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On a quantitative form of Bézivin's method for q-series

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Abstract: In this work we obtain rather good linear independence measures for the values of a certain class of q-series. The result is obtained by using a quantitative refinement of Bézivin's method for q-series. This refinement was recently developed by Rochev in rational case and the present paper extends this to general algebraic number fields both in the Archimedean and p-adic case.

Keywords: q-series; linear independence measure; Bézivin's method

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1. Introduction and results

Let K denote an algebraic number field of degree $d = [K : \mathbb{Q}]$, and denote by O_K the ring of integers of K. For any place w of K let $|.|_w$ be the corresponding valuation normalized in such a way that $|p|_w = p^{-1}$ for a finite place w lying over p, and $|x|_w = |x|$ $(x \in \mathbb{Q})$ for an infinite place w. In the present paper we are interested in the class of q-series of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{k=1}^{n} P(q^k)}, \quad P(q^k) \neq 0, k = 1, 2, \dots, \tag{1}$$

where $P(x) \in K[x]$, deg $P = D \geqslant 1$. We now fix a place v of K and $q \in K$ satisfying $|q|_v > 1$. Then the value $f(\alpha)$ at any point $\alpha \in K$ is defined in K_v, the completion of K with respect to v. Our main aim here is to give a general linear independence measure for the values of the function $f(z)$ and its derivatives.

The most simple cases of $f(z)$ are the Tschakaloff function $(P(x) = x)$ and q-exponential function $(P(x) = x - 1)$,

$$T_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{q^{\binom{n+1}{2}}}, \quad E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q-1)\cdots(q^n-1)} = \prod_{n=1}^{\infty}\left(1+\frac{z}{q^n}\right).$$

The study of arithmetic properties of these functions has a long history, we refer to [6] and [15] for this.

In 1988 Bézivin [4] introduced a new method for considering the linear independence of the values of series

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{k=1}^{n} A(k)},$$

where $\{A(n)\}$ is a linear recurrence sequence satisfying certain conditions. Essential in this method is the use of rationality criterions for functions, see also André [3] for this kind of ideas. Bézivin's result applies to $f(z)$ and gives the following theorem.

Theorem Be. *Let K be $\mathbb{Q}$ or an imaginary quadratic field and v the infinite place, and let q be an integer in K with $|q| > 1$. Assume that $\alpha_1, \ldots, \alpha_m \in K^*$ satisfy*

$$\frac{\alpha_i}{\alpha_j} \neq q^n \quad \forall\, i \neq j; \quad n \in \mathbb{Z}, \tag{2}$$

$$\alpha_j \neq P(0)q^n \quad \forall\, j = 1, ..., m; \quad n \in \mathbb{Z}. \tag{3}$$

Then, for any positive integer s, the $1 + ms$ numbers

$$1, f^{(\sigma)}(\alpha_j), \quad j = 1, \ldots, m; \quad \sigma = 0, 1, \ldots, s-1,$$

are linearly independent over K.

In [1] this result was extended to general K, v and to the more general set of $1 + msD$ numbers

$$1, f^{(\sigma)}(q^k\alpha_j), \quad j = 1, \ldots, m; k = 0, 1, \ldots, D-1; \sigma = 0, 1, \ldots, s-1. \tag{4}$$

Again the proof uses Bézivin's method. This result with its quantitative refinement was then obtained in [2] by using an analogue of Siegel's method for a system of linear q-difference equations. To present this statement we define the height of $\alpha \in K^*$ by

$$h(\alpha) = \prod_w \max(1, |\alpha|_w{}^{d_w/d}),$$

where the product is over all places w of K and $d_w = [K_w : \mathbb{Q}_w]$. Furthermore, for nonzero $\underline{a} = (a_1, ..., a_m) \in K^m$, let

$$h(\underline{a}) = \prod_w \max(1, |\underline{a}|_w{}^{d_w/d}), \quad |\underline{a}|_w = \max_i(|a_i|_w).$$

The above condition $|q|_v > 1$ is not enough in the following, but we have to assume a further restriction to q given in terms of

$$\lambda = \lambda(q, v) = \frac{d \log h(q)}{d_v \log |q|_v}.$$

Note that $\lambda \geqslant 1$ always, and $\lambda = 1$ for example in the following cases: $K = \mathbb{Q}$ or an imaginary quadratic field, v is infinite and q is an integer of K; $K = \mathbb{Q}$, $v = p$, and $q = p^{-\ell}$ with some positive integer ℓ; $K = \mathbb{Q}(q)$, where q is a so-called PV-number, v is infinite and $|q|_v > 1$.

The following result is [2, Corollary 5.2].

Theorem AMV. *Assume that $q \in K$ satisfies $|q|_v > 1$, and let $\alpha_1, ..., \alpha_m$ be nonzero elements of K satisfying the conditions* (2) *and*

$$\alpha_j \neq P(0)q^n, \quad j = 1, ..., m; n = 1, 2, ... \tag{5}$$

Then there exists an effectively computable constant $\Lambda > 1$ depending on m, s and D such that if

$$1 \leqslant \lambda < \Lambda,$$

then the $1 + msD =: 1 + N$ numbers (4) *belonging to K_v are linearly independent over K. Further, there exist positive constants C and H_0 depending on P, q, v, s and*

α_j *such that for all non-zero* $\underline{A} = (A_0, ..., A_{j,k,\sigma}, ...) \in K^{1+N}$ *we have*

$$|A_0 + \sum_{j=1}^{m} \sum_{k=0}^{D-1} \sum_{\sigma=0}^{s-1} A_{j,k,\sigma} f^{(\sigma)}(q^k \alpha_j)|_v > H^{-\mu d/d_v - C/\sqrt{\log H}}, \tag{6}$$

where $H = \max(h(\underline{A}), H_0)$ *and* $\mu = \Lambda/(\Lambda - \lambda)$.

