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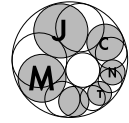
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# The winning property of mixed badly approximable numbers

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**Abstract:** For any pair of real numbers  $(i, j)$  with  $0 < i, j < 1$  and  $i + j = 1$ , we prove that the set of  $p$ -adic mixed  $(i, j)$ -badly approximable numbers  $\mathbf{Bad}_p(i, j)$  is  $1/2$ -winning in the sense of Schmidt's game. This improves a recent result of Badziahin, Levesley, and Velani on mixed Schmidt conjecture.

**Keywords:** mixed badly approximable, Schmidt's game, winning set, winning dimension

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## 1. Introduction

Consider a pair of real numbers  $(i, j)$  such that

$$0 < i, j < 1 \quad \text{and} \quad i + j = 1. \quad (1.1)$$

Let  $\mathbf{Bad}(i, j)$  denotes the set of  $(i, j)$ -badly approximable vectors in  $\mathbb{R}^2$ , that is,

$$\mathbf{Bad}(i, j) := \left\{ (x, y) \in \mathbb{R}^2 : \exists c(x, y) > 0 \text{ such that} \right. \\ \left. \max\{\|qx\|^{1/i}, \|qy\|^{1/j}\} > \frac{c(x, y)}{q}, \forall q \in \mathbb{N} \right\}, \quad (1.2)$$

where  $\|\cdot\|$  denotes the distance of a number to its nearest integer. The Schmidt conjecture says that for any two pairs of real numbers  $(i_1, j_1)$  and  $(i_2, j_2)$  satisfying (1.1), we have

$$\mathbf{Bad}(i_1, j_1) \cap \mathbf{Bad}(i_2, j_2) \neq \emptyset.$$

Recently, Badziahin, Pollington, and Velani proved Schmidt conjecture in [4] by showing that the intersection of countably many  $\mathbf{Bad}(i_n, j_n)$  is of full Hausdorff dimension. In two recent papers, An improved BPV's theorem by showing that every  $\mathbf{Bad}(i, j)$  is a winning set in the sense of Schmidt's game, see [1, 2]. Recall that any countable intersection of  $\alpha$ -winning sets is also  $\alpha$ -winning, and an  $\alpha$ -winning set is of full Hausdorff dimension, see [7, 8]. Hence An indeed improved BPV's theorem.

We now consider the case in dimension one, the set (1.2) is then reduced to the classical set of *badly approximable numbers*.

$$\mathbf{Bad} := \left\{ x \in \mathbb{R} : \exists c(x) > 0 \text{ such that } \|qx\| > \frac{c(x)}{q}, \forall q \in \mathbb{N} \right\}. \quad (1.3)$$

Given  $0 < \alpha, \beta < 1$ , let  $\gamma := 1 - 2\alpha + \alpha\beta$ , we say the pair  $(\alpha, \beta)$  is *admissible* if  $\gamma > 0$ . A classical result of Schmidt says that  $\mathbf{Bad}$  is  $(\alpha, \beta)$ -winning for all admissible  $(\alpha, \beta)$ . Observe that  $(\frac{1}{2}, \beta)$  is always admissible for every  $0 < \beta < 1$ , so in particular,  $\mathbf{Bad}$  is  $1/2$ -winning.

In [5], the  $\mathcal{D}$ -adic mixed Diophantine problems were studied. Let  $\mathcal{D}$  be a bounded sequence of positive integers  $(d_k)_{k=1}^{\infty}$ , where every  $d_k \geq 2$ . Let  $D_0 := 1$ ,  $D_n := \prod_{k=1}^n d_k$ . For every natural number  $q \in \mathbb{N}$ , define the  $\mathcal{D}$ -adic pseudo absolute value as follows,

$$|q|_{\mathcal{D}} := \inf \left\{ \frac{1}{D_n} : q \in D_n \mathbb{Z} \right\}.$$

