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Universality results for the Riemann zeta-function

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Abstract: In 1975, S. M. Voronin proved that the Riemann zeta-function $\zeta(s)$ is universal in the sense that any analytic function can be approximated with a given accuracy uniformly on compact subsets of the critical strip by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. In the paper, we present a short survey on the Voronin theorem. The main attention is devoted to effectivization problem of the universality theorem as well as to the universality of $F(\zeta(s))$ for some classes of functions F.

Keywords: effectivization problem, Riemann zeta-function, space of analytic functions, universality, Voronin theorem, weak convergence

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1. Introduction

Until nowadays, the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, which is defined, for $\sigma > 1$, by a very simple Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and is analytically continuable to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1, remains one of the most interesting and mysterious mathematical objects. Value-distribution of the function $\zeta(s)$, in particular, the location of non-trivial zeros keeps still many secrets.

Already H. Bohr observed that the set of values of $\zeta(s)$ is very rich. He proved, see [33], that, in the strip $\{s \in \mathbb{C} : \ 1 < \sigma < 1 + \delta\}$ with any $\delta > 0$, $\zeta(s)$ takes every non-zero value infinitely many times. Moreover, jointly with R. Courant, he obtained [4] that, for $\frac{1}{2} < \sigma < 1$, the set

$$\{\zeta(\sigma + i\tau) : \ t \in \mathbb{R}\} \tag{1}$$

is dense in $\mathbb{C}$. S. M. Voronin extended [34] the above result to the space $\mathbb{C}^N$, $N \in \mathbb{N}$, proving the denseness in $\mathbb{C}^N$ of the set

$$\left\{\left(\zeta(\sigma + it), \zeta'(\sigma + it), \ldots, \zeta^{(N-1)}(\sigma + it)\right) : \ t \in \mathbb{R}\right\} \tag{2}$$

with $1/2 < \sigma < 1$. Of course, the latter result is very deep, however, it is not so important than the following Voronin theorem [35] which reveals one more denseness property of the Riemann zeta-function.

THEOREM 1. *Suppose that* $0 < r < \frac{1}{4}$. *Let* $f(s)$ *be a continuous non-vanishing function on the disc* $|s| \leqslant r$ *which is analytic in the interior of this disc. Then, for every* $\varepsilon > 0$, *there exists a real number* $\tau = \tau(\varepsilon)$ *such that*

$$\max_{|s| \leqslant r} \left|\zeta\left(s + \frac{3}{4} + i\tau\right) - f(s)\right| < \varepsilon. \tag{3}$$

Voronin called a property of $\zeta(s)$ in Theorem 1 the universality. This name is quite precise because really the function $\zeta(s)$ is universal in the sense that, roughly speaking, any analytic function uniformly on the disc $|s| \leqslant r$, $r < \dfrac{1}{4}$, can be approximated with a desired accuracy by a shift $\zeta\left(s + \dfrac{3}{4} + i\tau\right)$ with some $\tau \in \mathbb{R}$.

On the other hand, since the space of analytic functions is infinite-dimensional, Theorem 1 can be considered as an infinite-dimensional version of the above Bohr—Courant result on denseness of the set (1).

Theorem 1 can be improved in two directions. Firstly, the disc $|s| \leqslant r$ can be replaced by any compact subset. Secondly, it can be proved that there are infinitely many shifts $\zeta(s + i\tau)$ approximating a given analytic function. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and, for $K \in \mathcal{K}$, let $H_0(K)$ be the class of continuous non-vanishing functions on K which are analytic in the interior of K. Moreover, let meas A stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then a modern version of Theorem 1 is of the form.

THEOREM 2. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

The inequality of Theorem 2 shows that the set of shifts $\zeta(s+i\tau)$ approximating the function $f(s)$ is infinite and has a positive lower density.

The main ingredient in the initial Voronin proof of Theorem 1 is an analogue of Riemann's theorem on rearrangement of terms for series in Hilbert spaces [26], [27]. This analogue is applied to prove the main lemma on the approximation of analytic functions by finite Euler products. Using this and the Kronecker approximation theorem, Voronin approximates $\zeta(s)$ in the mean by a finite Euler product, and obtains Theorem 1.

A. Reich by Voronin's method extended [28] Theorem 1 for general Euler products, and observed that the set of numbers τ satisfying inequality (3) has a positive lower density. We note that this also follows from Voronin's paper [35].