Remark 1.1. If $\lambda = 1$, then we have

$$\mu \leqslant \frac{8N}{8N-1}(8sN^2 + (s+4)N + s/3 + 2)$$

in Theorem AMV. Furthermore, Λ is of the form $1 + c/(sN^2)$ with a positive constant c.

The proof of Theorem AMV uses Siegel's lemma in the construction of needed approximation forms. There are also several works considering special cases of this theorem by using explicit Padé approximations of the second kind, see Stihl [13] in the case $K = \mathbb{Q}$ or an imaginary quadratic field, v infinite and $P(0) = 0$ without derivatives, and Katsurada [9] the same case with derivatives, and [14] with general K, v and $P(0) = 0$. In all these cases the measure is better than (6) above, $\mu = cN$ with some constant $c > 0$. Here we would like to point out that a partial quantitative form of Bézivin's results was also obtained using Hilbert—Perron—Skolem method by Bundschuh and Wallisser [7], [8], but their results do not apply to the consideration of our $f(z)$.

For a long time Bézivin's method itself was thought to be only of qualitative nature, but recently the first author [11], [12] developed a quantitative refinement of this method leading to the following improvement of the case $K = \mathbb{Q}$, v infinite of Theorem AMV, see [12, Theorem 1].

THEOREM R. *Let* $q, \alpha_1, ..., \alpha_m \in \mathbb{Q}^*$ *satisfy the conditions* (2) *and* (5). *Let* $s_1, \ldots, s_m$ *be positive integers and put*

$$S = s_1 + \ldots + s_m, \tag{7}$$

$$M = \begin{cases} DS + 1/2 + \sqrt{D^2S^2 + 1/4}, & \textit{if} \quad P(x) = p_D x^D,\ p_D \in \mathbb{Q}^*, \\ DS + 1 + \sqrt{DS(DS+1)}, & \textit{otherwise}. \end{cases} \tag{8}$$

If

$$\lambda < 1 + \frac{1}{M-1},$$

then the real numbers

$$1, f^{(\sigma)}(q^k\alpha_j), \quad j = 1, \ldots, m; \quad k = 0, 1, \ldots, D-1; \quad \sigma = 0, 1, \ldots, s_j - 1 \quad (9)$$

are linearly independent over $\mathbb{Q}$. *Moreover, there exists a positive constant* C_0 *depending on* P, q, m, α_j, *and* s_j *such that for any non-zero* $\underline{A} = (A_0, \ldots, A_{j,k,\sigma}, \ldots) \in \mathbb{Z}^{1+DS}$ *we have*

$$|A_0 + \sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1} A_{j,k,\sigma} f^{(\sigma)}(q^k\alpha_j)| \geqslant H^{-\mu - C_0/\sqrt{\log H}} \quad (10)$$

with $H = \max(2, |\underline{A}'|)$, *where* $\underline{A}'$ *is* $\underline{A}$ *without* A_0, *and* $\mu = (M-1)/(M/\lambda - (M-1))$.

In [11] the first author considered also general K, v under a milder condition on q but with the lower estimate of the form $\exp(-C(\log H)^{3/2})$, where $C > 0$ is a constant like C_0 above. The proof of Theorem R is a modification of the method of [11].

The main result of our paper is the following improvement and generalization of Theorem AMV and Theorem R.

THEOREM 1. *Let* $q, \alpha_1, ..., \alpha_m \in K^*$ *satisfy the conditions* (2) *and* (5), *and let* S *and* M *be as in Theorem R. If*

$$\lambda < 1 + \frac{1}{M-1},$$

then the numbers (9) *belonging to* K_v *are linearly independent over* K. *Moreover, there exists a positive constant* C_1 *depending on* P, q, v, m, α_j, *and* s_j *such that for any non-zero* $\underline{A} = (A_0, \ldots, A_{j,k,\sigma}, \ldots) \in K^{1+DS}$ *we have*

$$|A_0 + \sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1} A_{j,k,\sigma} f^{(\sigma)}(q^k\alpha_j)|_v \geqslant H^{-\mu d/d_v - C_1/\sqrt{\log H}} \max(1, |\underline{A}|_v) \quad (11)$$

where $H = \max(2, h(\underline{A}))$ *and* $\mu = M/(M - \lambda(M-1))$.

In the case $K = \mathbb{Q}$ and v infinite our Theorem 1 implies Theorem R, and the analogous result holds also in any imaginary quadratic field.

We note that the measure of Theorem 1 is well comparable to the measures obtained by using Padé approximations in special cases of $f(z)$. The proof of our theorem follows the main lines of the proof of Theorem R in [12]. For the sake of completeness we shall give a self-contained proof in the following sections 2–4. The final section 5 is devoted to some applications.

