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Smooth Sums over Smooth k-Free Numbers and Statistical Mechanics

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Abstract: We provide an asymptotic estimate for certain sums over k-free integers with small prime factors. These sums depend upon a complex parameter α and involve a smooth cut-off f. They are a variation of several classical number-theoretical sums. One term in the asymptotics is an integral operator whose kernel is the α-convolution of the Dickman-de Bruijn distribution, and the other term is explicitly estimated. The trade-off between the value of α and the regularity of f is discussed. This work generalizes the results of [6, 7], where $k = 2$ and $\alpha = 1$.

Keywords: k-free numbers, smooth-numbers, average order of arithmetic functions, convolutions of the Dickman-de Bruijn distribution, weak convergence of complex measures.

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1. Introduction

The study of the typical behavior of arithmetic functions has a long history in number theory. Let n denote a positive integer. Let $\omega(n)$ (resp. $\Omega(n)$) denote the number of prime divisors of n, counted without (resp. with) multiplicity. If $d(n)$ denotes the number of divisors of n, then clearly $\Omega(n) \leqslant \log n/\log 2$, and $2^{\omega(n)} \leqslant d(n) \leqslant 2^{\Omega(n)} \leqslant n$. Notice that $2^{\omega(n)}$ equals the number of square-free divisors of n.

We are interested in k-free numbers, i. e. integers such that $p^k \nmid n$ for every prime p. Notice that for $k = 2$ the set of square-free numbers is characterized by the condition $\Omega(n) = \omega(n)$. This paper is devoted to the study of the asymptotic behavior of certain sums over k-free numbers, with an additional restriction on the size of their prime factors. Let us fix an integer $k \geqslant 2$, $\alpha \in \mathbb{C}$, and a bounded function $f : \mathbb{R} \to \mathbb{C}$. Define the sum

$$S_{\Omega,f}(k,\alpha;N) = \sum_{\substack{n \;\; k\text{-free} \\ p|n \Rightarrow p \leqslant N}} f\left(\frac{\log n}{\log N}\right) \frac{\alpha^{\Omega(n)}}{n}. \tag{1}$$

We shall refer to (1) as a *smooth* sum because the regularity of f plays an important role in our analysis. The k-free numbers involved in the sum classically called *smooth* because their prime factors are considerably smaller then the numbers themselves.

We shall always assume that $\alpha \neq 0$, otherwise the sum (1) is identically zero. Let us define the function space

$$\mathcal{S}_\eta(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \exists\, C > 0 \text{ s.t. } \left|\widehat{f}(\lambda)\right| \leqslant \frac{C}{1+|\lambda|^\eta} \;\; \forall\, \lambda \in \mathbb{R} \right\},$$

where $\widehat{f}$ is the Fourier transform of f. Notice that, for example, $\mathbf{1}_{[0,1]} \in \mathcal{S}_1$ and that the Schwartz space $\mathcal{S} \subset \mathcal{S}_\eta$ for every η. The main result of our paper is the following

Theorem 1. *Let $|\alpha| < 2$ and let $f \in \mathcal{S}_\eta$ with $\eta > |\alpha| - \operatorname{Re}\alpha + 1$. Then there exists a non-zero constant $C = C(k,\alpha) \in \mathbb{C}$ such that, for every $R = R(N)$ satisfying*

$$\frac{R}{\log N} \to 0 \quad \textit{and} \quad \frac{(\log N)^{|\alpha| - \operatorname{Re}\alpha}}{R^{\eta-1}} \to 0 \quad \textit{as } N \to \infty,$$

we have

$$S_{\Omega,f}(k,\alpha;N) = C \cdot (\log N)^\alpha \cdot \left(\int\limits_{|\lambda| \leqslant R} \varphi^{(\alpha)}(\lambda) \widehat{f}(\lambda) \mathrm{d}\lambda + \varepsilon_N \right), \tag{2}$$

where

$$\varphi^{(\alpha)}(\lambda) = \exp\left\{ \alpha \int\limits_0^1 \frac{e^{i\lambda v} - 1}{v} \mathrm{d}v \right\} \tag{3}$$

and $\varepsilon_N = \varepsilon_N(k, \alpha, f, R) \to 0$ *as* $N \to \infty$ *satisfies*

$$\varepsilon_N = O\left(\frac{\log\log N}{\log N}\right) + O\left(\frac{(\log N)^{|\alpha| - \operatorname{Re}\alpha}}{R^{\eta-1}}\right). \tag{4}$$

The constants implied by the O*-notation in* (4) *depend on* k, α, f, *and* R.

Remark. The function $\varphi^{(\alpha)}(\lambda)$ in (3) is the characteristic function (inverse Fourier transform) of the α-convolution of the Dickman-de Bruijn probability distribution on $\mathbb{R}_{\geqslant 0}$. Notice that $\varphi^{(\alpha)}(\lambda)$ need not be bounded.

Remark. The integral in (2) is $O(1)$. However — depending on the function f — it may tend to zero as $N \to \infty$ and, if this is the case, it may be dominated by the error term ε_N. In Section 6 we discuss a concrete example where the integral term is bounded away from zero. A recent preprint by M. Avdeeva, D. Li and Ya. G. Sinai [3] gives new information on the integral term in (2) when α is a negative real number.

Remark. Our methods allow us to enlarge the set of $\alpha \in \mathbb{C}$ for which Theorem 1 holds, provided we modify (1) by considering the sum only over the k-free integers that are not divisible by a finite number of primes. For example, (2) holds for smooth sums over smooth *odd* k-free integers for $|\alpha| < 3$.

Theorem 1 shows that there is a competition between the magnitude of $R(N)$ and the regularity parameter η for the function f. Two natural choices for $R(N)$ are $\log N/\log\log N$ and $(\log N)^\tau$, $0 < \tau < 1$, and are covered by the following corollaries.

Corollary 1. *Let* $R(N) = \dfrac{\log N}{\log\log N}$ *in Theorem* 1. *Then*

$$\varepsilon_N = \begin{cases} O\left(\dfrac{\log\log N}{\log N}\right), & \text{if } \eta > |\alpha| - \operatorname{Re}\alpha + 2; \\ O\left(\dfrac{(\log\log N)^{\eta-1}}{(\log N)^{\eta-(|\alpha|-\operatorname{Re}\alpha+1)}}\right), & \text{if } |\alpha| - \operatorname{Re}\alpha < \eta \leqslant |\alpha| - \operatorname{Re}\alpha + 2. \end{cases}$$

COROLLARY 2. *Let* $R(N) = (\log N)^{1-\tau}$ *in Theorem* 1. *Then*

$$\varepsilon_N = \begin{cases} O\left(\dfrac{\log\log N}{\log N}\right), & \text{if } 0<\tau\leqslant\dfrac{\eta-(|\alpha|-\mathrm{Re}\,\alpha+2)}{\eta-1} \text{ — case } (a); \\ O\left((\log N)^{-\eta(1-\tau)-\tau+(|\alpha|-\mathrm{Re}\,\alpha+1)}\right), & \text{if } 0\vee\dfrac{\eta-(|\alpha|-\mathrm{Re}\,\alpha+2)}{\eta-1}<\tau< \\ & <\dfrac{\eta-(|\alpha|-\mathrm{Re}\,\alpha+1)}{\eta-1} \text{ — case } (b). \end{cases}$$

The two cases (a) and (b) are summarized in Figure 1, where the trade-off between τ and η is apparent.

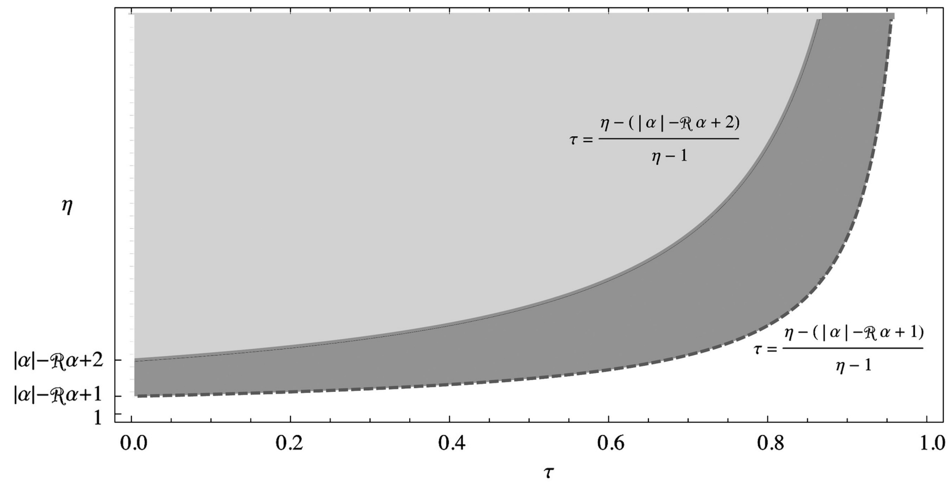

Fig. 1. Plot of the two regions (a) (light grey) and (b) (dark grey), wherein the error terms $O\left(\dfrac{\log\log N}{\log N}\right)$ and $O\left((\log N)^{-\eta(1-\tau)-\tau+(|\alpha|-\mathrm{Re}\,\alpha+1)}\right)$ in Corollary 2 dominate respectively. The boundary of the two regions are hyperbolæ, whose equations are also shown

The paper is organized as follows. Section 2 serves as context and motivation; it includes results concerning the average of certain arithmetic functions, some asymptotic results about smooth numbers, and convolutions of the Dickman-De Bruijn distribution. In Section 3 we introduce a complex measure and we rewrite the sum (1) by means of a random[1)] variable, following ideas from Statistical Mechanics already

[1)] We borrow the classical probabilistic terminology, although for most values of the parameter α, we are not dealing with a probability measure.

used in [6, 7]. The main results in Section 4 are Theorem 4, that is devoted to the pointwise approximation of the characteristic function of the above random variable, and Proposition 4, dealing with some integral estimates. Theorem 1 follows from these results. The proof of Theorem 4 occupies Section 5 and Appendix A. Section 6 discusses a concrete example where the function f is C^∞ with compact support.

2. Motivation

2.1. Normal and Average Orders or Arithmetic Functions

Hardy and Ramanujan [19] proved that the *normal order* of $\omega(n)$ and $\Omega(n)$ is $\log\log n$, i. e. for every $\varepsilon > 0$, the set of $n \leqslant x$ such that the inequalities

$$(1-\varepsilon)\log\log n \leqslant \omega(n) \leqslant (1+\varepsilon)\log\log n$$

fail to hold has cardinality $o(x)$ as $x \to \infty$ (and the same statement is true for $\Omega(n)$ in place of $\omega(n)$). Erdős and Kac [15] improved on this result and established a Central Limit Theorem:

$$\frac{1}{x}\left|\left\{n \leqslant x:\ a \leqslant \frac{\omega(n)-\log\log n}{\sqrt{\log\log n}} \leqslant b\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-t^2/2}\mathrm{d}t + O\left(\frac{1}{\sqrt{\log\log x}}\right). \tag{5}$$

Dirichlet proved that the *average order* order of $d(n)$ is $\log n$, i. e. $\sum_{n\leqslant x} d(n) \sim$ $\sim x\log x$. More precisely

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma-1)x + \Delta(x),$$

where γ is Euler-Mascheroni's constant and $\Delta(x) = O(x^{1/2})$. Finding the smallest $\theta > 0$ such that $\Delta(x) = O(x^{\theta+\varepsilon})$ for every $\varepsilon > 0$ is known as the *Dirichlet divisor problem*. Several authors have improved Dirichlet's bound $\theta \leqslant \frac{1}{2}$ towards the conjectured valued $\theta = 1/4$, the current record being $\theta \leqslant 131/416$ (Huxley [25]).