S. M. Gonek in his thesis [7] proposed a method different from that of Voronin. This method, in a certain sense, is related to an approach of A. Good [8] who combined Voronin's method with a method of H. L. Montgomery [23] for Ω-estimates of log $\zeta(s)$, and obtained interesting quantitative results including Theorem 1. Moreover, Gonek's version of Theorem 1 is proved, in place of a disc, for simply connected compact subsets K of the strip D. Let $f(s)$ be a continuous function on K and analytic in the interior of K. First, Gonek shows that if n is sufficient large, then there exist a $N > n$ and real numbers θ_p (p denotes a prime number) such that

$$\max_{s\in K} \left| \sum_{n<p<N} \frac{e^{2\pi i\theta_p}}{p^s} - f(s) \right| < \frac{\varepsilon}{2}.$$

For this, the fundamental lemma on the approximation of analytic functions by partial sums of general Dirichlet series, different from the main lemma of Voronin, is applied. Further, combining mean-value estimates and zero-density estimates for the function $\zeta(s)$, and applying the Kronecker approximation theorem, he deduces that, for a suitably chosen branch of the logarithm, there is a $\tau \in \mathbb{R}$ such that

$$\max_{s\in K}\left|\log\zeta(s+i\tau)-\sum_{n<p<N}\frac{e^{2\pi i\theta_p}}{p^s}\right|<\frac{\varepsilon}{2}.$$

Two latter inequalities imply the existence of $\tau \in \mathbb{R}$ such that

$$\max_{s\in K}|\log\zeta(s+i\tau)-f(s)|<\varepsilon.$$

Hence, Theorem 1 follows.

B. Bagchi proposed [1] another proof of the universality theorem for $\zeta(s)$. His approach is based on probabilistic limit theorems for weakly convergent probability measures in the space of analytic functions. Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic functions on G endowed with the topology of uniform convergence on compacta. Let $\mathcal{B}(X)$ stand for the class of Borel sets of the space X. Define the torus

$$\Omega=\prod_p \gamma_p,$$

where γ_p, for all primes p, is the unit circle on the complex plane. Since the unit circle is compact, the torus Ω, in view of the Tikhonov theorem, with the product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on the measurable space $(\Omega,\mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and this leads to the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. For each prime p, let $\omega(p)$ denote the projection of $\omega\in\Omega$ to the circle γ_p. Now, on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$, define the $H(D)$-valued random element $\zeta(s,\omega)$ by the formula

$$\zeta(s,\omega)=\prod_p\left(1-\frac{\omega(p)}{p^s}\right)^{-1}.$$

The latter infinite product, for almost all $\omega\in\Omega$ with respect to the Haar measure m_H, converges uniformly on compact subsets of the strip D, and therefore, defines a $H(D)$-valued random element. Let P_ζ be the distribution of $\zeta(s,\omega)$, i. e., P_ζ is a probability measure on $(H(D), \mathcal{B}(H(D)))$ defined by

$$P_\zeta(A) = m_H\left(\omega \in \Omega: \ \zeta(s,\omega) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

The first step in the Bagchi method is a limit theorem on the weak convergence of

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T]: \ \zeta(s+i\tau) \in A\right\}, \quad A \in \mathcal{B}(H(D)), \tag{4}$$

to the measure P_ζ as $T \to \infty$.

The second step consists of finding the support of the measure P_ζ. By using some elements of Hilbert spaces and of the theory of exponential functions, it is proved that the support of P_ζ is the set

$$S \overset{def}{=} \left\{g \in H(D): \ g(s) \neq 0 \text{ or } g(s) \equiv 0\right\}.$$

Now it is easy to obtain Theorem 2. The Mergelyan theorem on the approximation of analytic functions by polynomials [22], see also [37], is applied: *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and $f(s)$ be a continuous on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Since $f(s) \neq 0$ on K, by the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} \left|f(s) - e^{p(s)}\right| < \frac{\varepsilon}{2}. \tag{5}$$

Let

$$G = \left\{g \in H(D): \ \sup_{s\in K}\left|g(s) - e^{p(s)}\right| < \frac{\varepsilon}{2}\right\}.$$

Then G is an open neighbourhood of the function $e^{p(s)}$ which is an element of the support of P_ζ. Thus, $P_\zeta(G) > 0$. This and the weak convergence of the measure (4) show that

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T]: \ \sup_{s\in K}\left|\zeta(s+i\tau) - e^{p(s)}\right| < \frac{\varepsilon}{2}\right\} \geqslant P_\zeta(G) > 0.$$

Combining this with (5) gives Theorem 2.