Let us sketch the main ideas of the proof. In Section 2 two sequences $(u_n(\underline{x}))$ and $(v_n(\underline{x}))$ of linear forms in the components of a variable vector $\underline{x} = (x_0, \dots, x_{j,k,\sigma}, \dots)$ are defined by

$$\sum_{j=1}^{m} \sum_{k=0}^{D-1} \sum_{\sigma=0}^{s_j-1} x_{j,k,\sigma} f^{(\sigma)}(q^k \alpha_j) = \sum_{n=0}^{\infty} \frac{u_n(\underline{x})}{\prod_{k=1}^{n} P(q^k)},$$

$$v_n(\underline{x}) = \prod_{k=1}^{n} P(q^k) \cdot \left(x_0 + \sum_{\ell=0}^{n} \frac{u_\ell(\underline{x})}{\prod_{k=1}^{\ell} P(q^k)} \right).$$

Assume that at least one component $A_{j,k,\sigma}$ of $\underline{A}$ is nonzero and define the vector $\underline{\omega}$ by $\omega_{j,k,\sigma} = A_{j,k,\sigma}$ and

$$\omega_0 = -\sum_{j=1}^{m} \sum_{k=0}^{D-1} \sum_{\sigma=0}^{s_j-1} A_{j,k,\sigma} f^{(\sigma)}(q^k \alpha_j) \in K_v.$$

Then

$$\omega_0 + \sum_{n=0}^{\infty} \frac{u_n(\underline{\omega})}{\prod_{k=1}^{n} P(q^k)} = \omega_0 + \sum_{j=1}^{m} \sum_{k=0}^{D-1} \sum_{\sigma=0}^{s_j-1} \omega_{j,k,\sigma} f^{(\sigma)}(q^k \alpha_j) = 0,$$

and therefore $v_n(\underline{\omega})$ is given by using ‘the tail‘

$$\frac{v_n(\underline{\omega})}{\prod_{k=1}^{n} P(q^k)} = -\sum_{\ell=n+1}^{\infty} \frac{u_\ell(\underline{\omega})}{\prod_{k=1}^{\ell} P(q^k)}.$$

A further construction starting from $v_n(\underline{x})$ gives, for an integer $\ell \geqslant 0$ and $n \geqslant S\ell$, more linear forms $v_{\ell,n}(\underline{x})$ in such a way that $\left|v_{\ell,n}(\underline{\omega})\right|_v$ is ‘small‘ (Lemma 2), $\left|v_{\ell,n}(\underline{A})\right|_v$ is ‘not too small‘ (which follows from non-vanishing of $v_{\ell,n}(\underline{A})$ (Lemma 3) and the product formula) and $v_{\ell,n}(\underline{A}) - v_{\ell,n}(\underline{\omega}) = C(\ell, n)(A_0 - \omega_0)$, where $|C(\ell, n)|_v$

is well controlled from above (Lemma 1). By a suitable choice of ℓ, n we then obtain

$$\frac{1}{2}\left|v_{\ell,n}(\underline{A})\right|_v < \left|v_{\ell,n}(\underline{A}) - v_{\ell,n}(\underline{\omega})\right|_v \leqslant |C(\ell,n)|_v\,|A_0 - \omega_0|_v =$$
$$= |C(\ell,n)|_v \left|A_0 + \sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1} A_{j,k,\sigma} f^{(\sigma)}(q^k\alpha_j)\right|_v ,$$

which leads to (11).

2. Preliminary notation and basic construction

Let $\mathbb{C}_v$ denote the completion of the algebraic closure of K_v. For a function $\xi(n)$ of an integer argument n, we denote by B the backward shift operator

$$B(\xi(n)) = \xi(n-1).$$

For $a \in \mathbb{C}_v$ introduce the difference operator

$$D_a = I - aB,$$

where I is the identity operator: $I\big(\xi(n)\big) = \xi(n)$. Note that these operators commute with each other. It is well known that for $a \in \mathbb{C}_v^*$ and $p(z) \in \mathbb{C}_v[z]$ with $\deg p \leqslant$ $\leqslant t \in \mathbb{Z}_{\geqslant 0}$ we have

$$D_a^{t+1}\big(p(n)a^n\big) = 0, \quad n \in \mathbb{Z}. \tag{12}$$

Also, it is readily seen that for $a, b \in \mathbb{C}_v$ with $b \neq 0$ we have

$$D_a\big(b^n\xi(n)\big) = b^n D_{ab^{-1}}\big(\xi(n)\big). \tag{13}$$

If $a, \xi(n) \in K$ and $|\xi(n)|_w \leqslant \eta(n)$ for some place w of K, then

$$\left|D_a\big(\xi(n)\big)\right|_w = \left|I\big(\xi(n)\big) - aB\big(\xi(n)\big)\right|_w \leqslant \eta(n) + |a|_w\,\eta(n-1) = D_{-|a|_w}\big(\eta(n)\big). \tag{14}$$

Assume now that $q, \alpha_1, \ldots, \alpha_m \in K^*$. Let $\underline{x}$ denote the vector of variables $\underline{x} = (x_0, \ldots, x_{j,k,\sigma}, \ldots)$, $j = 1, \ldots, m;\ k = 0, 1, \ldots, D-1;\ \sigma = 0, 1, \ldots, s_j - 1$.

We shall consider sequences of linear forms in these variables

$$
\begin{aligned}
u_n = u_n(\underline{x}) &= \sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1}\Big(\left(\frac{d}{dz}\right)^{\sigma} z^n\Big)\Big|_{z=\alpha_j q^k} x_{j,k,\sigma} = \\
&= \sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1} \sigma!\binom{n}{\sigma}(\alpha_j q^k)^{n-\sigma} x_{j,k,\sigma} \in K[\underline{x}], \quad n \in \mathbb{Z},
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
v_n = v_n(\underline{x}) &= \prod_{k=1}^{n} P(q^k)\cdot\left(x_0 + \sum_{\ell=0}^{n}\frac{u_\ell(\underline{x})}{\prod_{k=1}^{\ell} P(q^k)}\right) = \\
&= x_0\prod_{k=1}^{n} P(q^k) + \sum_{\ell=0}^{n} u_\ell(\underline{x}) \prod_{k=\ell+1}^{n} P(q^k) \in K[\underline{x}], \quad n \geqslant 0.
\end{aligned} \tag{16}
$$