Mertens [27, 28] proved that the average order of $2^{\omega(n)}$ is $\frac{1}{\zeta(2)}\log n$. More precisely

$$\sum_{n\leqslant x} 2^{\omega(n)} = \frac{1}{\zeta(2)}x\log x + \left(\frac{2\gamma-1}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)}\right)x + \Delta^{(2)}(x), \tag{6}$$

where $\Delta^{(2)}(x) = O(x^{1/2}\log x)$. It is also conjectured that $\Delta^{(2)}(x) = O(x^{\theta+\varepsilon})$ with $\theta = 1/4$, and the best known result (assuming the Riemann Hypothesis) is $\theta \leqslant 4/11$ (Baker [4]).

Grosswald [17] proved that the average order of $2^{\Omega(n)}$ is $A\log^2 n$, where $A \approx 0.27317$ is an explicit constant. More precisely

$$\sum_{n\leqslant x} 2^{\Omega(n)} = Ax\log^2 x + Bx\log x + O(x), \tag{7}$$

where B is another explicit constant, and the error term is optimal (Bateman [5]).

It is interesting to generalize the sums (6) and (7) by replacing 2 by an arbitrary complex number z. Moreover, one can restrict the summation to the set of square-free integers (characterized by the condition $\omega(n) = \Omega(n)$) by multiplying the summand by $\mu^2(n)$, where μ is the Möbius function. Selberg [29] proved that for every $z \in \mathbb{C}$, as $x \to \infty$

$$\sum_{n\leqslant x} z^{\omega(n)} = F(z)\,x(\log x)^{z-1} + O\left(x(\log x)^{\operatorname{Re} z-2}\right), \tag{8}$$

$$\sum_{n\leqslant x} \mu^2(n) z^{\omega(n)} = G(z)\,x(\log x)^{z-1} + O\left(x(\log x)^{\operatorname{Re} z-2}\right), \tag{9}$$

and, under the assumption that $|z| < 2$,

$$\sum_{n\leqslant x} z^{\Omega(n)} = H(z)\,x(\log x)^{z-1} + O\left(x(\log x)^{\operatorname{Re} z-2}\right), \tag{10}$$

where

$$F(z) = \frac{1}{\Gamma(z)}\prod_p \left(1 + \frac{z}{p-1}\right)\left(1 - \frac{1}{p}\right)^z,$$

$$G(z) = \frac{1}{\Gamma(z)}\prod_p \left(1 + \frac{z}{p}\right)\left(1 - \frac{1}{p}\right)^z,$$

$$H(z) = \frac{1}{\Gamma(z)}\prod_p \left(1 - \frac{z}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^z. \tag{11}$$

The convergence in (8)-(10) with respect to z is uniform on compact sets. Notice that (8) yields the first two terms in (6), and that the condition $|z| < 2$ in (10) can not be relaxed since, for example, for $z = 2$, (10) and (7) are different. These results were further improved and generalized by Delange [12]. He proved the following

THEOREM 2. *Let f be a non-negative, integer-valued, additive function such that $f(p) = 1$, and let χ be a bounded, multiplicative function such that $\chi(p) = 1$. For $\varrho \geqslant 0$ let*

$$\sigma_0(\varrho) = \inf \left\{ \sigma > \frac{1}{2} : \sum_{k\geqslant 2} \sum_{p} \frac{|\chi(p^k)|\varrho^{f(p^k)}}{p^{k\sigma}} < \infty \right\}$$

(let us set $\inf \varnothing = +\infty$*). Let $E = \{\varrho \geqslant 0 : \sigma_0(\varrho) < 1\}$ and set $R = \sup E \geqslant 1$. Then there exists a sequence $\{A_j(z)\}_{j\geqslant 0}$ of holomorphic functions on $|z| < R$ (and continuous on $|z| \leqslant R$ if $R \in E$) such that for every $q \geqslant 0$*

$$\sum_{n\leqslant x} \chi(n) z^{f(n)} = x(\log x)^{z-1} \left(\sum_{j=0}^{q} \frac{A_j(z)}{(\log x)^j} + O\left(\frac{1}{(\log x)^{q+1}} \right) \right) \quad \text{as } x \to \infty. \tag{12}$$

The constant implied by the O-notation is uniform in z on compact sets.

In other words, one can write an asymptotic expansion for the sum in (12) in powers of $\log x$ to arbitrary order. The functions $A_j(z)$ in (12) can be constructed explicitly, in particular

$$A_0(z) = \frac{1}{\Gamma(z)} \prod_{p} \left(1 + \sum_{k=1}^{\infty} \frac{\chi(p^k) z^{f(p^k)}}{p^k} \right) \left(1 - \frac{1}{p} \right)^{z}.$$

Notice that the results by Selberg follow from Theorem 2, namely $(f, \chi) = (\omega, 1)$ gives (8), $(f, \chi) = (\omega, \mu^2)$ gives (9), and $(f, \chi) = (\Omega, 1)$ gives (10). Theorem 2 implies a general result concerning the average order of χ along the level sets of f:

THEOREM 3. *Let f, χ be as in Theorem 2. For every $m \geqslant 1$, there exists a sequence $\{P_j\}_{j\geqslant 0}$ of polynomials of degree $\leqslant m - 1$ such that for every $q \geqslant 0$*

$$\sum_{\substack{n \leqslant x \\ f(n) = m}} \chi(n) = \sum_{j=1}^{q} \frac{x P_j(\log\log x)}{(\log x)^{j+1}} + O\left(\frac{x(\log\log x)^{m-1}}{(\log x)^{q+2}} \right) \quad \text{as } x \to \infty.$$

The coefficient of the monomial of degree $m-1$ *in* P_j *is*

$$\frac{(-1)^j}{(m-1)!}\,\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\frac{1}{s}\prod_p\left(1+\sum_{f(p^k)=0}\frac{\chi(p^k)}{p^{ks}}\right)\right)\bigg|_{s=1}.$$

In particular Theorem 3 implies the results by Landau [26]: as $x\to\infty$

$$\sum_{\substack{n\leqslant x\\ \omega(n)=m}}1\sim\sum_{\substack{n\leqslant x\\ \Omega(n)=m}}1\sim\sum_{\substack{n\leqslant x\\ \omega(n)=\Omega(n)=m}}1\sim\frac{x}{\log x}\frac{(\log\log x)^{m-1}}{(m-1)!}$$

by taking $(f,\chi)=(\omega,1)$, $(f,\chi)=(\Omega,1)$, and $(f,\chi)=(\omega,\mu^2)$ respectively. Let us also point out that the error terms coming from Theorem 3 (i. e. $O\Big(\frac{x(\log\log x)^{m-1}}{(\log x)^2}\Big)$) are better than the ones given by Landau (i. e. $O\Big(\frac{x(\log\log x)^{m-2}}{\log x}\Big)$). Recall that the integers for which $\Omega(n)=m$ are called m*-almost primes*.

Ordinary and Logarithmic Mean Values

Given an arithmetic function g, one can define the (*ordinary*) *mean value of* g as $M[g]=\lim\limits_{x\to\infty}\frac{1}{x}\sum\limits_{n\leqslant x}g(n)$, and the *logarithmic mean value of* g as $L[g]=$ $=\lim\limits_{x\to\infty}\frac{1}{\log x}\sum\limits_{n\leqslant x}\frac{g(n)}{n}$.

It is a classical fact that, if $M[g]$ exists, then $L[g]$ also exists and they are equal. The other implication is in general not true. For example, $M[\mu]=L[\mu]=0$ and $M[\mu^2]=L[\mu^2]=6/\pi^2$.

The sums (1) we are concerned with can be seen as partial sums for some weighted logarithmic averages.

2.2. Sums over Smooth Integers

Sums of type (8-12) can be further generalized by setting a constraint on the size of primes in the factorization of n. Integers whose prime factors are all $\leqslant y$ are called y*-smooth*, while those whose prime factors are $\geqslant c$ are called c*-rough* or

c-*jagged*. Generalizing (10), we can introduce the sum

$$\Psi(z;x,y,c)=\sum_{\substack{n\leqslant x\\ p|n\Rightarrow c<p\leqslant y}} z^{\Omega(n)}. \tag{13}$$

Clearly, if $c<2$ we set no restriction on the roughness of the integers in the sum, and if $y\geqslant x$ we set no restriction for their smoothness. The inequality $|z|<2$ in (10) can be replaced by $|z|<c$ for $c\geqslant 2$, and (using the notation (13) we just introduced) we have

$$\Psi(z;x,x,c)=H_c(z)\,x(\log x)^{z-1}+O(x(\log x)^{\mathrm{Re}\,z-2})\qquad\text{as } x\to\infty, \tag{14}$$

where

$$H_c(z)=\frac{1}{\Gamma(z)}\prod_{p\leqslant c}\left(1-\frac{1}{p}\right)^{z}\prod_{p>c}\left(1-\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{z},\qquad\text{cfr. (11).} \tag{15}$$

Moreover, the results by Delange (e. g. Theorem 2) allow to write the asymptotic of $\Psi(z;x,x,c)$ in powers of $\log x$, where the functions $z\mapsto A_j(z)$ (see (12)) are holomorphic for $|z|<p'$, where $p'=p'(c)$ is the least prime larger than c.

For $z=1$ we have the well-known counting of y-smooth integers:

$$\Psi(x,y)=\Psi(1;x,y,1)=|\{n\leqslant x:\ n \text{ is } y\text{-smooth}\}|. \tag{16}$$

Dickman [13] proved that $\Psi(x,y)\sim x\rho(u)$ as $x\to\infty$ when for $y=x^{1/u}$ for some $u\geqslant 1$, where ρ is the Dickman-de Bruijn function, i. e. the solution of the delay differential equation $u\rho'(u)+\rho(u-1)=0$ with initial condition $\rho(u)=1$ for $0\leqslant u\leqslant 1$. The range of u's for which the above asymptotic result is valid has been significantly enlarged, and explicit error terms are known. Namely

$$\Psi(x,y)=x\rho(u)\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right),\qquad\text{where } y=x^{1/u}, \tag{17}$$

and $y>\exp((\log x)^{5/8+\varepsilon})$ (de Bruijn [8–10]), or $y>\exp((\log\log x)^{5/3+\varepsilon})$ (Hildebrand [22]). Moreover, (17) holds uniformly for $y\geqslant(\log x)^{2+\varepsilon}$ if and only if the

Riemann Hypothesis is true (Hildebrand [22]). For smaller values of y the asymptotic result (17) is not true anymore. One has, for example,

$$\Psi(x, \log^A x) = x^{1-1/A+O(1/\log\log x)} \qquad \text{for } A > 1$$

(Granville [16], p. 291) and

$$\log \Psi(x, \kappa \log x) = (\log(1+\kappa) + \kappa \log(1+1/\kappa)) \frac{\log x}{\log\log x} \left(1 + O\left(\frac{1}{\log\log x}\right)\right)$$

(Granville [16], Erdős [14]).

One can further restrict the sums (13-16) to square-free integers (for which $\omega(n) = \Omega(n)$) (cfr. (9)). If y is not too small compared to x (namely $\log y/\log\log x \to \infty$), then the rhs of (16) is simply multiplied by the density of square-free numbers, i. e. $6/\pi^2$. In the case we shall consider this will not be the case, since we will deal with $y \sim \log n$.

It is also worthwhile to mention that Alladi [2], Hensley [21], and Hildebrand [23, 24] proved an analog of the Erdős-Kac result (5) for y-smooth integers, with the same mean and variance ($\sim \log\log x$) for $u = \log x/\log y = o(\log\log x)$, while the mean is $\sim u$ and the variance is $\sim u/(\log u)^2$ whenever $\log x \ll y \ll \exp((\log x)^{1/21})$.

A large amount of work on averages of multiplicative functions over smooth integers has been done by G. Tenenbaum and J. Wu [31–33] and G. Hanrot, Tenenbaum and Wu [18]. The reader can refer to [31] for an historical introduction to the subject.

2.3. Convolutions of the Dickman-de Bruijn Distribution

The case of general $\Psi(z, x, y, c)$ has been studied by DeKoninck and Hensley [11]. They proved an approximation of Ψ by means of another function ψ which, in turn, is close to $x(\log x)^{z-1}\rho_z(u)$, where ρ_z is a close relative of the Dickman-de Bruijn function.