The full proof of Theorem 2 can be found in [13].

Note that in [1], Theorem 2 is proved in the case when K is a compact simply connected and locally path connected subset of the strip D, however, a simple modification of the proof gives the statement for $K \in \mathcal{K}$.

The Voronin theorem has many applications. We mention one theoretical and one practical of them. The universality theorem implies the functional independence of $\zeta(s)$. The problem of the independence of functions comes back to O. Hölder and D. Hilbert. In [10], Hölder proved the algebraic-differential independence for the gamma-function $\Gamma(s)$. This means that there exists no polynomial $p \not\equiv 0$ such that, for $N \in \mathbb{N}_0$,

$$p\left(s, \Gamma(s), \Gamma'(s), \dots, \Gamma^{(N)}(s)\right) = 0$$

for all s. Hilbert during the ICM in Paris in 1900 noted that the Riemann zeta-function also does not satisfy any algebraic-differential equation, and that this can be obtained by using the functional equation for $\zeta(s)$ and Hölder's result on the algebraic-differential independence of $\Gamma(s)$. Moreover, he conjectured the algebraic-differential independence of the function

$$\zeta(s, x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}.$$

These problems were solved by A. Ostrowski in [24]. Using the universality theorem, Voronin obtained in [36] the functional independence of the Riemann zeta-function. He proved that the function $\zeta(s)$ does not satisfy any differential equation

$$\sum_{m=0}^{n} s^m F_m\left(y(s), y'(s), \dots, y^{(N-1)}(s)\right) = 0,$$

where $F_0, \dots, F_n$ are continuous functions not all identically zero.

In [2], the Voronin theorem was applied for estimation of complicated integrals over analytic curves which are used in quantum mechanics.

The aim of this paper is to give a short survey of recent results related to the universality of $\zeta(s)$ in the frame of Theorems 1 and 2. We will discuss the effectivization problem of the Voronin theorem as well as will present some results on the extension of Theorem 2 for some classes of functions $F(\zeta(s))$.

2. Effectivization problem

Before Voronin, universal in a certain sense objects in analysis also were known, however, they were not explicitly given. The first universal object was found by M. Fekete, see [25]. He proved that there exists a real power series

$$\sum_{m=1}^{\infty} a_m x^m$$

such that, for every continuous function $f(x)$ on the interval $[-1, 1]$, $f(0) = 0$, there exists an increasing sequence of positive integers n_k for which

$$\lim_{k\to\infty} \sum_{m\leqslant n_k} a_m x^m = f(x)$$

uniformly on $[-1, 1]$. We note that only the existence of the above series was proved, examples were not presented. Several other universal but not explicitly given objects are described in a very informative paper [9]. We mention once one of them which has a certain similarity to the Voronin universality theorem. In [3], G. D. Birkhoff proved a theorem on universality of an entire function. From his theorem, it follows that there exists an entire function $g(s)$ such that, for any entire function $f(s)$, a compact subset $K \subset \mathbb{C}$ and arbitrary $\varepsilon > 0$, there exists a complex number a such that

$$\sup_{s\in K} |g(s+a) - f(s)| < \varepsilon.$$

The Birkhoff theorem is non-effective, it gives only the existence of the function $g(s)$. Also, the number $a = a(f, K, \varepsilon)$ is not effectively defined. However, a common feature of the Birkhoff and Voronin theorems is the involving of shifts in the definitions of universal properties.

All other universal objects known before Voronin also were non-explicit. Thus, the Riemann zeta-function is the first explicitly given universal object. Appendix of the monograph [31] contains more of information on universal objects.

Unfortunately, the Voronin theorem is also non-effective in the following sense. Though, by Theorem 2, the set of shifts $\zeta(s + i\tau)$approximating a given analytic function is infinite, we do not know any concrete value τ with approximation property. For some applications, it is sufficient even to know an interval $[0, T]$

containing such a τ that, for given $\varepsilon > 0$, K and $f(s)$ satisfying the hypotheses of Theorem 2,

$$\sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon.$$

Voronin knew this short-coming of his theorem, and invited the author to prove an effective version of the universality theorem.