It is readily seen that

$$
v_n = P(q^n)v_{n-1} + u_n, \quad n \geqslant 1, \tag{17}
$$

with $v_0 = x_0 + u_0 = x_0 + \sum_{j=1}^{m}\sum_{k=0}^{D-1} x_{j,k,0}$. Further, for $\ell, n \geqslant 0$ with $n \geqslant S\ell$, where S is given in (7), put

$$
v_{\ell,n} = v_{\ell,n}(\underline{x}) = \prod_{k=1}^{\ell}\prod_{j=1}^{m} D^{s_j}_{\alpha_j q^{-k}}\big(v_n(\underline{x})\big) := \left(\prod_{k=1}^{\ell}\prod_{j=1}^{m} D^{s_j}_{\alpha_j q^{-k}}\right)\big(v_n(\underline{x})\big) \in K[\underline{x}]. \tag{18}
$$

For a linear form L with coefficients from K we use the notation $H_w(L) := := |\underline{L}|_w$, where $\underline{L}$ is the coefficient vector of L. Let

$$
P(x) = \sum_{j=0}^{D} p_j x^j \in K[x],
$$

and denote $\underline{\gamma} = (1, \alpha_1, \dots, \alpha_m, p_0, \dots, p_D)$. Then the following lemma is valid.

Lemma 1. *For any place w of K we have*

$$
H_w(v_{\ell,n}) \leqslant c_1^{\delta(w)(n+S\ell)} \left|\underline{\gamma}\right|_w^{2n+S\ell} |q|_w^{*Dn(n+3)/2} \left|q^{-1}\right|_w^{*S\ell(\ell+1)/2},
$$

where c_1 *(as also* $c_2, \ldots$ *later) is a positive constant depending only on* $P, q, v, m, \alpha_j, s_j$, *and* $\delta(w) = 1$ *for infinite* w *and* $\delta(w) = 0$ *otherwise, and* $|a|_w^* = \max(1, |a|_w)$.

Proof. Clearly, by the definitions of u_n and v_n,

$$H_w(u_n) = \max_{j,k,\sigma}\left(\left|\sigma!\binom{n}{\sigma}(\alpha_j q^k)^{n-\sigma}\right|_w\right) \leqslant c_2^{\delta(w)n}\left|\underline{\gamma}\right|_w^n |q|_w^{*Dn},$$

$$H_w(v_n) \leqslant c_3^{\delta(w)n}\left|\underline{\gamma}\right|_w^n |q|_w^{*Dn(n+1)/2} \max(1, \max_{0\leqslant \ell \leqslant n} H_w(u_\ell)) \leqslant c_4^{\delta(w)n}\left|\underline{\gamma}\right|_w^{2n} |q|_w^{*Dn(n+3)/2}.$$

Then the use of (14) and (18) gives the truth of Lemma 1. □

To state the following lemma we introduce

$$\epsilon_0 = \begin{cases} 1, & \text{if} \quad P(x) = p_D x^D,\ p_D \in K^*, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2. *Let* $|q|_v > 1$ *and* $\underline{\omega} = (\omega_0, \ldots, \omega_{j,k,\sigma}, \ldots) \in \mathbb{C}_v^{1+DS}$ *be such that*

$$\omega_0 + \sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0. \tag{19}$$

For $\ell \geqslant 0$ *and* $n \geqslant S\ell$ *we then have*

$$\left|v_{\ell,n}(\underline{\omega})\right|_v \leqslant \left|\underline{\omega}'\right|_v |q|_v^{-\ell n + (S-\epsilon_0/D)\ell^2/2 + c_5(n+1)},$$

where $\underline{\omega}'$ *is* $\underline{\omega}$ *without* ω_0.

Proof. By (15) and (19) we get

$$\omega_0 + \sum_{n=0}^{\infty} \frac{u_n(\underline{\omega})}{\prod_{k=1}^{n} P(q^k)} = \omega_0 + \sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0,$$

and therefore (16) implies

$$v_n(\underline{\omega}) = -\sum_{\tau=n+1}^{\infty} \frac{u_\tau(\underline{\omega})}{\prod_{k=n+1}^{\tau} P(q^k)}.$$

Since

$$|u_n(\underline{\omega})|_v \leqslant c_6^n \left|\underline{\omega}'\right|_v,$$

we obtain

$$|v_n(\underline{\omega})|_v \leqslant \left|\underline{\omega}'\right|_v c_7 c_6^n.$$

Consequently for $0 \leqslant \nu < D$ and $n \geqslant S\nu$ we have

$$|v_{\nu,n}(\underline{\omega})|_v \leqslant |\underline{\omega}'|_v \cdot |q|_v^{c_8(n+1)} \leqslant |\underline{\omega}'|_v \cdot |q|_v^{-\nu n+(S-\epsilon_0/D)\nu^2/2+(c_8+D)n+c_8}. \tag{20}$$

Next we shall prove the following inductive step: Let $\ell \geqslant D$. Assume that for $0 \leqslant \nu < \ell$ and $n \geqslant S\nu$ we have

$$|v_{\nu,n}(\underline{\omega})|_v \leqslant |\underline{\omega}'|_v \cdot |q|_v^{-\nu n+(S-\epsilon_0/D)\nu^2/2+an+b},$$

where $a > (\log|\underline{\alpha}|_v)/(\log|q|_v)$, $\underline{\alpha} = (1, \alpha_1, \ldots, \alpha_m)$, and b are independent of ν and n. Then, for all $n \geqslant S\ell$,

$$|v_{\ell,n}(\underline{\omega})|_v \leqslant |\underline{\omega}'|_v \cdot |q|_v^{-\ell n+(S-\epsilon_0/D)\ell^2/2+an+b+a+c_9}.$$

Before the proof of this step we note that it together with (20) implies, for all $\ell \geqslant 0$ and $n \geqslant S\ell$,

$$|v_{\ell,n}(\underline{\omega})|_v \leqslant |\underline{\omega}'|_v \cdot |q|_v^{-\ell n+(S-\epsilon_0/D)\ell^2/2+(c_8+D)n+c_8+(c_8+D+c_9)\ell},$$

and since $\ell \leqslant n/S$, this proves Lemma 2. So it remains to prove the truth of the above inductive step.

