/5}}$ ,  $f_2 = \frac{N^{2/3-3\epsilon_0}}{q^2}$ ,  $m_1 = \frac{N^{2/5-2\epsilon_0}}{Q^{6/5}}$ ,  $m_2 = \frac{N^{2/3-3\epsilon_0}}{Q_1^2}$ . Applying Lemma 6.4 we obtain

$$\begin{aligned} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \sum_{TN_i \leq |l| \leq TN_{i+1}} \left| S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right|^2 &\ll |\Omega_N|^2 T \Lambda^{3\gamma-1+30\epsilon_0} Q^{6\gamma-1+44\epsilon_0} \frac{Q^2 \Lambda^\delta}{Q_1^2 \Lambda_1^\delta} \ll \\ &\ll |\Omega_N|^2 \Lambda^{3\gamma-1+32\epsilon_0} Q^{6\gamma-1+46\epsilon_0}. \end{aligned}$$

Using Lemma 7.5 we have

$$\begin{aligned} & \sum_{\substack{c_1 \leq q \leq c_2 \\ 1 \leq a \leq q}}^* \int_{f_1 \leq |K| \leq f_2} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \\ & \ll \frac{1}{T} \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \sum_{i: m_1^{1-\epsilon_0} \leq N_i \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \sum_{TN_i \leq |l| \leq TN_{i+1}} \left| S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right|^2 \ll \\ & \ll |\Omega_N|^2 \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \sum_{i: m_1^{1-\epsilon_0} \leq N_i \leq m_2} \Lambda^{3\gamma-1+32\epsilon_0} Q^{6\gamma-1+46\epsilon_0}. \end{aligned}$$

For the sums over  $i$  and  $j$  to be bounded by a constant, it is sufficient to have  $\gamma \leq \frac{1}{6} - 6\epsilon_0$ . The lemma is proved.  $\square$

LEMMA 7.10. For  $\gamma \leq \frac{1}{6} - 10\epsilon_0$ ,  $\epsilon_0 \in (0, 1/2500)$  the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq \xi_1 \\ q > N^{(1-3\gamma)/3-7\epsilon_0}}}^* \int_{\frac{N^{2/3-3\epsilon_0}}{q^2} \leq |K| \leq \frac{N_{1/2}^*}{q}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \tag{7.20}$$

PROOF. We use Lemma 7.5 with

$$c_1 = N^{(1-3\gamma)/3-7\epsilon_0}, c_2 = \xi_1, f_1 = \frac{N^{2/3-3\epsilon_0}}{q^2}, f_2 = \frac{N_{1/2}^*}{q}, m_1 = \frac{N^{2/3-3\epsilon_0}}{Q^2}, m_2 = \frac{N_{1/2}^*}{Q_1}. \tag{7.21}$$

Applying Lemma 6.9, we obtain

$$\begin{aligned} & \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \sum_{TN_i \leq |l| \leq TN_{i+1}} \left| S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right|^2 \ll |\Omega_N|^2 T \left( \frac{N^{2\gamma+4\epsilon_0} (\Lambda Q)^{4\epsilon_0}}{\Lambda_1 Q_1} + N^{2\gamma-1+4\epsilon_0} \times \right. \\ & \left. \times (\Lambda Q)^{1+4\epsilon_0} + N^{3\gamma-1/2+8\epsilon_0} \frac{(\Lambda Q)^{1/2-\gamma+6\epsilon_0} Q^{2\epsilon_0}}{Q_1} + N^{3\gamma-3/2+8\epsilon_0} (\Lambda Q)^{3/2-\gamma+6\epsilon_0} Q^{1+2\epsilon_0} \right). \end{aligned}$$