The  $\mathcal{D}$ -adic pseudo absolute value reduces to the usual  $p$ -adic norm if we let  $\mathcal{D}$  be the constant sequence consisting of a prime number  $p$ . Recently, Badziahin, Levesley, and Velani initiated the study of mixed Schmidt conjecture in [3]. Let

$$\mathbf{Bad}_{\mathcal{D}}(i, j) := \left\{ x \in \mathbb{R} : \exists c(x) > 0 \text{ such that } \max \left\{ |q|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j} \right\} > \frac{c(x)}{q}, \forall q \in \mathbb{N} \right\}. \quad (1.4)$$

We call this set as *the set of mixed  $(i, j)$ -badly approximable numbers*. The mixed Schmidt conjecture is then stated as follows: for any two pairs of real numbers

$(i_1, j_1)$  and  $(i_2, j_2)$  satisfying (1.1), we have

$$\mathbf{Bad}_{\mathcal{D}}(i_1, j_1) \cap \mathbf{Bad}_{\mathcal{D}}(i_2, j_2) \neq \emptyset.$$

In [3], they proved  $\mathbf{Bad}_{\mathcal{D}}(i, j)$  is  $1/4$ -winning, thus resolved the mixed Schmidt conjecture affirmatively.

Now we recall the notion of winning dimension, which is introduced in [7]. The definition of  $(\alpha, \beta)$ -winning and  $\alpha$ -winning will be reviewed in Section 2. Let  $S \subset \mathbb{R}^n$ , the *winning dimension* of  $S$ , denoted by  $\text{windim } S$ , is defined as follows

$$\text{windim } S = \sup\{0 < \alpha < 1 : S \text{ is } \alpha\text{-winning}\}.$$

A result in [7] says that if  $S$  is a nontrivial subset, then  $0 \leq \text{windim } S \leq 1/2$ . For example,  $\text{windim } \mathbf{Bad} = 1/2$ . BLV's theorem says that  $\text{windim } \mathbf{Bad}_{\mathcal{D}}(i, j) \geq 1/4$ . Moshchevitin asked whether  $\text{windim } \mathbf{Bad}_{\mathcal{D}}(i, j) = 1/2$  in his recent survey on Diophantine problems, see [6]. This paper answers this question affirmatively. In fact, our theorem is a natural generalization of Schmidt's classical result on the set  $\mathbf{Bad}$ .

**THEOREM 1.** *The set  $\mathbf{Bad}_{\mathcal{D}}(i, j)$  is  $(\alpha, \beta)$ -winning for all admissible  $(\alpha, \beta)$ , in particular,  $\mathbf{Bad}_{\mathcal{D}}(i, j)$  is  $1/2$ -winning.*

As a result,  $\text{windim } \mathbf{Bad}_{\mathcal{D}}(i, j) = \frac{1}{2}$ . A result in [8] says a set is  $(\alpha, \beta)$ -winning for  $\gamma \leq 0$  if and only if this set is the whole set, accordingly, the winning property of  $\mathbf{Bad}_{\mathcal{D}}(i, j)$  is the best possible, so we give the best improvement of BLV's result in the sense of winning dimension.

Given a prime number  $p$ , we use  $\mathbf{Bad}_p(i, j)$  to denote the set defined by (1.4) where the  $\mathcal{D}$ -adic is replaced by  $p$ -adic. Theorem 1 has the following corollary.

**COROLLARY 1.** *For any two different prime numbers  $p, q$ , for any two pairs of real numbers  $(i_1, j_1)$  and  $(i_2, j_2)$  satisfying (1.1), the set  $\mathbf{Bad}_p(i_1, j_1) \cap \mathbf{Bad}_q(i_2, j_2)$  is  $1/2$ -winning. In particular,*

$$\mathbf{Bad}_p(i_1, j_1) \cap \mathbf{Bad}_q(i_2, j_2) \neq \emptyset.$$

Note that the "In particular" part could also be deduced from BLV's result.

The rest of the paper is organized as follows: in Section 2, we introduce the notion of Schmidt's game and establish some notations, then we give two useful lemmas; the proof of Theorem 1 will be given in Section 3.