Let us consider $\alpha \geqslant 1$ and let us use the notation $\Psi_\alpha(x, y) = \Psi(\alpha; x, y, \alpha)$. The asymptotic of $\Psi_\alpha(x, x)$ follows from Theorem 2, see (14-15). The first result for y-smooth numbers is due to Hensley [20], who gave the asymptotic of $\Psi_\alpha(x, x^{1/u})$, similarly to (17). He proved that for every $\alpha \geqslant 1$ and every $0 < \varepsilon < 1$,

$$\Psi_\alpha(x, x^{1/u}) = \rho_\alpha(u)\, x(\log x)^{\alpha-1}(1 + o(1)),$$

uniformly in $1 \leqslant u \leqslant (1-\varepsilon)\log\log x/\log\log\log x$ as $x \to \infty$, where $\rho_\alpha(u)$ satisfies the delay differential equation involving the function A_0 (see (13)) from Theorem 2.

$$\begin{cases} \rho_\alpha(u) = 0, & u < 0, \\ \rho_\alpha(u) = A_0(\alpha), & 0 \leqslant u \leqslant 1, \\ -u^\alpha \rho'_\alpha(u) = \alpha(u-1)^{\alpha-1}\rho_\alpha(u-1), & u > 1. \end{cases} \tag{18}$$

For $\alpha = 1$ the function (18) agrees with the Dickman-de Bruijn function (see Section 2.2). It is convenient to introduce a probability distribution on $\mathbb{R}_{\geqslant 0}$, whose density $w_\alpha(u)$ satisfies,

$$w_\alpha(u) = \frac{e^{\alpha\gamma}}{\Gamma(\alpha)} u^{\alpha-1}\rho_\alpha(u).$$

Notice that for $\alpha = 1$ the two functions $\rho(u)$ and $w_1(u)$ differ by a multiplicative constant, but for general α this is not the case. The density $w_\alpha(u)$ decays faster than exponentially as $u \to \infty$ (see [8]). It is known that the characteristic function $\varphi^{(\alpha)}$ of w_α is given by

$$\varphi^{(\alpha)}(\lambda) = \int_0^\infty e^{i\lambda u} w_\alpha(u)\mathrm{d}u = \exp\left(\alpha \int_0^1 \frac{e^{i\lambda v}-1}{v}\mathrm{d}v\right). \tag{19}$$

In other words, since $\varphi^{(\alpha)}(\lambda) = (\varphi^{(1)}(\lambda))^\alpha$, w_α is the density of the α*-convolution of the Dickman-de Bruijn distribution*. Later we shall use the fact that

$$\varphi^{(1)}(\lambda) \sim \frac{ie^{-\gamma}}{\lambda} \quad \text{as } |\lambda| \to \infty, \tag{20}$$

where γ is Euler-Mascheroni's constant. In our analysis, we shall consider $\alpha \in \mathbb{C}$. In this case α-convolutions of the Dickman-de Bruijn distribution cannot be considered as probability distributions on $\mathbb{R}_{\geqslant 0}$, but only as *distributions in the sense of Schwartz*. More precisely, they will be elements of $(\mathcal{S}_\eta(\mathbb{R}))^*$ for $\eta > |\alpha| - \operatorname{Re}\alpha + 1$.

3. Reformulaton of the Problem. The Ensemble $\mathfrak{X}_N^{(k)}$

Recall the prime counting function $\pi(N) = |\{p \in \mathcal{P} : p \leqslant N\}|$. The classical Prime Number Theorem gives $\pi(N) \sim \frac{N}{\log N}$ as $N \to \infty$ Let us consider the set

$$\mathfrak{X}_N^{(k)} = \left\{ x = \prod_{p \leqslant N} p^{\nu(p)} : 0 \leqslant \nu(p) \leqslant k-1 \right\}$$

consisting of all k-free integers whose prime factors do not exceed N. Notice that $|\mathfrak{X}_N^{(k)}| = k^{\pi(N)}$ and $\max \mathfrak{X}_N^{(k)} = \prod_{p \leqslant N} p^{k-1} = e^{(k-1)\pi(N) \log \pi(N)(1+o(1))}$. This means that $\mathfrak{X}_N^{(k)}$ is a sparse set. For $x \in \mathfrak{X}_N^{(k)}$ we have $\Omega(x) = \sum\limits_{p \leqslant N} \nu(p)$ and $\omega(x) = |\{p \leqslant N : \nu(p) > 0\}|$. By definition, all $x \in \mathfrak{X}_N^{(k)}$ are k-free, and it easy to check that all k-free $x \leqslant p_{\pi(N)}$ belong to $\mathfrak{X}_N^{(k)}$.

Let us introduce a complex measure $P_\Omega^{(\alpha)}$ on $\mathfrak{X}_N^{(k)}$: for every $X \subseteq \mathfrak{X}_N^{(k)}$, let

$$P_\Omega^{(\alpha)}(X) = \sum_{x \in X} \frac{\alpha^{\Omega(x)}}{x}.$$

For example $\mathfrak{X}_5^{(2)} = \{1, 2, 3, 5, 6, 10, 15, 30\}$, and $P_\Omega^{(\alpha)}(\{1, 3, 10\}) = \dfrac{\alpha^0}{1} + \dfrac{\alpha^1}{3} + \dfrac{\alpha^2}{10}$. Another example is $\mathfrak{X}_4^{(3)} = \{1, 2, 3, 4, 6, 9, 18, 36\}$, where $P_\Omega^{(\alpha)}(\{1, 9, 18\}) = \dfrac{\alpha^0}{1} + {} + \dfrac{\alpha^2}{9} + \dfrac{\alpha^3}{18}$.

Using the terminology of Statistical Mechanics, we introduce the *partition function*

$$Z_{\Omega,N}^{(k,\alpha)} = P_\Omega^{(\alpha)}(\mathfrak{X}_N^{(k)}).$$

We prove an asymptotic result for $Z_{\Omega,N}^{(k,\alpha)}$ as $N \to \infty$. In the case of $\operatorname{Re} \alpha > 0$ we have that $|Z_N^{(k,\alpha)}| \to \infty$ as $N \to \infty$ and the analogy with Statistical Mechanics suggest the term "*thermodynamical limit*". Let us define a set for the parameter α for which the partition function vanishes for sufficiently large N. This set is responsible for the restriction $|\alpha| < 2$ in our Theorem 1. Let

$$\mathcal{A}_\Omega^{(k)} = \left\{ z \in \mathbb{C} : z = pe^{2\pi i l/k}, \ p \in \mathcal{P}, \ 1 \leqslant l \leqslant k-1 \right\}.$$

Notice that $-\mathcal{P} \subseteq \mathcal{A}_\Omega^{(k)}$ if k is even and $\mathcal{A}_\Omega^{(2)} = -\mathcal{P}$. We have the following

LEMMA 1. *There exist a constants $C_\Omega = C_\Omega(k,\alpha) \in \mathbb{C}$ such that, as $N \to \infty$,*

$$Z_{\Omega,N}^{(k,\alpha)} = \begin{cases} 0, & \alpha \in \mathcal{A}_\Omega^{(k)}; \\ C_\Omega (\log N)^\alpha \left(1 + O\left(\frac{1}{\log N}\right)\right), & \textit{otherwise}; \end{cases} \tag{21}$$

where the constant implied by the O-notation depends on k and α.

PROOF. We can write

$$Z_{\Omega,N}^{(k,\alpha)} = \prod_{p \leqslant N} \left(1 + \frac{\alpha}{p} + \frac{\alpha^2}{p^2} + \ldots + \frac{\alpha^{k-1}}{p^{k-1}}\right). \tag{22}$$

Notice that $\sum_{l=0}^{k-1} (\alpha/u)^l = 0$ if and only if $u = \alpha e^{2pi\frac{l}{k}}$, $l = 1, \ldots, k-1$, and there is a prime p of this form if and only if $\alpha \in \mathcal{A}_\Omega^{(k)}$.

Now, let $\alpha \notin \mathcal{A}_\Omega^{(k)}$, $\alpha \neq 0$. Then there exist a constant $d = d(\alpha)$ such that $z(p) = \sum_{l=0}^{k-1} (\alpha/p)^l \in B_1(1/2) = \{z \in \mathbb{C} : |z-1| < 1/2\}$ for every $p > d$. For all such p's we can write $\log z(p) = \log |z(p)| + i \arg(z(p))$ and choose the same branch of the logarithm. From (22) we get

$$Z_{\Omega,N}^{(k,\alpha)} = C_1 \cdot \prod_{d<p\leqslant N} \left(1 + \frac{\alpha}{p}\right) \cdot \prod_{d<p\leqslant N} \left(1 + \frac{\alpha^2/p^2 + \ldots + \alpha^{k-1}/p^{k-1}}{1 + \alpha/p}\right), \tag{23}$$

where $C_1 = C_1(\alpha) = \prod_{p \leqslant d} z(p) \neq 0$. The second factor in (23) gives the asymptotic $(\log N)^\alpha$. In fact, by taking the logarithm, we have

$$\log Z_{\Omega,N}^{(k,\alpha)} = \log C_1 + \alpha \sum_{d<p\leqslant N} \frac{1}{p} - \sum_{d<p\leqslant N} \left(\log\left(1 + \frac{\alpha}{p}\right) - \frac{\alpha}{p}\right) +$$

$$+ \sum_{d<p\leqslant N} \log\left(1 + \frac{\alpha^2/p^2 - \alpha^k/p^k}{1 - \alpha^2/p^2}\right) = \log C_1 + \alpha(\log\log N + C_2) +$$

$$+ C_3 + C_4 + O\left(\frac{1}{\log N}\right),$$

where $C_2 = C_2(\alpha)$, $C_3 = C_3(\alpha)$ and $C_4 = C_4(k, \alpha)$ do not depend on N. We then see immediately that (21) holds with $C_\Omega(k, \alpha) = C_1 e^{\alpha C_2 + C_3 + C_4}$. $\square$

Remark. A classical theorem by Mertens [28] gives $C_\Omega(2, 1) = e^{-\gamma}$.

Let us observe that the sum (1) can be written as

$$S_{\Omega,f}(k, \alpha; N) = \sum_{x \in \mathfrak{X}_N^{(k)}} f\left(\frac{\log x}{\log N}\right) P_\Omega^{(\alpha)}(\{x\}).$$

We can write

$$\log x = \sum_{p \leqslant N} \nu(p) \log p$$

and introduce the function

$$\mathfrak{X}_N^{(k)} \ni x \mapsto \xi_N(x) = \frac{\log x}{\log N} = \sum_{p \leqslant N} \nu(p) \frac{\log p}{\log N}.$$

Our main theorem will follow from the study of the distribution of ξ_N with respect to the measures $P_\Omega^{(\alpha)}$. It is convenient for us to introduce the *normalized* measure $P_{\Omega,N}^{(k,\alpha)}$ on $\mathfrak{X}_N^{(k)}$, i. e.

$$P_{\Omega,N}^{(\alpha,k)} = (Z_{\Omega,N}^{(k,\alpha)})^{-1} P_\Omega^{(\alpha)}.$$

For every $p \leqslant N$, the measure $P_{\Omega,N}^{(k,\alpha)}$ on $\mathfrak{X}_N^{(k)}$ induces a measure on $\{0, 1, \dots, k-1\}$ via the function

$$\mathfrak{X}_N^{(k)} \ni x \mapsto \nu(p) = \nu(p, x) = \max\{l \geqslant 0 : \ p^l | x\}. \tag{24}$$

This means that for each $0 \leqslant t \leqslant k-1$ we are given the 'probability' $P_{\Omega,N}^{(k,\alpha)}(\{x \in \in \mathfrak{X}_N^{(k)} : \ \nu(p, x) = t\})$. We shall use the distributions of the $\nu(p)$'s to compute the ones of ξ_N. We have the following simple