By Lemma 7.5 we have

$$\begin{aligned} & \sum_{\substack{c_1 \leq q \leq c_2 \\ 1 \leq a \leq q}}^* \int_{f_1 \leq |K| \leq f_2} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \\ & \ll \frac{1}{T} \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \sum_{i: m_1^{1-\epsilon_0} \leq N_i \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \sum_{TN_i \leq |l| \leq TN_{i+1}} \left| S_N \left( \frac{a}{q} + \frac{l}{TN} \right) \right|^2 \ll \\ & \ll |\Omega_N|^2 \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \sum_{i: m_1^{1-\epsilon_0} \leq N_i \leq m_2} \left( \frac{N^{2\gamma+7\epsilon_0}}{\Lambda Q} + N^{2\gamma-1+4\epsilon_0} (\Lambda Q)^{1+4\epsilon_0} + \right. \\ & \left. + N^{3\gamma-1/2+8\epsilon_0} \frac{(\Lambda Q)^{1/2-\gamma+6\epsilon_0} Q^{3\epsilon_0}}{Q} + N^{3\gamma-3/2+8\epsilon_0} (\Lambda Q)^{3/2-\gamma+6\epsilon_0} Q^{1+2\epsilon_0} \right). \end{aligned}$$

Using the following estimates (we recall that  $\Lambda = N_{i+1}$ )

$$\sum_{i: m_1^{1-\epsilon_0} \leq N_i \leq m_2} \Lambda^\alpha \ll \begin{cases} N^{\epsilon_0} m_2^\alpha, & \text{if } \alpha > 0; \\ N^{\epsilon_0} m_1^\alpha, & \text{if } \alpha < 0. \end{cases} \tag{7.22}$$

we obtain

$$\begin{aligned} & \sum_{\substack{c_1 \leq q \leq c_2 \\ 1 \leq a \leq q}}^* \int_{f_1 \leq |K| \leq f_2} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \\ & \ll |\Omega_N|^2 \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \left( N^{2\gamma-2/3+12\epsilon_0} Q + N^{2\gamma-1/2+14\epsilon_0} + N^{5\gamma/2-1/4+14\epsilon_0} Q^{-1+10\epsilon_0} + \right. \\ & \left. + N^{5\gamma/2-3/4+14\epsilon_0} Q^{1+6\epsilon_0} \right) \ll |\Omega_N|^2 \left( N^{2\gamma-2/3+12\epsilon_0} N^{2\gamma+12\epsilon_0} + N^{2\gamma-1/2+15\epsilon_0} + \right. \\ & \left. + N^{5\gamma/2-1/4+14\epsilon_0} N^{-(1-3\gamma)/3+7\epsilon_0} + N^{5\gamma/2-3/4+14\epsilon_0} N^{2\gamma+14\epsilon_0} \right) \ll |\Omega_N|^2 N^{-\epsilon_0}. \end{aligned}$$

The last inequality holds if  $\gamma \leq \frac{1}{6} - 10\epsilon_0$ . The Lemma is proved.  $\square$

## 8. The proof of Theorem 1.3

Let  $\gamma < \frac{1}{6}$ . We choose  $\epsilon_0$  such that  $\epsilon_0 \in (0, 1/2500)$  and  $\gamma \leq \frac{1}{6} - 10\epsilon_0$ . Then it follows from Lemmas 7.6–7.10 that the first integral in the right side of (7.2) is less than  $\frac{|\Omega_N|^2}{N}$ , that is,

$$\frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq N_{1/2} \\ q > Q_0}}^* \int_{\substack{Q_0 \leq |K| \leq \frac{N_{1/2}^*}{q}}} \left| S_N \left( \frac{a}{q} + \frac{K}{N} \right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (8.1)$$

Substituting (8.1) and the results of Lemma 7.3 in Lemma 7.2, we obtain

$$\int_0^1 |S_N(\theta)|^2 d\theta \ll \frac{|\Omega_N|^2}{N} \quad \text{when } \gamma < \frac{1}{6}. \quad (8.2)$$

Thus inequality (4.1) is proved. But Theorem 1.3 is an easy consequence of (4.1) (see [2, Sec. 2.3]). This completes the proof of the theorem.

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