Since $\ell \geqslant D$, it follows from (12) and (15) that

$$\prod_{k=1}^{\ell}\prod_{j=1}^{m} D_{\alpha_j q^{D-k}}^{s_j}\big(u_n(\underline{\omega})\big) = 0, \quad n \in \mathbb{Z}. \tag{21}$$

Therefore, from (17) we get

$$\prod_{k=1}^{\ell}\prod_{j=1}^{m} D_{\alpha_j q^{D-k}}^{s_j}\left(v_{n+1}(\underline{\omega}) - P(q^{n+1})v_n(\underline{\omega})\right) = 0, \quad n \geqslant S\ell. \tag{22}$$

By using (13) this relation can be rewritten in the form

$$
\begin{aligned}
p_D v_{\ell,n}(\underline{\omega}) = q^{-D(n+1)} \prod_{k=1}^{\ell} \prod_{j=1}^{m} D^{s_j}_{\alpha_j q^{D-k}} \big(v_{n+1}(\underline{\omega})\big) = \\
- \sum_{\nu=1}^{D} p_{D-\nu} q^{-\nu(n+1)} \prod_{k=1}^{\ell} \prod_{j=1}^{m} D^{s_j}_{\alpha_j q^{\nu-k}} \big(v_n(\underline{\omega})\big).
\end{aligned}
\tag{23}
$$

It follows from (13), (14) and the assumptions of the inductive step that for $1 \leqslant \nu \leqslant D$ we have

$$
\begin{aligned}
&\left| q^{-\nu n} \prod_{k=1}^{\ell} \prod_{j=1}^{m} D^{s_j}_{\alpha_j q^{\nu-k}} \big(v_{n+\epsilon_0}(\underline{\omega})\big) \right|_v = \left| q^{-\nu n} \prod_{k=0}^{\nu-1} \prod_{j=1}^{m} D^{s_j}_{\alpha_j q^{k}} \big(v_{\ell-\nu,n+\epsilon_0}(\underline{\omega})\big) \right|_v \leqslant \\
&\leqslant |q|_v^{-\nu n} \prod_{k=0}^{\nu-1} \prod_{j=1}^{m} D^{s_j}_{-|\alpha_j q^k|_v} \left(|\underline{\omega}'|_v \cdot |q|_v^{-(\ell-\nu)(n+\epsilon_0)+(S-\epsilon_0/D)(\ell-\nu)^2/2+a(n+\epsilon_0)+b} \right) = \\
&= |\underline{\omega}'|_v \cdot |q|_v^{-\nu n-(\ell-\nu)(n+\epsilon_0)+(S-\epsilon_0/D)(\ell-\nu)^2/2+a(n+\epsilon_0)+b} \prod_{k=0}^{\nu-1} \prod_{j=1}^{m} \big(1+|\alpha_j q^{k+\ell-\nu-a}|_v\big)^{s_j} \leqslant \\
&\leqslant |\underline{\omega}'|_v \cdot |q|_v^{-\nu n-(\ell-\nu)(n+\epsilon_0)+(S-\epsilon_0/D)(\ell-\nu)^2/2+a(n+\epsilon_0)+b+S\ell\nu} \prod_{k=0}^{\nu-1} \prod_{j=1}^{m} \big(1+|\alpha_j q^{k-\nu}|_v\big)^{s_j} = \\
&= |\underline{\omega}'|_v \cdot |q|_v^{-\ell n+(S-\epsilon_0/D)\ell^2/2+a(n+\epsilon_0)+b-(1-\nu/D)\epsilon_0\ell+(S-\epsilon_0/D)\nu^2/2+\epsilon_0\nu} \times \\
&\quad \times \prod_{k=1}^{\nu} \prod_{j=1}^{m} \big(1+|\alpha_j q^{-k}|_v\big)^{s_j} \leqslant |\underline{\omega}'|_v \cdot |q|_v^{-\ell n+(S-\epsilon_0/D)\ell^2/2+a(n+\epsilon_0)+b+c_{10}}.
\end{aligned}
\tag{24}
$$

The truth of the inductive step follows from (23) and (24). Thus Lemma 2 is proved. □

3. Non-vanishing lemma

The following non-vanishing lemma is of crucial importance for our considerations.

LEMMA 3. *Let* $\alpha_1, \dots, \alpha_m$ *satisfy the conditions of Theorem* 1, *and let* $\underline{\omega} \in \mathbb{C}_v^{1+DS}$ *be such that* $\underline{\omega}' \neq \underline{0}$. *Then, for any non-negative integers* ℓ_0, n_0 *with* $n_0 \geqslant S\ell_0$, *there exists an integer n with* $n_0 \leqslant n \leqslant n_0 + DS$ *such that* $v_{\ell_0,n}(\underline{\omega}) \neq 0$.

This follows immediately from Lemmas 4 and 5 below.