LEMMA 2. *For every* $0 \leqslant t \leqslant k-1$

$$P_{\Omega,N}^{(k,\alpha)}(\{\nu(p) = t\}) = \frac{\alpha^t p^{k-1-t}(p-\alpha)}{p^k - \alpha^k}.$$

Proof. The result follows from the straightforward computation

$$P_{\Omega,N}^{(k,\alpha)}(\{\nu(p)=t\}) = \frac{1}{Z_{\Omega,N}^{(k,\alpha)}}\frac{\alpha^t}{p^t}\prod_{p'\leqslant N,\, p'\neq p}\left(1+\frac{\alpha}{p'}+\ldots+\frac{\alpha^{k-1}}{p'^{k-1}}\right) =$$
$$= \frac{\alpha^t/p^t}{1+\alpha/p+\ldots+\alpha^{k-1}/p^{k-1}} = \frac{\alpha^t}{p^t}\frac{1-\alpha/p}{1-\alpha^k/p^k} = \frac{\alpha^t p^{k-1-t}(p-\alpha)}{p^k-\alpha^k},$$

and the normalization of the measure $P_{\Omega,N}^{(k,\alpha)}$. □

Remark. The functions $\nu(p)$ are independent with respect to the measure $P_{\Omega,N}^{(k,\alpha)}$, i. e. for every $r\geqslant 1$, every $p_{j_1}<p_{j_2}<\ldots<p_{j_r}\leqslant N$, and every $(\epsilon_1,\epsilon_2,\ldots,\epsilon_r)\in$ $\in\{0,1,\ldots,k-1\}^r$, we have

$$P_{\Omega,N}^{(k,\alpha)}\left(\{\nu(p_{j_1})=\epsilon_1,\ \nu(p_{j_2})=\epsilon_2,\ldots,\nu(p_{j_r})=\epsilon_r\}\right) = \prod_{l=1}^{r} P_{\Omega,N}^{(k,\alpha)}(\{\nu(p_{j_l})=\epsilon_l\}),$$

i. e. joint 'probabilities' factor completely. On the other hand, by Lemma 2, the $\nu(p)$'s are not identically distributed with respect to $P_{\Omega,N}^{(k,\alpha)}$. Notice, for instance, that $P_{\Omega,N}^{(k,\alpha)}(\{\nu(p_{\pi(N)})=0\}) = 1-\frac{\alpha}{N}+O\big(\frac{1}{N^k}\big)$ as $N\to\infty$. The dependence of the "probabilities" $P_{\Omega,N}^{(k,\alpha)}(\{\nu(p)=t\})$ upon N will play a crucial role in the proof of our main theorem.

Let us define the rational functions $F_{\Omega,t}^{(k,\alpha)}\in\mathbb{C}(u)$ such that $F_{\Omega,t}^{(k,\alpha)}(p) =$ $= P_{\Omega,N}^{(k,\alpha)}(\{\nu(p)=t\})$:

$$F_{\Omega,t}^{(k,\alpha)}(u) = \frac{\alpha^t u^{k-1-t}(u-\alpha)}{u^k-\alpha^k} = \\ = \frac{\alpha^t u^{k-1-t}}{u^{k-1}+\alpha u^{k-2}+\alpha^2 u^{k-3}+\ldots+\alpha^{k-2}u+\alpha^{k-1}}, \tag{25}$$

where $0\leqslant t\leqslant k-1$. Notice that the poles of $F_{\Omega,t}^{(k,\alpha)}(u)$ are $-\alpha e^{2\pi i\frac{l}{k}}$, $1\leqslant l\leqslant k-1$. Let $b_{\Omega,t}^{(k,\alpha)}(l)$, $l\geqslant 0$ the coefficient of the Laurent series for $F_{\Omega,t}^{(k,\alpha)}(u)$ on the neighborhood of infinity $|u|>|\alpha|$,

$$F_{\Omega,t}^{(k,\alpha)}(u) = \sum_{l=0}^{\infty}\frac{b_{\Omega,t}^{(k,\alpha)}(l)}{u^l}.$$

Notice that $F^{(k,\alpha)}_{\Omega,t}(u)$ has no positive powers in its expansion at ∞ since $\frac{\alpha^t u^{k-1-t}}{u^{k-1}} \sim \frac{\alpha^t}{u^t}$. Set $b^{(k,\alpha)}_{\Omega,0}(l) = 0$ for $l < 0$. We have the following two simple lemmata:

LEMMA 3.

$$b^{(k,\alpha)}_{\Omega,0}(l) = \begin{cases} \alpha^l, & l \equiv 0 \bmod k; \\ -\alpha^l, & l \equiv 1 \bmod k; \\ 0, & l \equiv 2, \dots, k-1 \bmod k. \end{cases}$$

LEMMA 4.

$$b^{(k,\alpha)}_{\Omega,t}(l) = \alpha^t b^{(k,\alpha)}_{\Omega,0}(l-t), \qquad t \geqslant 0.$$

4. Limit theorem for ξ_N

As commonly done in Probability Theory, weak convergence of measures is obtained by showing the pointwise convergence of the corresponding characteristic functions (inverse Fourier transforms). Let us define, for $\lambda \in \mathbb{R}$,

$$\varphi^{(k,\alpha)}_{\Omega,N}(\lambda) = \mathbb{E}_{P^{(k,\alpha)}_{\Omega,N}}(e^{i\lambda\xi_N}).$$

Notice that the chosen normalizations for $P^{(k,\alpha)}_{\Omega,N}$ gives $\varphi^{(k,\alpha)}_{\Omega,N}(0) = 1$. Moreover, when $P^{(k,\alpha)}_{\Omega,N}$ is a *probability* measure (i. e. when α is a positive real number), then we have the inequality $|\varphi^{(k,\alpha)}_{\Omega,N}(\lambda)| \leqslant 1$, but this is in general not true. Instead we have, for $|\alpha| < 2$,

$$\begin{aligned} \left|\varphi^{(k,\alpha)}_{\Omega,N}(\lambda)\right| &\leqslant \mathbb{E}_{|P^{(k,\alpha)}_{\Omega,N}|} 1 = \frac{Z^{(k,|\alpha|)}_{\Omega,N}}{\left|Z^{(k,\alpha)}_{\Omega,N}\right|} = \frac{C_\Omega(k,|\alpha|)}{|C_\Omega(k,\alpha)|} \frac{(\log N)^{|\alpha|}\,(1 + O(1/\log N))}{(\log N)^{\mathrm{Re}\,\alpha}\,(1 + O(1/\log N))} = \\ &= O\Big((\log N)^{|\alpha| - \mathrm{Re}\,\alpha}\Big), \end{aligned} \tag{26}$$

where the constant implied by the O-notation depends only on k, and α, uniformly in λ.

Recall $\varphi^{(\alpha)}(\lambda)$ from (19), i. e. the characteristic function of the α-convolution of the Dickman-de Bruijn distribution. We prove the following

THEOREM 4. *Let $|\alpha| < 2$, and assume that $\lambda = o(\log N)$ as $N \to \infty$. Then*

$$\varphi_{\Omega,N}^{(k,\alpha)}(\lambda) = \varphi^{(\alpha)}(\lambda)\left(1 + O\left(\frac{1}{\log N}\right) + O(\varepsilon \log|\varepsilon|)\right),$$

where $\varepsilon = \frac{\lambda}{\log N} = o(1)$ as $N \to \infty$. The constants implied by the O-notation depend only on k and α.

Remark. The case of $\lambda \to 0$ is of no harm in dealing with characteristic functions of normalized measures (recall that $\varphi_{\Omega,N}^{(k,\alpha)}(0) = 1$), and the error-term bounds obtained are the same as for $\lambda = O(1)$. The term $O(1/\log N)$ in the theorem above prevents an underestimate of the error term in the (uninteresting) case when $\lambda \to 0$, i. e. $\varepsilon = o(1/\log N)$. The interesting application of the above result is for $|\lambda| \to \infty$ more slowly than $\log N$, and the corresponding error term can be simply written as $O(\varepsilon \log|\varepsilon|)$.

Remark. Theorem 4 generalizes the main result in citeCellarosi-Sinai-Mobius-1, Cellarosi-Sinai-Mobius-2 (where the case $(k, \alpha) = (2, 1)$ is addressed), and provides an explicit error term (as opposed to simply an error term $o(1)$).

A consequence of Theorem 4 is the following

PROPOSITION. *Let us fix $k \geqslant 2$, $|\alpha| < 2$, $f \in \mathcal{S}_\eta$ with $\eta \geqslant |\alpha| - \operatorname{Re}\alpha + 1$. Then for every $R = R(N)$ such that*

$$\frac{R}{\log N} \to 0 \quad \textit{and} \quad \frac{(\log N)^{|\alpha| - \operatorname{Re}\alpha}}{R^{\eta-1}} \to 0 \quad \textit{as } N \to \infty$$

$$\int_{\mathbb{R}} \varphi_{\Omega,N}^{(k,\alpha)}(\lambda)\widehat{f}(\lambda)\mathrm{d}\lambda = \int_{|\lambda| \leqslant R} \varphi^{(\alpha)}(\lambda)\widehat{f}(\lambda)\mathrm{d}\lambda + \varepsilon_N,$$

where $\varepsilon_N = \varepsilon_N(k, \alpha, f, R) \to 0$ as $N \to \infty$ satisfies

$$\varepsilon_N = O\left(\frac{\log\log N}{\log N}\right) + O\left(\frac{(\log N)^{|\alpha| - \operatorname{Re}\alpha}}{R^{\eta-1}}\right).$$

PROOF. For every $R = R(N)$ such that $R(N) = o(\log N)$ as $N \to \infty$, let us write

$$\int_{\mathbb{R}} \varphi_{\Omega,N}^{(k,\alpha)}(\lambda)\widehat{f}(\lambda)\mathrm{d}\lambda = \int_{|\lambda|\leqslant R} + \int_{|\lambda|>R} = \mathcal{I}_1 + \mathcal{I}_2.$$

Now we can use Theorem 4:

$$\mathcal{I}_1 = \int_{|\lambda|\leqslant R} \varphi^{(\alpha)}(\lambda)\widehat{f}(\lambda)\mathrm{d}\lambda + O\left(\int_{|\lambda|<R} \varphi^{(\alpha)}(\lambda)\frac{\lambda}{\log N}\log\left|\frac{\lambda}{\log N}\right|\widehat{f}(\lambda)\mathrm{d}\lambda\right) +$$
$$+O\left(\int_{|\lambda|\leqslant R} \varphi^{(\alpha)}(\lambda)\frac{1}{\log N}\widehat{f}(\lambda)\mathrm{d}\lambda\right) = \mathcal{I}_{1,1} + \mathcal{I}_{1,2} + \mathcal{I}_{1,3}.$$

Notice that, by (20) and the fact that $\varphi^{(\alpha)}(\lambda) = (\varphi^{(1)}(\lambda))^{\alpha}$, we have $\varphi^{(\alpha)}(\lambda) = O(\lambda^{-\operatorname{Re}\alpha})$ as $\lambda \to \infty$. We can write

$$\mathcal{I}_{1,2} = O\left(\int_{|\lambda|\leqslant 1} \left|\frac{\lambda}{\log N}\log\left|\frac{\lambda}{\log N}\right|\right|\mathrm{d}\lambda\right) + O\left(\int_{1<|\lambda|\leqslant R} \frac{\lambda^{1-\operatorname{Re}\alpha-\eta}}{\log N}\log\left|\frac{\lambda}{\log N}\right|\mathrm{d}\lambda\right) =$$
$$= O\left(\frac{\log\log N}{\log N}\right) + \begin{cases} O\left(\frac{\log R}{\log N}\log\left(\frac{\log^2 N}{R}\right)\right), & \operatorname{Re}\alpha+\eta=2; \\ O\left(\frac{\log\log N}{\log N}\right) + O\left(\frac{R^{2-\operatorname{Re}\alpha-\eta}}{\log N}\log\left(\frac{\log N}{R}\right)\right), & \text{otherwise;} \end{cases}$$

$$\mathcal{I}_{1,3} = O\left(\frac{1}{\log N}\right) + O\left(\int_{1<\lambda\leqslant R} \frac{\lambda^{-\operatorname{Re}\alpha-\eta}}{\log N}\mathrm{d}\lambda\right) = \begin{cases} O\left(\frac{\log R}{\log N}\right), & \operatorname{Re}\alpha+\eta=1; \\ O\left(\frac{R^{1-\operatorname{Re}\alpha-\eta}}{\log N}\right), & \text{otherwise.} \end{cases}$$