The first attempt in the effectivization of Voronin's theorem was made by A. Good in already mentioned paper [8]. He considered the logarithm $\log\zeta(s)$ which denotes the unique analytic function on the open region in the half-plane $\sigma > 1/2$ defined as

$$\log\zeta(s) = \sum_{m=1}^{\infty} \frac{\Lambda_1(m)}{m^s}$$

for $\sigma > 1$, where

$$\Lambda_1(m) = \begin{cases} \dfrac{1}{k} & \text{if} \quad m = p^k,\ k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The results of [8] give quantitative versions of Theorem 1 and of earlier Voronin's results on the denseness of the set of values of $\zeta(s)$. The statements of theorems from [8] are rather complicated, therefore, we present only a part of one of them.

For a small positive ε, let $\frac{1}{2} + \varepsilon < \sigma_1 < 1 - \varepsilon$. Let β, r_1, and R_1 satisfy the inequalities

$$0 < \beta, r_1, R_1 < 1, \quad \varepsilon \leqslant \beta + R_1 < 1, \quad r_1 < \frac{\delta}{\mathrm{e}}\mathrm{e}^{-1/\delta},$$

where

$$\delta = \frac{1 - R_1 - \beta}{\log\left(\dfrac{2\mathrm{e}}{R_1}\right)}.$$

Define $r = r_1(1-\sigma_0)$, $R = R_1(1-\sigma_0)$ and $\frac{1}{2}+\varepsilon+r \leqslant \sigma_0 < 1-\varepsilon-r$. As usual, let

$$\pi(x) = \sum_{p\leqslant x} 1.$$

Then, Good obtained the following assertion [8].

THEOREM 3. *There are the positive constants* c_1, c_2 *and* c_3, *and a function* $f_0(s)$ *which is analytic in the half-plane* $\sigma > \frac{1}{2}$ *such that, for* $T \geqslant c_1$,

$$c_2 \leqslant V \leqslant \left(\frac{\rho}{\log \rho}\right)^{1/2}, \quad \rho \leqslant c_3 \left(\log T \log^{1/2} V\right)^{1/2} \left(V \log^{1/2} V\right)^{-5/(2(2\sigma_1-1))}$$

and a function

$$f(z) = \sum_{m=0}^{\infty} w_k z^k$$

with complex numbers w_k *satisfying*

$$|w_k| \leqslant \frac{\rho^{\beta(1-\sigma_0)}}{\log^4 \rho} R^{-k}, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

the number of positive integers $n \leqslant T$ *such that*

$$\log \zeta(s + ina) = f_0(s) + f(s - \sigma_0) + \mathrm{O}\left(\frac{\rho^{1-\sigma}}{V \log \rho} + \frac{\rho^{(1-\delta)(\frac{1}{2}-\sigma)}}{\log \rho} + \frac{\rho^{-a(1-\sigma_0)}}{\log \rho}\right)$$

with

$$a = \delta \log\left(\frac{\delta}{\mathrm{e} r_1}\right) - 1 > 0,$$

for $|s - s_0| \leqslant r$, *exceeds* $\frac{1}{5} T V^{-\pi(\rho)}$.

For sufficiently small r, Theorem 3 improves Theorem 1. Moreover, the interval $[0, c_1]$ contains at least one n with approximation property of $\log \zeta(s + ina)$. Thus, it remains to estimate the constant c_1.

The Good method does not use a non-effective analogue of the Riemann theorem on rearrangement of series in Hilbert spaces. Instead of that theorem, differently from Voronin, Good deals with some convexity results including the tree-circle theorem, see [33], and the Hahn—Banach theorem, see [29].

Important effective results concerning Theorem 1 were obtained by R. Garunkštis [5], however, in a very narrow region. Refining the Good method, Garunkštis indicated an interval containing a number τ with approximation property. We state his particular result.

THEOREM 4 ([5]). *Let $0 < \varepsilon < \frac{1}{2}$. Suppose that the function $f(s)$ is analytic in the disc $|s| \leqslant 0.06$, and that $\max_{|s|\leqslant 0.06} |f(s)| \leqslant 1$. Then there exists $\tau \in \mathbb{R}$,*

$$0 < \tau \leqslant \exp\left\{\exp\left\{10\varepsilon^{-13}\right\}\right\},$$

such that

$$\max_{s\leqslant 0.0001} \left|\log\zeta\left(s+\frac{3}{4}+i\tau\right) - f(s)\right| < \varepsilon.$$

Moreover,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau\in[0,T] : \max_{s\leqslant 0.0001}\left|\log\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon\right\} \geqslant$$
$$\geqslant \exp\{-\varepsilon^{-13}\}.$$

Now we state a recent effective result related to the Voronin theorem. This result is based on the effective version of a statement on the denseness of the set (2) [11].