LEMMA 4. *Let* $\underline{\omega} \in \mathbb{C}_v^{1+DS}$ *be such that for some non-negative integers* ℓ_0, n_0 *with* $n_0 \geqslant S\ell_0$ *we have*

$$v_{\ell_0,n_0}(\underline{\omega}) = v_{\ell_0,n_0+1}(\underline{\omega}) = \ldots = v_{\ell_0,n_0+DS}(\underline{\omega}) = 0. \tag{25}$$

Then the generating function

$$F(z) = \sum_{n=0}^{\infty} v_n(\underline{\omega}) z^n \in \mathbb{C}_v[[z]]$$

of the sequence $v_n(\underline{\omega})$ *is rational.*

PROOF. We consider the linear recurrence sequence $\{w_n\}_{n\geqslant 0}$ given by

$$w_n = v_{n_0 - S\ell_0 + n}(\underline{\omega}), \quad 0 \leqslant n < S\ell_0,$$
$$\prod_{k=1}^{\ell_0} \prod_{j=1}^{m} D_{\alpha_j q^{-k}}^{s_j}(w_n) = 0, \quad n \geqslant S\ell_0.$$

From (18) and (25) it follows that

$$w_n = v_{n_0 - S\ell_0 + n}(\underline{\omega}), \quad 0 \leqslant n \leqslant S(\ell_0 + D). \tag{26}$$

By (13) and (25), for all $\nu \in \mathbb{Z}$ we have

$$\prod_{k=1}^{\ell_0} \prod_{j=1}^{m} D_{\alpha_j q^{\nu-k}}^{s_j}\big(q^{\nu n} w_n\big) = q^{\nu n} \prod_{k=1}^{\ell_0} \prod_{j=1}^{m} D_{\alpha_j q^{-k}}^{s_j}(w_n) = 0, \quad n \geqslant S\ell_0.$$

Hence the sequence

$$z_n = w_{n+1} - P(q^{n_0 - S\ell_0 + n + 1})w_n - u_{n_0 - S\ell_0 + n + 1}(\underline{\omega}), \quad n \geqslant 0,$$

satisfies the linear recurrence relation

$$\prod_{k=-\ell_0}^{D-1} \prod_{j=1}^{m} D_{\alpha_j q^k}^{s_j}(z_n) = 0, \quad n \geqslant S(\ell_0 + D),$$

of order $S(\ell_0 + D)$.

On the other hand, it follows from (17) and (26) that $z_n = 0$ for $0 \leqslant n < < S(\ell_0 + D)$. Hence z_n is identically zero. This implies that $w_n = v_{n_0 - S\ell_0 + n}(\underline{\omega})$ for all $n \geqslant 0$, i.e., $v_n(\underline{\omega})$ is a linear recurrence sequence and

$$F(z) = \sum_{n \geqslant 0} v_n(\underline{\omega}) z^n \in \mathbb{C}_v(z).$$

This completes the proof of Lemma 4. □

LEMMA 5. *Let* $\alpha_1, \ldots, \alpha_m$ *satisfy the conditions of Theorem* 1, *and let* $\underline{\omega} \in \mathbb{C}_v^{1+DS}$ *be such that* $\underline{\omega}' \neq \underline{0}$. *Then the generating function* $F(z)$ *of the sequence* $v_n(\underline{\omega})$ *is not rational.*

PROOF. We assume that $F(z)$ is rational and deduce a contradiction by considering the poles of it. From (17) it follows that $F(z)$ satisfies the functional equation

$$(1 - p_0 z)F(z) = \sum_{\nu=1}^{D} p_\nu q^\nu z F(q^\nu z) + R(z), \tag{27}$$

where

$$P(z) = \sum_{\nu=0}^{D} p_\nu z^\nu,$$

$$R(z) = \omega_0 + \sum_{n=0}^{\infty} u_n(\underline{\omega}) z^n = \omega_0 + \sum_{j=1}^{m} \sum_{k=0}^{D-1} \sum_{\sigma=0}^{s_j - 1} \frac{\omega_{j,k,\sigma} \sigma! z^\sigma}{(1 - \alpha_j q^k z)^{\sigma+1}} \in \mathbb{C}_v(z).$$

Condition (2) implies that all $\alpha_j q^k$ are different. Since $\underline{\omega}' \neq \underline{0}$, the function $R(z)$ has at least one pole. It follows from (27) that $F(z)$ also has a non-zero pole.

We claim that any pole of $F(z)$ is of the form $\alpha_j^{-1}q^n$ with a positive integer n. Assume the contrary. Let β be a pole that cannot be represented in this form with the least $|\beta|_v$. Then $R(z)$ doesn't have a pole at the point βq^{-D}. It follows from (27) that one of the functions $F(q^\nu z)$ with $0 \leqslant \nu < D$ has a pole at βq^{-D}. Hence we have $\beta = \beta' q^{D-\nu}$ for some pole β' of $F(z)$. But then $|\beta'|_v < |\beta|_v$. Consequently β' can be represented in the required form as well as β. This contradiction proves our claim about poles of $F(z)$. In particular, it follows from condition (2) that $F(z)$ and $R(z)$ do not have common poles.

Now suppose β is a pole of $F(z)$ with maximal $|\beta|_v$. It follows from (27) and the above that the function $(1-p_0 z)F(z)$ does not have a singularity at the point β. Hence $p_0\beta = 1$. Since $\beta = \alpha_j^{-1}q^n$ with $n > 0$, this contradicts condition (5). This contradiction proves the lemma. □

4. Proof of Theorem 1

Let the assumptions of Theorem 1 be satisfied, and let $\underline{A} = (A_0, \ldots, A_{j,k,\sigma}, \ldots) \in$ $\in K^{1+DS}$ be non-zero. If $\underline{A'} = \underline{0}$, then the estimate (11) holds, since $|A_0|_v \geqslant$ $\geqslant h(A_0)^{-d/d_v}$. So we may assume in the following that $\underline{A'} \neq \underline{0}$.