Let us observe that $\mathcal{I}_{1,3} = O(\mathcal{I}_{1,2})$. To estimate $\mathcal{I}_2$ we use the trivial estimate (26) and the fact that $\eta > 1$:

$$\mathcal{I}_2 = O\left((\log N)^{|\alpha|-\operatorname{Re}\alpha}\int_{\lambda>R}\frac{\mathrm{d}\lambda}{\lambda^{\eta}}\right) = O\left(\frac{(\log N)^{|\alpha|-\operatorname{Re}\alpha}}{R^{\eta-1}}\right).$$

Summarizing,

$$\int_{\mathbb{R}} \varphi_{\Omega,N}^{(k,\alpha)}(\lambda)\widehat{f}(\lambda)\mathrm{d}\lambda = \int_{|\lambda|\leqslant R} \varphi^{(\alpha)}(\lambda)\widehat{f}(\lambda)\mathrm{d}\lambda +$$

$$+ O\left(\frac{\log\log N}{\log N}\right) + O\left(\frac{(\log N)^{|\alpha|-\mathrm{Re}\,\alpha}}{R^{\eta-1}}\right) + \tag{27}$$

$$+ \begin{cases} O\left(\dfrac{\log R}{\log N}\log\left(\dfrac{\log^2 N}{R}\right)\right), & \mathrm{Re}\,\alpha+\eta=2; \\ O\left(\dfrac{R^{2-\mathrm{Re}\,\alpha-\eta}}{\log N}\log\left(\dfrac{\log N}{R}\right)\right), & \text{otherwise.} \end{cases} \tag{28}$$

One can check that the term in (28) is dominated by those in (27). □

Theorem 1 follows now immediately. In fact,

$$S_{\Omega,f}(k,\alpha;N) = Z_{\Omega,N}^{(k,\alpha)} \int_{\mathbb{R}} \varphi_{\Omega,N}^{(k,\alpha)}(\lambda)\widehat{f}(\lambda)\mathrm{d}\lambda,$$

and Lemma 1 and Proposition 4 give the desired statements (2)–(4).

5. Proof of Theorem 4

We shall need the following result

LEMMA 5. *Let*

$$\mathcal{E}_{\Omega}^{(k,\alpha)}(u,\lambda,N) = \tfrac{\mathrm{d}}{\mathrm{d}u}F_{\Omega,0}^{(k,\alpha)}(u) - \tfrac{\alpha}{u^2} +$$

$$+ \left(\tfrac{\mathrm{d}}{\mathrm{d}u}F_{\Omega,1}^{(k,\alpha)}(u) + \tfrac{\alpha}{u^2}\right) e^{i\lambda\frac{\log u}{\log N}} + \sum_{t=2}^{k-1} \tfrac{\mathrm{d}}{\mathrm{d}u}F_{\Omega,t}^{(k,\alpha)}(u) e^{i\lambda t\frac{\log u}{\log N}},$$

where $|u| > |\alpha|$. *Then*

$$\mathcal{E}_{\Omega}^{(k,\alpha)}(u,\lambda,N) = O\left(\sum_{j=1}^{k-1} \frac{\alpha^j\left(e^{i\lambda j\frac{\log u}{\log N}}-1\right)}{u^{j+2}}\right),$$

where the constant implied by the O*-notation depends on* j, k *and* α.

PROOF. We have

$$\begin{aligned}
&\mathcal{E}_\Omega^{(k,\alpha)}(u,\lambda,N) = \\
&= \sum_{l=2}^{\infty} \frac{-l\, b_{\Omega,0}^{(k,\alpha)}(l)}{u^{l+1}} + e^{i\lambda\frac{\log u}{\log N}} \sum_{l=2}^{\infty} \frac{-l\alpha\, b_{\Omega,0}^{(k,\alpha)}(l-1)}{u^{l+1}} + \\
&+ \sum_{t=2}^{k-1} e^{i\lambda t\frac{\log u}{\log N}} \sum_{l=t+1}^{\infty} \frac{-l\alpha^t\, b_{\Omega,0}^{(k,\alpha)}(l-t)}{u^{l+1}} = \\
&= \sum_{l=2}^{\infty} \frac{-l}{u^{l+1}} \sum_{j=0}^{l\wedge(k-1)} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) e^{i\lambda j\frac{\log u}{\log N}} = \\
&= \sum_{l=2}^{\infty} \frac{-l}{u^{l+1}} \sum_{j=0}^{l\wedge(k-1)} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) \sum_{s=0}^{\infty} \frac{\left(i\lambda j\frac{\log u}{\log N}\right)^s}{s!}, \qquad (29)
\end{aligned}$$

where $l\wedge(k-1)$ denotes the minimum of l and $k-1$. The terms corresponding to $s=0$ in the sum (29) can be removed, as Lemmata 3-4 yield

$$\sum_{j=0}^{l\wedge(k-1)} \alpha^j b_{\Omega,0}(l-j) = 0.$$

In fact, if $2 \leqslant l \leqslant k-1$, then $\sum_{j=0}^{l} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) = \alpha^{l-1} b_{\Omega,0}^{(k,\alpha)}(1) + \alpha^l b_{\Omega,0}^{(k,\alpha)}(0) = 0$. If $l \geqslant k$, say $l = ck+d$, with $c \geqslant 1$ and $0 \leqslant d \leqslant k-1$, then

$$\sum_{j=0}^{k-1} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) = \begin{cases} b_{\Omega,0}^{(k,\alpha)}(ck) + \alpha^{k-1} b_{\Omega,0}^{(k,\alpha)}((c-1)k+1) = 0, & \text{if } d = 0; \\ \alpha^{d-1} b_{\Omega,0}^{(k,\alpha)}(ck+1) + \alpha^d b_{\Omega,0}^{(k,\alpha)}(ck) = 0, & \text{if } d \geqslant 1. \end{cases}$$

We proceed from (29), after noticing that the terms corresponding to $j=0$ can be removed too:

$$\mathcal{E}_\Omega^{(k,\alpha)}(u,\lambda,N) = \sum_{l=2}^{\infty} \frac{-l}{u^{l+1}} \sum_{j=0}^{l\wedge(k-1)} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) \sum_{s=1}^{\infty} \frac{\left(i\lambda j\frac{\log u}{\log N}\right)^s}{s!} =$$

$$= \sum_{l=2}^{\infty} \frac{-l}{u^{l+1}} \sum_{j=1}^{l\wedge(k-1)} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) \left(e^{i\lambda j \frac{\log u}{\log N}} - 1\right) =$$

$$= \sum_{j=1}^{k-1} \alpha^j \left(e^{i\lambda j \frac{\log u}{\log N}} - 1\right) \sum_{l=j+1}^{\infty} \frac{-l\, b_{\Omega,0}^{(k,\alpha)}(l-j)}{u^{l+1}} =$$

$$= \sum_{j=1}^{k-1} \frac{\alpha^j \left(e^{i\lambda j \frac{\log u}{\log N}} - 1\right)}{u^{j+2}} O(1),$$

where the constant implied by the O-notation depends on j, k and α. □

5.1. The main step

We can write

$$\varphi_{\Omega,N}^{(k,\alpha)}(\lambda) = \sum_{x\in\mathfrak{X}_N^{(k)}} \exp\left(i\lambda \sum_{p\leqslant N} \nu(p) \frac{\log p}{\log N}\right) P_{\Omega,N}^{(k,\alpha)}(\{x\}) =$$

$$= \sum_{\epsilon(p_1),\ldots,\epsilon(p_{\pi(N)})\in\{0,1,\ldots,k-1\}} \prod_{p\leqslant N} \exp\left(i\lambda\epsilon(p)\frac{\log p}{\log N}\right) P_{\Omega,N}^{(k,\alpha)}(\{\nu(p)=\epsilon(p)\}) =$$

$$= \prod_{p\leqslant N} \sum_{t=0}^{k-1} e^{i\lambda t \frac{\log p}{\log N}} P_{\Omega,N}^{(k,\alpha)}(\{\nu(p)=t\}).$$

Now, by Lemma 2 and (25), we get

$$\varphi_{\Omega,N}^{(k,\alpha)}(\lambda) = \prod_{p\leqslant N}\left(F_{\Omega,0}^{(k,\alpha)}(p) + F_{\Omega,1}^{(k,\alpha)}(p) e^{i\lambda\frac{\log p}{\log N}} + \sum_{t=2}^{k-1} e^{i\lambda t\frac{\log p}{\log N}} F_{\Omega,t}^{(k,\alpha)}(p)\right) =$$

$$= \prod_{p\leqslant N}\left(1 + \frac{\alpha}{p}\left(e^{i\lambda\frac{\log p}{\log N}} - 1\right) + \sum_{t=2}^{k-1} G_{\Omega,t,N}^{(k,\alpha)}(p,\lambda)\right), \tag{30}$$

where

$$G_{\Omega,t,N}^{(k,\alpha)}(u,\lambda) = \begin{cases} F_{\Omega,0}^{(k,\alpha)}(u) - 1 + \dfrac{\alpha}{u} + \left(F_{\Omega,1}^{(k,\alpha)}(u) - \dfrac{\alpha}{u}\right) e^{i\lambda\frac{\log u}{\log N}} + F_{\Omega,2}^{(k,\alpha)}(u) e^{2i\lambda\frac{\log u}{\log N}}, & t=2; \\ F_{\Omega,t}^{(k,\alpha)}(u) e^{i\lambda t\frac{\log u}{\log N}}, & 3\leqslant t\leqslant k-1. \end{cases}$$

Lemmata 3 and 4 imply that $G^{(k,\alpha)}_{\Omega,t,N}(u,\lambda)=O(1/u^t)$, for $2\leqslant t\leqslant k-1$, where the constants implied by the O-notations depend only on k and α. This means, in particular, that there exists $d_*=d_*(k,\alpha)$ such that for every $\lambda\in\mathbb{R}$, every $N\geqslant 2$ and every $p>d_*$, the complex numbers

$$z(p)=z^{(k,\alpha)}(p,\lambda,N)=1+\frac{\alpha}{p}\left(e^{i\lambda\frac{\log p}{\log N}}-1\right)+\sum_{t=2}^{k-1}G^{(k,\alpha)}_{\Omega,t,N}(p,\lambda)$$

belong to the open disk of radius $1/2$, centered at 1. Thus we can write $\log z(p)=$ $=\log|z(p)|+i\arg z(p)$ and choose a branch of the logarithm consistently for all $p>d_*$. Let us also assume that $d_*>|\alpha|$ (we shall use this fact later). We get

$$\varphi^{(k,\alpha)}_{\Omega,N}(\lambda)=\prod_{p\leqslant d_*}z^{(k,\alpha)}(p,\lambda,N)\cdot\widetilde{\varphi}^{(k,\alpha)}_{\Omega,N}(\lambda) \tag{31}$$

where, after taking the logarithm,

$$\log\widetilde{\varphi}^{(k,\alpha)}_{\Omega,N}(\lambda)=\sum_{d_*<p\leqslant N}\log z^{(k,\alpha)}(p,\lambda,N).$$

Abel summation yields

$$\log\widetilde{\varphi}^{(k,\alpha)}_{\Omega,N}(\lambda)=\pi(N)\mathcal{L}(N,\lambda,N)-\pi(d_*)\mathcal{L}(d_*,\lambda,N)-\int_{d_*}^{N}\pi(u)\frac{\partial}{\partial u}\mathcal{L}(u,\lambda,N)\mathrm{d}u, \tag{32}$$

where $\mathcal{L}(u,\lambda,N)=\mathcal{L}^{(k,\alpha)}(u,\lambda,N)=\log z^{(k,\alpha)}(u,\lambda,N)$.