THEOREM 5. *Let $\sigma \in (\frac{1}{2}, 1)$, $\varepsilon \in (0,1)$ and $\boldsymbol{b} = (b_0, b_1, \dots, b_{N-1}) \in \mathbb{C}^N$ with $|b_0| > \varepsilon$ be fixed numbers. Then the system of inequalities*

$$\left|\zeta^{(k)}(\sigma+it) - b_k\right| < \varepsilon, \quad k = 0, 1, \dots, N-1,$$

has a solution $t \in [T, 2T]$ provided that

$$T \geqslant C(N,\sigma)\exp\left\{\exp\left\{5A(n,\boldsymbol{b},\varepsilon)^{\frac{8}{1-\sigma}+\frac{8}{\sigma-\frac{1}{2}}}\right\}\right\},$$

where $C(N,\sigma)$ is a positive effectively computable constant depending only on N and σ, and

$$A(n,\boldsymbol{b},\varepsilon) = |\log|b_0|| + \left(\frac{||\boldsymbol{b}||}{\varepsilon}\right)^{N^2}$$

with $||\boldsymbol{b}|| = \sum_{k=0}^{N-1} |b_k|$.

Let r be a positive real number, $\sigma_0 \in (1/2, 1)$, $s_0 = \sigma_0 + it_0$ and $K=\{s\in\mathbb{C}: |s-s_0|\leqslant r\}$. Define

$$M(\tau) = \max_{|s-s_0|<r} |\zeta(s+i\tau)| .$$

Moreover, for any analytic function $f(s)$ on $|s-s_0|<r$, we put

$$\boldsymbol{f} = \Big(f(s_0), f'(s_0), \dots, f^{(N-1)}(s_0)\Big) .$$

Then we have the following effective result [6].

THEOREM 6. *Let $f: K \to \mathbb{C}$ be a continuous, $f(s_0) \neq 0$ and analytic for $|s-s_0|<r$ function. Then, for any $\varepsilon \in (0, |f(s_0)|)$, there exist real numbers $\tau \in [T, 2T]$ and $\delta = \delta(\varepsilon, f, \tau) > 0$ defined by*

$$M(\tau)\frac{\delta^N}{1-\delta} = \frac{\varepsilon}{3}(2 - \exp\{\delta r\})$$

such that

$$\max_{|s-s_0|\leqslant\delta r} |\zeta(s+i\tau) - f(s)| < \varepsilon,$$

where $T = T(f, \varepsilon, \sigma_0)$ satisfies

$$T \geqslant C(N, \sigma_0) \exp\left\{\exp\left\{5A\left(N, \boldsymbol{f}, \frac{\varepsilon}{3}\right)^{\frac{8}{1-\sigma_0}+\frac{8}{\sigma_0-\frac{1}{2}}}\right\}\right\}$$

and $T > r$, and $C(N, \sigma_0)$ and δ are effectively computable.

We note that the assertion of Theorem 6 also remains valid in the case $f(s_0) = 0$ if to replace $A\,(N, \boldsymbol{f}, \varepsilon/3)$ in the lower bound for T by $A\,(N, \boldsymbol{f}_\varepsilon, \varepsilon/3)$, where

$$\boldsymbol{f}_\varepsilon = \Big(\frac{\varepsilon}{2}, f'(s_0), \dots, f^{(N-1)}(s_0)\Big) .$$

Thus, Theorem 6 is a satisfactory effectivization of Theorem 1.

Also, it is known [14] that the effectivization problem of the Voronin theorem can be reduced to the estimation of the rate of convergence in a limit theorem for the probability measure (4). Suppose that a polynomial $p(s)$ satisfies inequality (5).