Take now

$$n_0 = \left\lceil \frac{DS - \epsilon_0/2 + \sqrt{(DS)^2 + (1-\epsilon_0)DS + \epsilon_0^2/4}}{D}\ell \right\rceil = \tag{28}$$
$$= \left\lceil \frac{(M-1)\ell}{D} \right\rceil \geqslant S\ell,$$

where M is given in (8) and the integer $\ell > 0$ will be chosen later. From Lemma 3 it follows that there exists an integer n, $n_0 \leqslant n \leqslant n_0 + DS$, such that $v_{\ell,n}(\underline{A}) \neq 0$. Clearly $v_{\ell,n}(\underline{A}) \in K$, and therefore, by the product formula and Lemma 1,

$$\frac{d_v}{d}\log\left|v_{\ell,n}(\underline{A})\right|_v = -\sum_{w\neq v}\frac{d_w}{d}\log\left|v_{\ell,n}(\underline{A})\right|_w \geqslant -\log h(\underline{A}) + \frac{d_v}{d}\log|\underline{A}|_v^* -$$
$$-\frac{Dn(n+3)}{2}\log h(q) + \frac{d_v}{d}\frac{Dn(n+3)}{2}\log|q|_v - \frac{S\ell(\ell+1)}{2}\log h(q^{-1}) - c_{11}\ell \geqslant$$

$$\geqslant -\log h(\underline{A}) + \frac{d_v}{d}\log|\underline{A}|_v^* - \frac{Dn^2+S\ell^2}{2}\log h(q) + \frac{d_v}{d}\frac{Dn(n+3)}{2}\log|q|_v - c_{12}\ell. \tag{29}$$

We now define $\underline{\omega}$ by $\underline{\omega}' = \underline{A}'$ and

$$\omega_0 = -\sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1} A_{j,k,\sigma} f^{(\sigma)}(q^k\alpha_j) \in K_v.$$

Then Lemma 2 gives an estimate

$$\log\left|v_{\ell,n}(\underline{\omega})\right|_v \leqslant \log\left|\underline{A}'\right|_v + (-\ell n + (S-\epsilon_0/D)\ell^2/2 + c_5(n+1))\log|q|_v \leqslant$$
$$\leqslant \log|\underline{A}|_v + (-\ell n + (S-\epsilon_0/D)\ell^2/2 + c_5(n+1))\log|q|_v\,.$$

If now

$$\left|v_{\ell,n}(\underline{A})\right|_v \leqslant 2\left|v_{\ell,n}(\underline{\omega})\right|_v, \tag{30}$$

then the definition of λ, (28), (29) and the above estimate imply

$$\log h(\underline{A}) \geqslant \Omega\ell^2 - c_{13}\ell, \tag{31}$$

where

$$\Omega = \frac{(M-1)^2+DS}{2D}\left(\frac{(M-1)^2+2(M-1)-DS+\epsilon_0}{(M-1)^2+DS} - \lambda\right)\frac{d_v}{d}\log|q|_v =$$
$$= \frac{(M-1)^2+DS}{2D}\left(1+\frac{1}{M-1}-\lambda\right)\frac{d_v}{d}\log|q|_v\,.$$

Note that $\Omega > 0$ by the assumptions of Theorem 1. Let now $H = \max(2, h(\underline{A}))$, and fix the integer ℓ by the equation

$$\ell = \left\lceil \sqrt{\frac{\log H}{\Omega}} + c_{14} \right\rceil,$$

where $c_{14} > c_{13}/\Omega$. Then

$$\log H < \Omega\ell^2 - c_{13}\ell,$$

which contradicts (31). Therefore with this choice of ℓ (30) cannot hold and we have

$$\left|v_{\ell,n}(\underline{A})\right|_v > 2\left|v_{\ell,n}(\underline{\omega})\right|_v .$$

The above inequality together with Lemma 1 and the definition of $\underline{\omega}$ give

$$\frac{1}{2}\left|v_{\ell,n}(\underline{A})\right|_v < \left|v_{\ell,n}(\underline{A}) - v_{\ell,n}(\underline{\omega})\right|_v \leqslant H_v(v_{\ell,n})\,|A_0-\omega_0|_v \leqslant$$
$$\leqslant |q|_v^{\frac{(M-1)^2}{2D}\ell^2+c_{15}\ell}\left|A_0+\sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1}A_{j,k,\sigma}f^{(\sigma)}(q^k\alpha_j)\right|_v .$$

Thus the use of (29) and the definition of ℓ imply

$$\left|A_0+\sum_{j=1}^{m}\sum_{k=0}^{D-1}\sum_{\sigma=0}^{s_j-1}A_{j,k,\sigma}f^{(\sigma)}(q^k\alpha_j)\right|_v >$$
$$> |q|_v^{-\frac{(M-1)^2}{2D}\ell^2-c_{16}\ell}\left|v_{\ell,n}(\underline{A})\right|_v \geqslant |\underline{A}|_v^*\,H^{-\mu d/d_v-c_{17}/\sqrt{\log H}},$$

where

$$\mu = 1+\lambda\frac{(M-1)^2+DS}{2D\Omega}\frac{d_v}{d}\log|q|_v = 1+\frac{\lambda}{1+\dfrac{1}{M-1}-\lambda} = \frac{M}{M-\lambda(M-1)}.$$

This proves Theorem 1.