Remark. As pointed out by A. J. Hildebrand to the author, one can try to estimate (30) using a result by Tenenbaum ([30], Chapter III.5). This approach will be the subject of future investigation. Likely, it will allow for a wider range for R in Theorem 1 and reduce the error term in the case of small η.

5.2. The boundary terms

Let us estimate the first two terms in the rhs of (32). By Lemmata 3-4

$$\pi(N)\mathcal{L}(N,\lambda,N)=\pi(N)\log\left(1+\frac{\alpha}{N}\left(e^{i\lambda}-1\right)+\sum_{t=2}^{k-1}G^{(k,\alpha)}_{\Omega,t,N}(N,\lambda)\right)=$$

$$= O\left(\frac{N}{\log N}\right)\log\left(1+O\left(\frac{1}{N}\right)(e^{i\lambda}-1)+O\left(\frac{1}{N^2}\right)(1+e^{i\lambda}+e^{2i\lambda})+\right.$$

$$\left.+\sum_{t=3}^{k-1}O\left(\frac{1}{N^t}\right)e^{it\lambda}\right)=O\left(\frac{1}{\log N}\right), \tag{33}$$

$$\pi(d_*)\mathcal{L}(d_*,\lambda,N)=\pi(d_*)\log\left(1+\frac{\alpha}{d_*}\left(e^{i\lambda\frac{d_*}{\log N}}-1\right)+O\left(\frac{1}{N^2}\right)\right)=$$

$$=O(\varepsilon)+O\left(\tfrac{1}{N^2}\right), \tag{34}$$

where d_* is as above, and the constants implied by the O-notation depend only on k and α. The two boundary terms in the rhs of (32) are therefore $O\left(\frac{1}{\log N}\right)+O(\varepsilon)$ as $N\to\infty$.

5.3. The integral term

Let us now estimate the integral in the rhs of (32). Recall the functions $G^{(k,\alpha)}_{\Omega,t,N}(u,\lambda)$ and $\mathcal{E}^{(k,\alpha)}_{\Omega}(u,\lambda,N)$ introduced above. We have

$$\frac{\partial}{\partial u}\mathcal{L}(u,\lambda,N)=\frac{1}{z^{(k,\alpha)}(u,\lambda,N)}\left(-\frac{\alpha}{u^2}\left(e^{i\lambda\frac{\log u}{\log N}}-1\right)+\frac{i\alpha\lambda}{u^2\log N}e^{i\lambda\frac{\log u}{\log N}}+\right.$$

$$+\frac{\mathrm{d}}{\mathrm{d}u}F^{(k,\alpha)}_{\Omega,0}(u)-\frac{\alpha}{u^2}+\left(\frac{\mathrm{d}}{\mathrm{d}u}F^{(k,\alpha)}_{\Omega,1}(u)+\frac{\alpha}{u^2}\right)e^{i\lambda\frac{\log u}{\log N}}+\frac{i\lambda}{u\log N}\left(F^{(k,\alpha)}_{\Omega,1}(u)-\frac{\alpha}{u}\right)e^{i\lambda\frac{\log u}{\log N}}+$$

$$\left.+\sum_{t=2}^{k-1}\left(\frac{\mathrm{d}}{\mathrm{d}u}F^{(k,\alpha)}_{\Omega,t}(u)e^{i\lambda t\frac{\log u}{\log N}}+\frac{i\lambda t}{u\log N}F^{(k,\alpha)}_{\Omega,t}(u)e^{i\lambda t\frac{\log u}{\log N}}\right)\right)=$$

$$=\left(1-\frac{\frac{\alpha}{u}\left(e^{i\lambda\frac{\log u}{\log N}}-1\right)+\sum_{t=2}^{k-1}G^{(k,\alpha)}_{\Omega,t,N}(u,\lambda)}{z^{(k,\alpha)}(u,\lambda,N)}\right)\left(-\frac{\alpha}{u^2}\left(e^{i\lambda\frac{\log u}{\log N}}-1\right)+\frac{i\alpha\lambda}{u^2\log N}e^{i\lambda\frac{\log u}{\log N}}+\right.$$

$$\left.+\mathcal{E}^{(k,\alpha)}_{\Omega}(u,\lambda,N)+\frac{\lambda}{u^3\log N}O(1)+\sum_{t=2}^{k-1}\frac{\lambda}{u^{t+1}\log N}O(1)\right). \tag{35}$$

Since $d_*\leqslant u\leqslant N$, let us notice that $\frac{\alpha}{u}\left(e^{i\lambda\frac{\log u}{\log N}}-1\right)\frac{1}{z(u,\lambda,N)}=\frac{1}{u}\left(e^{i\lambda\frac{\log u}{\log N}}-1\right)O(1)$. Moreover, $\frac{1}{z(u,\lambda,N)}\sum_{t=2}^{k-1}G^{(k,\alpha)}_{\Omega,t,N}(u,\lambda)=O\left(\frac{1}{u^2}\right)$. Thus, the first bracket in (35) can be written as $I_{1,1}+I_{1,2}+I_{1,3}$, where $I_{1,j}=I^{(k,\alpha)}_{\Omega,1,j}(u,\lambda,N)$, $j=1,2,3$, and

$$\begin{aligned}
I_{1,1} &= 1,\\
I_{1,2} &= \frac{O(1)}{u}\left(e^{i\lambda\frac{\log u}{\log N}}-1\right),\\
I_{1,3} &= O\left(\frac{1}{u^2}\right).
\end{aligned}$$

Let us look at the second bracket in (35). By Lemma (5) we get $k-1$ terms of the form $\frac{O(1)}{u^{j+2}}\left(e^{i\lambda j\frac{\log u}{\log N}}-1\right)$, $1\leqslant j\leqslant k-1$ (it would not be enough to estimate them as $O(1/u^{j+2})$ as we want a better control of error terms). The implied constants can be chosen in order to not depend upon j but only on k and α. Let us also combine the last two terms in (35) into $O\left(\frac{\varepsilon}{u^3}\right)$. This means that the the second bracket in (35) can be written as $I_{2,1}+I_{2,2}+\ldots+I_{2,k+2}$, where $I_{2,j}=I^{(k,\alpha)}_{\Omega,2,j}(u,\lambda,N)$, $1\leqslant j\leqslant k+2$, and

$$\begin{aligned}
I_{2,1} &= -\frac{\alpha}{u^2}\left(e^{i\lambda\frac{\log u}{\log N}}-1\right),\\
I_{2,2} &= \frac{i\alpha\varepsilon}{u^2}e^{i\lambda\frac{\log u}{\log N}},\\
I_{2,j} &= \frac{O(1)\left(e^{i\lambda(j-2)\frac{log u}{\log N}}-1\right)}{u^j}, \qquad 3\leqslant j\leqslant k+1,\\
I_{2,k+2} &= O\left(\frac{\varepsilon}{u^3}\right).
\end{aligned}$$

Recall that all the constants implied by the above O-notations depend only on k,α, and not on λ, u, and N. Let us also write

$$\pi(u)=\frac{u}{\log u}+O\left(\frac{u}{\log^2 u}\right)=I_{0,1}+I_{0,2}.$$

The integral in the rhs of (32) becomes now

$$-\int_{d_*}^{N}\pi(u)\frac{\partial}{\partial u}\mathcal{L}(u,\lambda,N)\mathrm{d}u=-\sum_{j=1}^{2}\sum_{j'=1}^{3}\sum_{j''=1}^{k+2}\int_{d_*}^{N}I_{0,j}I_{1,j'}I_{2,j''}\mathrm{d}u. \tag{36}$$

We claim that, amongst the $6k+12$ integrals in (36), the one corresponding to $j=j'=j''=1$ is the main term, and the remaining $6k+11$ integrals are $O(1/\log N)+O(\varepsilon\log|\varepsilon|)$. Let us perform the change of variables $v=\log u/\log N$. We have

$$J_{1,1,1}=-\int_{d_*}^{N} I_{0,1}I_{1,1}I_{2,1}\mathrm{d}u=\alpha\int_{d_*}^{N}\frac{e^{i\lambda\frac{\log u}{\log N}}-1}{u\log u}\mathrm{d}u=\alpha\int_{\log d_*/\log N}^{1}\frac{e^{i\lambda v}-1}{v}\mathrm{d}v=$$

$$=\alpha\int_{0}^{1}\frac{e^{i\lambda v}-1}{v}\mathrm{d}v+O(\varepsilon). \tag{37}$$

The fact that

$$J_{j,j',j''}=-\int_{d_*}^{N} I_{0,j}I_{1,j'}I_{2,j''}\mathrm{d}u=O\left(\frac{1}{\log N}\right)+O(\varepsilon\log|\varepsilon|) \tag{38}$$

for all other values of j,j',j'' is shown in Appendix A. By (32- 34, 36-38), we have that

$$\log\widetilde{\varphi}^{(k,\alpha)}_{*,N}(\lambda)=\alpha\int_{0}^{1}\frac{e^{i\lambda v}-1}{v}\mathrm{d}v+O\left(\frac{1}{\log N}\right)+O(\varepsilon\log|\varepsilon|)\,. \tag{39}$$

The first factor in the rhs of (31) can be written as follows as $N\to\infty$

$$\prod_{p\leqslant d_*} z^{(k,\alpha)}(p,\lambda,N)=\prod_{p\leqslant d_*}\sum_{t=0}^{k-1} e^{i\lambda\frac{\log p}{\log N}}F^{(k,\alpha)}_{\Omega,t}(p)=$$

$$=\prod_{p\leqslant d_*}\left(\sum_{t=0}^{k-1}F^{(k,\alpha)}_{\Omega,t}(p)+O(\varepsilon)\right)=1+O(\varepsilon), \tag{40}$$

where we used the fact that

$$\sum_{t=0}^{k-1}F^{(k,\alpha)}_{\Omega,t}(p)=\frac{\sum_{t=0}^{k-1}\alpha^t p^{k-1-t}}{p^{k-1}+\alpha p^{k-2}+\alpha^2p^{k-3}+\ldots+\alpha^{k-1}}=1.$$

Now, combining (39) and (40), we get

$$\varphi_{\Omega,N}^{(k,\alpha)}(\lambda) = (1+O(\varepsilon))\exp\left(\alpha\int_0^1 \frac{e^{i\lambda v}-1}{v}\mathrm{d}v + O\left(\frac{1}{\log N}\right) + O(\varepsilon\log|\varepsilon|)\right) =$$

$$= \exp\left(\alpha\int_0^1 \frac{e^{i\lambda v}-1}{v}\mathrm{d}v\right)\left(1+O\left(\frac{1}{\log N}\right)+O(\varepsilon\log|\varepsilon|)\right)$$

and this concludes the proof of Theorem 4.