## 2. Schmidt's Game and Two Lemmas

First we recall the notion of Schmidt's game, for details see [7, 8]. In this paper we only consider Schmidt's game on  $\mathbb{R}$ , so we restrict our description only in this situation. Given a set  $S \subset \mathbb{R}$ , given two real numbers  $0 < \alpha, \beta < 1$ , two players, say Alice and Bob, will play the game. The game is played as follows, Bob starts the game by choosing a closed interval  $\mathbf{B}_1 \subset \mathbb{R}$ , then Alice chooses a closed interval  $\mathbf{A}_1$  such that  $\mathbf{A}_1 \subset \mathbf{B}_1$  and  $\rho(\mathbf{A}_1) = \alpha\rho(\mathbf{B}_1)$ , then Bob chooses another closed interval  $\mathbf{B}_2$  such that  $\mathbf{B}_2 \subset \mathbf{A}_1$  and  $\rho(\mathbf{B}_2) = \beta\rho(\mathbf{A}_1)$ , then Alice chooses another closed interval  $\mathbf{A}_2$  such that  $\mathbf{A}_2 \subset \mathbf{B}_2$  and  $\rho(\mathbf{A}_2) = \alpha\rho(\mathbf{B}_2)$ , and so on. Here  $\rho(\mathbf{A}) = \frac{1}{2}|\mathbf{A}|$ , where  $|\mathbf{A}|$  denotes the length of the interval  $\mathbf{A}$ . We can see that intervals appearing in the game obey the following relation,  $\mathbf{B}_1 \supset \mathbf{A}_1 \supset \mathbf{B}_2 \supset \mathbf{A}_2 \supset \dots$ . We say Alice wins the game if she can play such that the single point in  $\bigcap_{k=1}^{\infty} \mathbf{A}_k = \bigcap_{k=1}^{\infty} \mathbf{B}_k$  lies in  $S$ , otherwise Bob wins. We say  $S$  is  $(\alpha, \beta)$ -winning if Alice can always win the game no matter how Bob plays, and  $S$  is  $\alpha$ -winning if it is  $(\alpha, \beta)$ -winning for every  $0 < \beta < 1$ .

Let  $\rho_k := \rho(\mathbf{B}_k)$ , then  $\rho_{k+t} = (\alpha\beta)^t \rho_k$ . We now give the first lemma.

LEMMA 1. *Assume  $(\alpha, \beta)$  is admissible. Let  $t \in \mathbb{N}$  be such that  $(\alpha\beta)^t < \gamma/2$ . Suppose an interval  $\mathbf{B}_k$  occurs in the  $(\alpha, \beta)$  game, and suppose  $y \in \mathbb{R}$  is an arbitrary fixed point, then Alice can play, no matter how Bob plays, such that for every point  $x \in \mathbf{B}_{k+t}$ ,*

$$|x - y| > \frac{1}{2}\gamma\rho_k.$$

This lemma is essentially due to Schmidt, we just write it in a slightly different form in order to facilitate the proof of our theorem. See Schmidt's book [8] p. 49 for a complete proof. Here we give only the proof's main idea.

PROOF. Without loss of generality, we could assume  $y$  be the middle point of  $\mathbf{B}_k$ . Alice adopts the strategy that always selecting the most left possible inscribed interval in each turn. Then after  $t$  turns, all points in  $\mathbf{B}_{k+t}$  will satisfy the property in the lemma.  $\square$

To give the next lemma we need some notation from [3], we put them here for completeness. For any real number  $c > 0$ , let

$$\mathbf{Bad}_{\mathcal{D}}(c; i, j) := \left\{ x \in \mathbb{R} : \max\{|q|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j}\} > \frac{c}{q}, \forall q \in \mathbb{N} \right\},$$

then we see

$$\mathbf{Bad}_{\mathcal{D}}(i, j) = \bigcup_{c>0} \mathbf{Bad}_{\mathcal{D}}(c; i, j).$$

Let

$$\mathcal{C}_c := \left\{ \frac{r}{q} \in \mathbb{Q} : (r, q) = 1, q > 0, \text{ and } |q|_{\mathcal{D}} \leq \left( \frac{c}{q} \right)^i \right\}.$$

Let  $P = r/q$ , and let

$$\Delta_c(P) := \left\{ x \in \mathbb{R} : |x - P| \leq \frac{c^j}{q^{1+j}} \right\},$$

then clearly we have

$$\mathbf{Bad}_{\mathcal{D}}(c; i, j) = \mathbb{R} \setminus \bigcup_{P \in \mathcal{C}_c} \Delta_c(P).$$