5. Two applications

Sometimes functions of other type are connected to $f(z)$ in such a way that Theorem 1 can be applied to get linear independence measures. Here we give two examples. To present the first one we define for $\alpha\in\mathbb{C}$ the function

$$F_\alpha(z) = \sum_{n=0}^{\infty}\frac{\alpha^n\prod_{j=1}^{n}(1+z/q^j)}{\prod_{j=1}^{n}(q^j-1)}.$$

Recently Bézivin [5] studied linear independence of the values of this function by using the identity, see [5, Lemma 2.4 and Lemma 2.5(a)],

$$F_\alpha(z) = E_q(\alpha) \sum_{n=0}^{\infty} \frac{\alpha^n z^n}{\prod_{j=1}^{n}(q^j - 1)(q^j + \alpha)}, \quad \alpha \neq -q^n, n = 1, 2, ... \tag{32}$$

Clearly Theorem 1 can be applied to the sum on the right-hand side of (32), and we get the following Theorems 2 and 3, where v is an archimedean place and we denote $|.|_v = |.|$.

THEOREM 2. *Assume that* $q, \alpha \in K, |q| > 1, \alpha \neq 0, -q^n, n = 1, 2, ...,$ *and let* $\alpha_1, ..., \alpha_m$ *be nonzero elements of* K *satisfying the conditions* (2) *and*

$$\alpha_j \neq -q^n \quad \forall\, j = 1, \dots, m; \quad n = 1, 2, \dots .$$

If

$$\lambda < 1 + \frac{1}{M-1}, \quad M = 2S + 1 + \sqrt{2S(2S+1)},$$

where S *is given in* (7), *then the* $1 + 2S$ *numbers*

$$E_q(\alpha), F_\alpha^{(\sigma)}(q^k\alpha_j), \quad j = 1, \dots, m; \quad k = 0, 1; \quad \sigma = 0, 1, \dots, s_j - 1,$$

belonging to K_v *are linearly independent over* K. *Moreover, there exists a positive constant* C_2 *depending on* $\alpha, q, v, m, \alpha_j$, *and* s_j *such that for any non-zero* $\underline{A} = (A_0, \dots, A_{j,k,\sigma}, \dots) \in K^{1+2S}$ *we have*

$$|A_0 E_q(\alpha) + \sum_{j=1}^{m} \sum_{k=0}^{1} \sum_{\sigma=0}^{s_j-1} A_{j,k,\sigma} F^{(\sigma)}(q^k\alpha_j)|_v \geqslant H^{-\mu d/d_v - C_2/\sqrt{\log H}} \max(1, |\underline{A}|_v) \tag{33}$$

where $H = \max(2, h(\underline{A}))$ *and* $\mu = M/(M - \lambda(M-1))$.

Theorem 2 is an improvement and generalization of [5, Theorem 1.2 and Theorem 1.6] and the result of the second author [16, Theorem 1] proved by using Theorem AMV. In the special case $m = s_1 = 1, \alpha_1 = -1$ Theorem 2 gives linear independence of 1, $T_q(\alpha)$ and $E_q(\alpha)$. By Theorem R and the remark after Theorem 1 we get the following linear independence measure.

THEOREM 3. *Assume that* $K = \mathbb{Q}$ *or an imaginary quadratic field. Let* $q \in O_K$, $|q| > 1$, $\alpha \in K$, $\alpha \neq 0, -q^n$, $n = 1, 2, \dots$. *Then there exists a positive constant* C_3 *depending on* q *and* α *such that for all nonzero* $(a_0, a_1, a_2) \in O_K^3$ *we have*

$$|a_0 + a_1 T_q(\alpha) + a_2 E_q(\alpha)| > H^{-(2+\sqrt{6}) - C_3/\sqrt{\log H}},$$

where $H = \max(2, |a_0|, |a_1|, |a_2|)$.

Our second example considers the function

$$D_b(z) = \sum_{n=0}^{\infty} (1-b)(1-b/q)\cdots(1-b/q^{n-1})z^n,$$

a q-analogue of divergent series

$$\sum_{n=0}^{\infty} n! z^n.$$

Linear independence properties of the values of

$$g(z) = D_z(a), \quad a \neq q^n, n = 0, 1, \dots$$

are studied by using Padé approximations in [10], where it is also proved that

$$g(z) = \frac{1}{1-a} \sum_{n=0}^{\infty} \frac{(-aqz)^n}{(q-a)\cdots(q^n-a)}, \quad a \neq q^n, \quad n = 0, 1, \dots .$$

Since the sum on the right-hand side is a special case of our $f(z)$, Theorem 1 gives immediately.

THEOREM 4. *Let* $q, a, \alpha_1, \dots, \alpha_m$ *be nonzero elements of* K *satisfying* $|q|_v > 1$, (2) *and the conditions*

$$a \neq q^n, \quad \alpha_j \neq q^n, \quad j = 1, \dots, m; n = 0, 1, \dots .$$

If

$$\lambda < 1 + \frac{1}{M-1}, \quad M = S + 1 + \sqrt{S(S+1)},$$

where S is given in (7), *then the* $1 + S$ *numbers*

$$1, g^{(\sigma)}(\alpha_j), \quad j = 1, \dots, m; \sigma = 0, 1, \dots, s_j - 1,$$

belonging to K_v *are linearly independent over* K. *Further, there exists a positive constant* C_4 *depending on* a, q, v, m, α_j, *and* s_j *such that for any non-zero* $\underline{A} = (A_0, \dots, A_{j,\sigma}, \dots) \in K^{1+S}$ *we have*

$$|A_0 + \sum_{j=1}^{m} \sum_{\sigma=0}^{s_j-1} A_{j,\sigma} g^{(\sigma)}(\alpha_j)|_v \geqslant H^{-\mu d/d_v - C_4/\sqrt{\log H}} \max(1, |\underline{A}|_v) \tag{34}$$

where $H = \max(2, h(\underline{A}))$ *and* $\mu = M/(M - \lambda(M-1))$.

In the special case $s_1 = \dots = s_m = 1$ this result is analogous to the estimate in [10, Theorem 1], but there the bound does not have the term $\max(1, |\underline{A}|_v)$.

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