6. An Example

We pointed out that, for general α and f, the integral

$$I = \int_{|\lambda|\leqslant R} \widehat{f}(\lambda)\varphi^{(\alpha)}(\lambda)\mathrm{d}\lambda$$

in (2) may tend to zero as $N\to\infty$ (recall: $R=R(N)$). In this section we discuss an example where $\alpha=-1$ and the above integral is bounded away from zero as $N\to\infty$. We have the following

Proposition. *Let $f(u)=\mathbf{1}_{[-1,1]}e^{-1/(1-u^2)}$. Then, for sufficiently large N,*

$$\left|\int_{|\lambda|\leqslant R} \widehat{f}(\lambda)\varphi^{(-1)}(\lambda)\mathrm{d}\lambda\right| \geqslant \frac{3}{100}. \qquad (41)$$

Proof. Since $S_{\Omega,f}(k,-1;N)$ is real, then by Theorem 1, $\operatorname{Im} I = o(1)$ as $N\to\infty$. Thus, for every $\delta>0$, there exists N_δ such that $|\operatorname{Im} I|\leqslant\delta$ for every $N\geqslant N_\delta$. Let us now focus on $\operatorname{Re} I$. Notice that $\widehat{f}$ is real-valued and thus $\operatorname{Re} I = \int_{|\lambda|\leqslant R}\widehat{f}(\lambda)\,\times$ $\times\operatorname{Re}\varphi^{(-1)}(\lambda)\mathrm{d}\lambda$. We can write

$$\operatorname{Re}\varphi^{(-1)}(\lambda) = e^{\gamma-\operatorname{Ci}(\lambda)}\lambda\cos(\operatorname{Si}(\lambda)) \qquad (42)$$

where $\mathrm{Ci}(\lambda) = -\int_\lambda^\infty \frac{\cos t\,dt}{t}$ and $\mathrm{Si}(\lambda) = \int_0^\lambda \frac{\sin t\,dt}{t}$. We can use the expression (42) to obtain the estimate[2)] $|\operatorname{Re}\varphi^{(-1)}(\lambda)| \leqslant e^\gamma$, valid for all $\lambda \in \mathbb{R}$. Let us write

$$\operatorname{Re} I = \int_{|\lambda|\leqslant R} \widehat{f}(\lambda) \operatorname{Re}\varphi^{(-1)}(\lambda)\mathrm{d}\lambda = \int_{|\lambda|\leqslant r} + \int_{r<|\lambda|\leqslant R} = I_1 + I_2 \tag{43}$$

where $r > 0$ does not depend on N and will be chosen later. The idea is that I_1 can be estimated numerically with arbitrary precision using, for example, a quadrature method for the integral. More precisely, let $F(\lambda) = \widehat{f}(\lambda) \operatorname{Re}\varphi^{(-1)}(\lambda)$ and observe that $F(\lambda) = F(-\lambda)$, so that $\int_{|\lambda|\leqslant r} F(\lambda)\mathrm{d}\lambda = 2\int_0^r F(\lambda)\mathrm{d}\lambda$. We have

$$\int_0^r F(\lambda)\mathrm{d}\lambda = h\sum_{m=1}^{M} F\big(h(m-\tfrac{1}{2})\big) + \mathcal{E}(r, M),$$

where $h = \frac{r}{M}$ and $\mathcal{E}(r, M) = \frac{rh^2}{24}F''(\rho)$ for some $0 < \rho < r$. The graphs of F, F' and F'' are shown in Figure 2. One can see that $|F''(\lambda)| \leqslant \frac{1}{2}$.

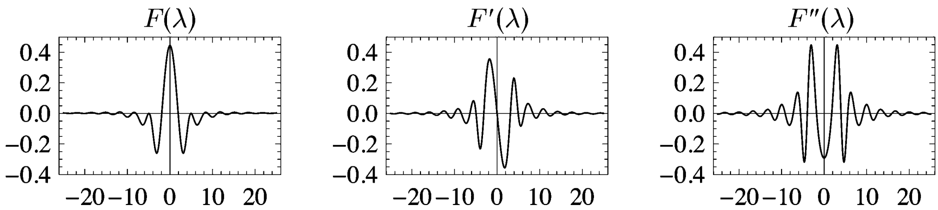

Fig. 2. The graphs of F, F' and F''

Since F can be estimated to an arbitrary precision, we can estimate the Riemann sum $h\sum_{m=1}^{M} F\big(h(m-\frac{1}{2})\big)$ arbitrarily well for fixed r and M, and we have the estimate $|\mathcal{E}(r, M)| \leqslant \frac{h^2 r}{48}$. The integral I_2 in (43) can be estimated explicitly as follows:

$$\Bigg|\int_{r<|\lambda|\leqslant R} F(\lambda)\mathrm{d}\lambda\Bigg| \leqslant 2\int_r^R |F(\lambda)|\mathrm{d}\lambda.$$

[2)] Although $|\varphi^{(-1)}(\lambda)| = O(\lambda)$ as $\lambda \to \infty$, only the imaginary part of the function $\varphi^{(-1)}$ is unbounded. This property holds true only for $\alpha = -1$.

By means of stationary phase method, it can be shown that the function $\widehat{f}$ satisfies the asymptotic

$$\widehat{f}(\lambda) \sim \frac{1}{\pi} \operatorname{Re}\left\{\sqrt{\frac{-i\pi}{\sqrt{2i}\lambda^{\frac{3}{2}}}} e^{i\lambda - \frac{1}{4} - \sqrt{2i\lambda}}\right\}$$

as $\lambda \to \infty$. Moreover, one can check that $|\widehat{f}(\lambda)| \leqslant \dfrac{3}{2\pi} e^{-\sqrt{\lambda}} \lambda^{-3/4}$ for $\lambda \geqslant 1$. This implies that $|F(\lambda)| \leqslant \frac{3e^{\gamma}}{2\pi} e^{-\sqrt{\lambda}} \lambda^{-3/4}$. We get

$$\left| \int\limits_{r<|\lambda|\leqslant R} F(\lambda) \mathrm{d}\lambda \right| \leqslant \frac{3e^{\gamma}}{\pi} \int\limits_{r}^{R} e^{-\sqrt{\lambda}} \lambda^{-3/4} \mathrm{d}\lambda = \frac{6e^{\gamma}}{\sqrt{\pi}} \left[\operatorname{erf}(\lambda^{1/4})\right]_{\lambda=r}^{\lambda=R} \leqslant \frac{6e^{\gamma}}{\sqrt{\pi}} \operatorname{erfc}(r^{1/4}),$$

where $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int\limits_{0}^{x} e^{-t^2} \mathrm{d}t$ and $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. Now, let $r = 5$ and $M = 1000$. We have

$$2h \sum_{m=1}^{M} F\big(h(m - \tfrac{1}{2})\big) = 0.23821680383626264857 \pm 10^{-20},$$

$$\frac{h^2 r}{24} = 5.208(3) \cdot 10^{-6},$$

$$\frac{6e^{\gamma}}{\sqrt{\pi}} \operatorname{erfc}(r^{1/4}) = 0.20771652138513808389 \pm 10^{-20},$$

and therefore

$$\left| \int\limits_{|\lambda|\leqslant R} F(\lambda) \mathrm{d}\lambda \right| \geqslant \left| 2h \sum_{m=1}^{M} F\big(h(m - \tfrac{1}{2})\big) \right| - |\mathcal{E}(r, M)| - \left| \int\limits_{r<|\lambda|\leqslant R} F(\lambda) \mathrm{d}\lambda \right| \geqslant \frac{3}{100}.$$

This shows that, for sufficiently large N,

$$\left| \int\limits_{|\lambda|\leqslant R(N)} \widehat{f}(\lambda) \varphi^{(-1)}(\lambda) \mathrm{d}\lambda \right| > \frac{3}{100} + \frac{1}{2500}. \tag{44}$$

Now let us choose $\delta = \dfrac{1}{2500}$ so that $|\operatorname{Im} I| \leqslant \delta$ for sufficiently large N. This, together with (44), yields the desired statement (41). $\square$

A. Estimate of the error terms

This appendix contains the estimates of the integrals $J_{j,j',j''}$ from (38) for $1 \leqslant j \leqslant 2$, $1 \leqslant j' \leqslant 3$ and $1 \leqslant j'' \leqslant k+2$, except for $(j, j', j'') = (1, 1, 1)$ already treated in (37). Recall that $\varepsilon = \frac{\lambda}{\log N}$ is assumed to tend to zero as $N \to \infty$. We will assume for simplicity that $\lambda > 0$. We group error terms in different sections, according to the methods used to estimate them.

A.1. $O(\varepsilon)$ terms

These error terms are very easy, due to the fact that $I_{2,2}$ and $I_{2,k+2}$ are $O(\varepsilon/u^2)$ and $O(\varepsilon/u^3)$ respectively and the corresponding error terms can be written as $O(\varepsilon\mathcal{I})$, where $\mathcal{I}$ is an absolutely convergent integral. We have

$$J_{1,2,2} = -\int_{d_*}^{N} I_{0,1}I_{1,2}I_{2,2}\mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right) e^{i\lambda\frac{\log u}{\log N}}}{u^2 \log u}\mathrm{d}u\right) = O(\varepsilon),$$

$$J_{1,3,2} = -\int_{d_*}^{N} I_{0,1}I_{1,3}I_{2,2}\mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{e^{i\lambda\frac{\log u}{\log N}}}{u^3 \log u}\mathrm{d}u\right) = O(\varepsilon),$$

$$J_{2,1,2} = -\int_{d_*}^{N} I_{0,2}I_{1,1}I_{2,2}\mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{e^{i\lambda\frac{\log u}{\log N}}}{u \log^2 u}\mathrm{d}u\right) = O(\varepsilon),$$

$$J_{2,2,2} = -\int_{d_*}^{N} I_{0,2}I_{1,2}I_{2,2}\mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right) e^{i\lambda\frac{\log u}{\log N}}}{u^2 \log^2 u}\mathrm{d}u\right) = O(\varepsilon),$$

$$J_{2,3,2} = -\int_{d_*}^{N} I_{0,2}I_{1,3}I_{2,2}\mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{e^{i\lambda\frac{\log u}{\log N}}}{u^3 \log^2 u}\mathrm{d}u\right) = O(\varepsilon),$$

$$J_{1,1,k+2} = -\int_{d_*}^{N} I_{0,1} I_{1,1} I_{2,k+2} \mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{\mathrm{d}u}{u^2 \log u}\right) = O(\varepsilon),$$

$$J_{1,2,k+2} = -\int_{d_*}^{N} I_{0,1} I_{1,2} I_{2,k+2} \mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1\right)}{u^3 \log u} \mathrm{d}u\right) = O(\varepsilon),$$

$$J_{1,3,k+2} = -\int_{d_*}^{N} I_{0,1} I_{1,3} I_{2,k+2} \mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{\mathrm{d}u}{u^4 \log u}\right) = O(\varepsilon),$$

$$J_{2,1,k+2} = -\int_{d_*}^{N} I_{0,2} I_{1,1} I_{2,k+2} \mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{\mathrm{d}u}{u^2 \log^2 u}\right) = O(\varepsilon),$$

$$J_{2,2,k+2} = -\int_{d_*}^{N} I_{0,2} I_{1,2} I_{2,k+2} \mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1\right)}{u^3 \log^2 u} \mathrm{d}u\right) = O(\varepsilon),$$

$$J_{2,3,k+2} = -\int_{d_*}^{N} I_{0,2} I_{1,3} I_{2,k+2} \mathrm{d}u = O\left(\varepsilon \int_{d_*}^{N} \frac{\mathrm{d}u}{u^4 \log^2 u}\right) = O(\varepsilon).$$

A.2. $O(\varepsilon \log \varepsilon)$ terms

The analysis of these error terms yields estimates of the form $O\left(\frac{1}{\log N}\right) + O(\varepsilon \log \varepsilon)$. We shall need the special function

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} \mathrm{d}t, \quad z \in \mathbb{R},$$

where the integral is in the sense principal value due to the singularity at zero. For complex arguments $\mathrm{Ei}(z)$ is defined by analytic continuation. Notice that $v \mapsto \mathrm{Ei}(iv)$ is the antiderivative of $v \mapsto e^{iv}/v$. Moreover, for $\tau, w \in \mathbb{R}$,

$$\mathrm{Ei}(\tau) = \gamma + \log \tau + O(\tau) \quad \text{as } \tau \to 0+; \tag{45}$$

$$\mathrm{Ei}(i\tau) = \gamma + \frac{i\pi}{2} + \log \tau + O(\tau) \quad \text{as } \tau \to 0+; \tag{46}$$

$$\mathrm{Ei}(w) = \frac{e^w}{w}\left(1 + O\left(\frac{1}{w}\right)\right) \quad \text{as } w \to \infty; \tag{47}$$

$$\mathrm{Ei}(iw) = i\pi - \frac{e^{iw}}{w}\left(1 + O\left(\frac{1}{w}\right)\right) \quad \text{as } w \to \infty; \tag{48}$$

see [1]. By (46, 48) we have

$$\begin{aligned} J_{1,1,2} &= -\int_{d_*}^{N} I_{0,1}I_{1,1}I_{2,2}\mathrm{d}u = -i\alpha\varepsilon \int_{d_*/\log N}^{1} \frac{e^{i\lambda v}}{v}\mathrm{d}v = -i\alpha\varepsilon\ \mathrm{Ei}(i\lambda v)\big|_{v=d_*/\log N}^{v=1} = \\ &= -i\alpha\varepsilon\left(\mathrm{Ei}(i\lambda) - \mathrm{Ei}(i\varepsilon d_*)\right) = \\ &= O\left(\varepsilon\left(O(1) - \gamma - \frac{i\pi}{2} - \log\varepsilon + \log d_* + O(\varepsilon)\right)\right) = O(\varepsilon\log\varepsilon). \end{aligned}$$