Let  $R \in \mathbb{R}, R > 1$ , let  $t \in \mathbb{N}$ , both of which will be fixed in Section 3. Define

$$\mathcal{C}_{c,k} := \left\{ P = \frac{r}{q} \in \mathcal{C}_c : R^{k-1} \leq q^{(1+j)/t} < R^k \right\},$$

then we have

$$\mathcal{C}_c = \bigcup_{k=1}^{\infty} \mathcal{C}_{c,k},$$

hence

$$\mathbf{Bad}_{\mathcal{D}}(c; i, j) = \mathbb{R} \setminus \bigcup_{k=1}^{\infty} \bigcup_{P \in \mathcal{C}_{c,k}} \Delta_c(P). \quad (2.1)$$

It is this relation that will be used in the proof in Section 3.

Now we give the following lemma, the idea of which is already in [3].

LEMMA 2. *For any two different points  $P_s = r_s/q_s \in \mathcal{C}_{c,k}$ ,  $s = 1, 2$ , we have*

$$|P_1 - P_2| > c^{-i} R^{\frac{i}{1+j}} t(k-1) - \frac{1}{1+j} 2tk.$$

PROOF. By  $P = r/q \in \mathcal{C}_{c,k} \subset \mathcal{C}_c$ , we have  $|q|_{\mathcal{D}} \leq (c/q)^i$ , by the definition of the norm  $|q|_{\mathcal{D}}$ , there is an appropriate  $n \in \mathbb{N}, q^* \in \mathbb{N}$  such that

$$q = D_n q^*, \text{ and } q \notin D_{n+1} \mathbb{Z},$$

hence,

$$D_n \geq \left(\frac{q}{c}\right)^i \geq c^{-i} R^{\frac{i}{1+j}} t(k-1).$$

Now there will be  $D_{n_1}$  and  $D_{n_2}$  respectively for  $P_1$  and  $P_2$ , and one of them will divide another by the definition of  $D_n$ , so  $(q_1, q_2) \geq \min\{D_{n_1}, D_{n_2}\} \geq c^{-i} R^{\frac{i}{1+j}} t(k-1)$ . Therefore,

$$|P_1 - P_2| \geq \frac{(q_1, q_2)}{q_1 q_2} > c^{-i} R^{\frac{i}{1+j}} t(k-1) - \frac{1}{1+j} 2tk.$$

□

### 3. Proof of Theorem 1

Now we prove Theorem 1. Let  $(\alpha, \beta)$  be admissible, then  $0 < \gamma < 1$ . Fix one  $t \in \mathbb{N}$  such that  $(\alpha\beta)^t < \gamma/2$ , which is used in Lemma 1. Our aim is to show that  $\mathbf{Bad}_{\mathcal{D}}(i, j)$  is  $(\alpha, \beta)$ -winning. Without loss of generality we can assume that  $\rho_1$  is very small, so we take the following constants,

$$\begin{cases} R = \frac{1}{\alpha\beta} > 1, \\ 0 < \rho_1 < \left(\frac{1}{4} R^{-2t/(1+j)}\right)^j, \\ 0 < c < \left(\frac{1}{2} \gamma \rho_1\right)^{1/j} < \rho_1^{1/j}. \end{cases} \quad (3.1)$$

As we pointed out in Section 2 that  $\mathbf{Bad}_{\mathcal{D}}(i, j) = \bigcup_{c>0} \mathbf{Bad}_{\mathcal{D}}(c; i, j)$ , hence it suffices to show that for the  $c$  satisfying (3.1),  $\mathbf{Bad}_{\mathcal{D}}(c; i, j)$  is  $(\alpha, \beta)$ -winning.

PROOF. We prove it by showing the following two facts.

**Fact 1.** For every  $k \geq 1$ ,

$$\#\{P \in \mathcal{C}_{c,k} : \Delta_c(P) \cap \mathbf{B}_{t(k-1)+1} \neq \emptyset\} \leq 1.$$

**Fact 2.** Suppose **Fact 1** holds, then Alice can play, no matter how Bob plays, such that for every  $k \geq 1$ ,

$$\#\{P \in \mathcal{C}_{c,k} : \Delta_c(P) \cap \mathbf{B}_{tk+1} \neq \emptyset\} = 0,$$

which is equivalent to

$$\Delta_c(P) \cap \mathbf{B}_{tk+1} = \emptyset, \quad \forall P \in \mathcal{C}_{c,k}.$$

Notice that the above equation implies

$$\Delta_c(P) \cap \mathbf{B}_{tk+1} = \emptyset, \quad \forall P \in \mathcal{C}_{c,l}, \quad l = 1, 2, \dots, k.$$

Recall the relation (2.1), then **Fact 2** is equivalent to say that Alice can play such that the single point in  $\bigcap_{k=1}^{\infty} \mathbf{B}_{tk+1}$  lies in  $\mathbf{Bad}_{\mathcal{D}}(c; i, j)$ , so Alice can always win the game and we are done. Hence we are only left to show the two facts.

Now we show **Fact 1**. Let  $z$  be the middle point of  $\mathbf{B}_{t(k-1)+1}$ . For those  $P \in \mathcal{C}_{c,k}$  satisfying  $\Delta_c(P) \cap \mathbf{B}_{t(k-1)+1} \neq \emptyset$ , let  $x \in \Delta_c(P) \cap \mathbf{B}_{t(k-1)+1}$ , then

$$|P - z| \leq |P - x| + |x - z| \leq \frac{c^j}{q^{1+j}} + \rho_{t(k-1)+1} < 2\rho_1 R^{-t(k-1)}.$$

Assume there are two points  $P_1, P_2 \in \mathcal{C}_{c,k}$ ,  $P_1 \neq P_2$ , and they satisfy  $\Delta_c(P_1) \cap \mathbf{B}_{t(k-1)+1} \neq \emptyset$ ,  $\Delta_c(P_2) \cap \mathbf{B}_{t(k-1)+1} \neq \emptyset$ . We apply Lemma 2 to see that

$$c^{-i} R^{\frac{i}{1+j}t(k-1)} - \frac{1}{1+j} 2tk < |P_1 - P_2| \leq |P_1 - z| + |z - P_2| < 4\rho_1 R^{-t(k-1)},$$

which is equivalent to

$$4\rho_1 c^i > R^{\left(\frac{i}{1+j} + 1\right)t(k-1)} - \frac{1}{1+j} 2tk = R^{-\frac{2t}{1+j}}.$$

Now use our assumption for  $c$  in (3.1), then

$$R^{-\frac{2t}{1+j}} < 4\rho_1 c^i < 4\rho_1^{1+i/j} = 4\rho_1^{1/j}.$$

This contradicts to our assumption on  $\rho$ .

Now we show **Fact 2**. We proceed by induction. The base is quite clear. Suppose that an interval  $\mathbf{B}_{t(k-1)+1}$  occurs in the game and it has empty intersection with all intervals  $\Delta_c(P)$  with  $P \in \mathcal{C}_{c,k-1}$ . By **Fact 1**, there is not more than one ‘‘dangerous’’ point  $P \in \mathcal{C}_{c,k}$  such that  $\Delta_c(P) \cap \mathbf{B}_{t(k-1)+1} \neq \emptyset$ . Consider this point  $P$  as the

point  $y$  in Lemma 1. Application of Lemma 1 shows that Alice can play, no matter how Bob plays, to ensure for all  $x \in \Delta_c(P) \cap \mathbf{B}_{tk+1}$  the inequality

$$\frac{c^j}{q^{1+j}} \geq |x - P| > \frac{1}{2}\gamma\rho_{t(k-1)+1} = \frac{1}{2}\gamma\rho_1 R^{-t(k-1)}.$$

Since  $P \in \mathcal{C}_{c,k}$ , this gives

$$c^j R^{-t(k-1)} \geq \frac{c^j}{q^{1+j}} > \frac{1}{2}\gamma\rho_1 R^{-t(k-1)}.$$

This reduces to

$$c^j > \frac{1}{2}\gamma\rho_1.$$

This contradicts to our assumption on  $c$ . So  $\Delta_c(P) \cap \mathbf{B}_{tk+1} = \emptyset$ .  $\square$

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