Let Ci and Si be the special functions introduced in Section 6. They satisfy the estimates

$$i\mathrm{Ci}(y) - \mathrm{Si}(y) = O(1) \quad \text{for } |y| \geqslant c > 0; \tag{49}$$

$$i\mathrm{Ci}(\tau) - \mathrm{Si}(\tau) = i(\gamma + \log\tau) - \tau + O(\tau^2) \quad \text{as } \tau \to 0; \tag{50}$$

see [1]. We have

$$J_{2,1,1} = -\int_{d_*}^{N} I_{0,2}I_{1,1}I_{2,1}\mathrm{d}u = O\left(\int_{d_*}^{N} \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right)}{u\log^2 u}\mathrm{d}u\right) = O\left(\widetilde{J}_{2,1,1}(x)\Big|_{\log d_*}^{\log N}\right),$$

where

$$\widetilde{J}_{2,1,1}(x) = \frac{1}{x} - \frac{e^{i\varepsilon x}}{x} + \varepsilon\left(i\mathrm{Ci}(\varepsilon x) - \mathrm{Si}(\varepsilon x)\right).$$

By (49, 50) we get

$$J_{2,1,1} = O\left(\frac{1}{\log N}\right) + O(\varepsilon\log\varepsilon).$$

A.3. Mixed terms

Here we present the estimates for error terms of order $O(\varepsilon) + O(1/\log N)$. In this section we assume that $3 \leqslant j \leqslant k+1$. The following integrals can be estimated using the properties (45, 47) of the exponential integral function Ei.

$$J_{2,1,j} = -\int_{d_*}^{N} I_{0,2} I_{1,1} I_{2,j} \mathrm{d}u = O\left(\int_{d_*}^{N} \frac{\left(e^{i\lambda (j-2)\frac{\log u}{\log N}} - 1\right)}{u^{j-1}\log^2 u} \mathrm{d}u \right) = O\left(\widetilde{J}_{2,1,j}(x)\Big|_{\log d_*}^{\log N} \right),$$

$$J_{2,2,j} = -\int_{d_*}^{N} I_{0,2} I_{1,2} I_{2,j} \mathrm{d}u = O\left(\int_{d_*}^{N} \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1\right)\left(e^{i\lambda (j-2)\frac{\log u}{\log N}} - 1\right)}{u^{j}\log^2 u} \mathrm{d}u \right) =$$

$$= O\left(\widetilde{J}_{2,2,j}(x)\Big|_{\log d_*}^{\log N} \right),$$

$$J_{2,3,j} = -\int_{d_*}^{N} I_{0,2} I_{1,3} I_{2,j} \mathrm{d}u = O\left(\int_{d_*}^{N} \frac{\left(e^{i\lambda (j-2)\frac{\log u}{\log N}} - 1\right)}{u^{j+1}\log^2 u} \mathrm{d}u \right) = O\left(\widetilde{J}_{2,3,j}(x)\Big|_{\log d_*}^{\log N} \right),$$

where

$$\widetilde{J}_{2,1,j}(x) = \frac{1}{x}\left(e^{-(j-2)x} - e^{-(j-2)(1-i\varepsilon)x}\right) + \\ +(j-2)\mathrm{Ei}(-(j-2)x) - (j-2)(1-i\varepsilon)\,\mathrm{Ei}(-(j-2)(1-i\varepsilon)\,x)\,,$$

$$\widetilde{J}_{2,2,j}(x) = \frac{1}{x}\left(e^{-(j-1-i\varepsilon)x} - e^{-(j-1)(1-i\varepsilon)x} + e^{-(j-1-(j-2)i\varepsilon)x} - e^{-(j-1)x}\right) + \\ + (j-1-i\varepsilon)\,\mathrm{Ei}(-\,(j-1-i\varepsilon)\,x) - \\ -(j-1)\,(1-i\varepsilon)\,\mathrm{Ei}(-(j-1)\,(1-i\varepsilon)\,x) + \\ + (j-1-(j-2)i\varepsilon)\,\mathrm{Ei}(-\,(j-1-(j-2)i\varepsilon)\,x) - \\ -(j-1)\mathrm{Ei}(-(j-1)x),$$

$$\widetilde{J}_{2,3,j}(x) = \frac{1}{x}\left(e^{-jx} - e^{-(j-(j-2)i\varepsilon)x}\right) + \\ +j\mathrm{Ei}(-jx) - (j-(j-2)i\varepsilon)\,\mathrm{Ei}(-\,(j-(j-2)i\varepsilon)\,x)\,.$$

By (45) and (47) we get

$$J_{2,1,j} = O\left(\frac{1}{N^{j-2}\log N}\right) + O(\varepsilon)\,,$$
$$J_{2,2,j} = O\left(\frac{1}{N^{j-1}\log N}\right) + O\left(\varepsilon^2\right)\,,$$
$$J_{2,3,j} = O\left(\frac{1}{N^{j}\log N}\right) + O(\varepsilon)\,.$$

For the remaining error terms we shall need the incomplete gamma function

$$\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}\mathrm{d}t, \quad z\in\mathbb{R}, \tag{51}$$

and defined for complex z by analytic continuation. We will only need the cases $a=0$ and $a=-1$. It satisfies, for $w,\tau\in\mathbb{R}$ and $c,z\in\mathbb{C}$,

$$\Gamma(0,w) = e^{-w}\left(\frac{1}{w} + O\left(\frac{1}{w^2}\right)\right), \qquad w\to\infty; \tag{52}$$
$$\Gamma(a,c+z\tau) = \Gamma(a,c) - \frac{e^{-c}}{c^{1-a}}z\tau + O(\tau^2), \qquad \tau\to 0; \tag{53}$$

see [1]. We have

$$J_{1,1,j} = -\int_{d_*}^{N} I_{0,1}I_{1,1}I_{2,j}\mathrm{d}u = O\left(\int_{d_*}^{N}\frac{\left(e^{i\lambda(j-2)\frac{\log u}{\log N}}-1\right)}{u^{j-1}\log u}\mathrm{d}u\right) = O\left(\widetilde{J}_{1,1,j}(x)\Big|_{\log d_*}^{\log N}\right),$$

$$J_{1,2,j} = -\int_{d_*}^{N} I_{0,1}I_{1,2}I_{2,j}\mathrm{d}u = O\left(\int_{d_*}^{N}\frac{\left(e^{i\lambda\frac{\log u}{\log N}}-1\right)\left(e^{i\lambda(j-2)\frac{\log u}{\log N}}-1\right)}{u^{j}\log u}\mathrm{d}u\right) =$$
$$= O\left(\widetilde{J}_{1,2,j}(x)\Big|_{\log d_*}^{\log N}\right),$$

$$J_{1,3,j} = -\int_{d_*}^{N} I_{0,1}I_{1,3}I_{2,j}\mathrm{d}u = O\left(\int_{d_*}^{N}\frac{\left(e^{i\lambda(j-2)\frac{\log u}{\log N}}-1\right)}{u^{j+1}\log u}\mathrm{d}u\right) = O\left(\widetilde{J}_{1,3,j}(x)\Big|_{\log d_*}^{\log N}\right),$$

$$J_{1,2,1} = -\int_{d_*}^{N} I_{0,1}I_{1,2}I_{2,1}\mathrm{d}u = O\left(\int_{d_*}^{N} \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right)^2}{u^2 \log u}\mathrm{d}u\right) = O\left(\widetilde{J}_{1,2,1}(x)\Big|_{\log d_*}^{\log N}\right),$$

where

$$\begin{aligned}
\widetilde{J}_{1,1,j}(x) &= \Gamma(0,(j-2)x) - \Gamma(0,(j-2)(1-i\varepsilon)x),\\
\widetilde{J}_{1,2,j}(x) &= -\Gamma(0,(j-1)x) - \Gamma(0,(j-1)(1-i\varepsilon)x) +\\
&\qquad + \Gamma(0,(j-1-i\varepsilon)x) + \Gamma(0,(j-1-(j-2)i\varepsilon)x),\\
\widetilde{J}_{1,3,j}(x) &= \Gamma(0,jx) - \Gamma(0,(j-(j-2)i\varepsilon)x),\\
\widetilde{J}_{1,2,1}(x) &= -\Gamma(0,x) + 2\Gamma(0,(1-i\varepsilon)x) - \Gamma(0,(1-2i\varepsilon)x).
\end{aligned}$$

From (52) and (53) we get

$$\begin{aligned}
J_{1,1,j} &= O\left(\frac{1}{N^{j-2}\log N}\right) + O(\varepsilon),\\
J_{1,2,j} &= O\left(\frac{1}{N^{j-1}\log N}\right) + O(\varepsilon).\\
J_{1,3,j} &= O\left(\frac{1}{N^{j}\log N}\right) + O(\varepsilon).\\
J_{1,2,1} &= O\left(\frac{1}{N\log N}\right) + O\left(\varepsilon^2\right).
\end{aligned}$$

The following error term estimates involve a combination of (45,47) and (52,53).

$$J_{1,3,1} = -\int_{d_*}^{N} I_{0,1}I_{1,3}I_{2,1}\mathrm{d}u = O\left(\int_{d_*}^{N} \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right)}{u^3 \log u}\mathrm{d}u\right) = O\left(\widetilde{J}_{1,3,1}(x)\Big|_{\log d_*}^{\log N}\right),$$

$$J_{2,2,1} = -\int_{d_*}^{N} I_{0,2}I_{1,2}I_{2,1}\mathrm{d}u = O\left(\int_{d_*}^{N} \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right)^2}{u^2 \log^2 u}\mathrm{d}u\right) = O\left(\widetilde{J}_{2,2,1}(x)\Big|_{\log d_*}^{\log N}\right),$$

$$J_{2,3,1} = -\int_{d_*}^{N} I_{0,2}I_{1,3}I_{2,1}\mathrm{d}u = O\left(\int_{d_*}^{N} \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right)}{u^3 \log^2 u}\mathrm{d}u\right) = O\left(\widetilde{J}_{2,2,1}(x)\Big|_{\log d_*}^{\log N}\right),$$

where

$$\widetilde{J}_{1,3,1} = -\mathrm{Ei}(-2x) - \Gamma(0,(2-i\varepsilon)\,x)\,,$$
$$\widetilde{J}_{2,2,1} = -\mathrm{Ei}(-x) + 2\,(1-i\varepsilon)\,\Gamma(-1,(1-i\varepsilon)\,x) - (1{-}2i\varepsilon)\,\Gamma(-1,(1{-}2i\varepsilon)\,x) - \frac{1}{xe^{x}}\,,$$
$$\widetilde{J}_{2,3,1} = 2\mathrm{Ei}(-2x) + \frac{1}{xe^{2x}} - (2-i\varepsilon)\,\Gamma(-1,(2-i\varepsilon)\,x)\,.$$

We get

$$J_{1,3,1} = O\left(\frac{1}{N\log N}\right) + O(\varepsilon)\,,$$
$$J_{1,2,2} = O\left(\frac{1}{N\log^2 N}\right) + O\left(\varepsilon^2\right)\,,$$
$$J_{2,3,1} = O\left(\frac{1}{N}\right) + O(\varepsilon)\,.$$

Combining all the error terms discussed in this appendix we obtain the desired estimate (38).

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