Define an open ball

$$A(\varepsilon) = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2} \right\},$$

and suppose that ε is such that the set $A(\varepsilon)$ is a continuity set of the limit measure P_ζ. Clearly, this is true for all $\varepsilon > 0$, except, maybe, for a countable set of values. Then we have that

$$\frac{1}{T} \mathrm{meas} \{ \tau \in [0, T] : \zeta(s + i\tau) \in A(\varepsilon) \} = P_\zeta(A(\varepsilon)) + R_T(\varepsilon) \tag{6}$$

with

$$\lim_{T \to \infty} R_T(\varepsilon) = 0.$$

Since $\mathrm{e}^{p(s)}$ is an element of the support of the measure P_ζ, and the set $A(\varepsilon)$ is open, we have that $P_\zeta(A(\varepsilon)) > 0$.

THEOREM 7. *Suppose that $T > 0$ satisfies the inequality*

$$|R_T(\varepsilon)| < P_\zeta(A(\varepsilon)). \tag{7}$$

Then there exists a real number $\tau \in [0, T]$ such that

$$\sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon. \tag{8}$$

PROOF. Inequality (7) shows that the right-hand side of (6) is positive. Thus, the left-hand side is positive, too. Therefore, there exist the numbers $\tau \in [0, T]$ such that

$$\sup_{s \in K} \left| \zeta(s + i\tau) - \mathrm{e}^{p(s)} \right| < \frac{\varepsilon}{2}.$$

This together with (5) shows that with the same τ inequality (8) is satisfied. □

We note that, because the strip D is not compact, the estimation of $R_T(\varepsilon)$ is a complicated problem.

Effective upper bounds for the upper density of universality also belong to a group of effective results related to Voronin's theorem, and were obtained by J. Steuding in [30]. His results are valid for a wide class of meromorphic functions, however, we present a corollary for $\zeta(s)$.

Let B_r denote a closed disc $|s| \leqslant r$. The mapping $g: \ B_r \to B_1$ is called analytic isomorphism if the inverse g^{-1} exists and is analytic. Clearly, g has exactly one simple zero ρ in the interior of B_r, and the Schwarz lemma [32] implies that

$$g(s) = r\mathrm{e}^{i\varphi} \frac{\rho - s}{r^2 - \overline{\rho} s}$$

with $\varphi \in \mathbb{R}$, $|s| < r$. Let $\mathcal{A}_r$ be the class of analytic isomorphisms $g: \ B_r \to B_1$, and $N(\sigma_1, \sigma_2, T)$ denote the number of zeros of $\log \zeta(s)$ lying in the rectangle $1/2 < \sigma_1 < \sigma < \sigma_2 < 1$, $0 \leqslant t < T$, according multiplicities.

THEOREM 8. ([30]) *Suppose that $g \in \mathcal{A}_r$. Then, for any $\varepsilon \in (0, (2r)^{-1}(\frac{1}{4} + |\mathrm{Re}\rho|)$,*

$$\limsup_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \ \max_{|s|\leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau \right) - \mathrm{e}^{g(s)} \right| < \varepsilon \right\} \leqslant$$
$$\leqslant \frac{8\mathrm{e}r^3\varepsilon}{r^2 - |\rho|^2} \lim_{T\to\infty} \frac{1}{T} N\left(\frac{3}{4} + \mathrm{Re}\rho - 2e\varepsilon, \frac{3}{4} + \mathrm{Re}\rho + 2e\varepsilon, T \right) = \mathrm{o}(\varepsilon)$$

as $\varepsilon \to 0$.

We see that Theorem 8 relates the upper density of universality to the value-distribution of $\zeta(s)$. Moreover, the decay of that density is more than linear as $\varepsilon \to 0$.

3. Extension of Theorem 2

In this section, we consider the universality of $F(\zeta(s))$ for some classes of operators $F: \ H(D) \to H(D)$.

Define the logarithm $\log \zeta(s)$ in the strip $\{s \in \mathbb{C} : \ 1/2 < \sigma < 1\}$ more precisely, namely, by continuous variation from the value $\log \zeta(2) \in \mathbb{R}$ along the line segments $[2, 2+it]$ and $[2+it, \sigma+it]$ provided that the path does not pass a possible zero or pole $s = 1$ of $\zeta(s)$. If it does, then we take

$$\log \zeta(\sigma + it) = \lim_{\varepsilon\to+0} \log \zeta(\sigma + i(t + \varepsilon)).$$

Then it is known [11] that the function $\log \zeta(s)$ is universal in the sense of Theorem 1. In Introduction, we have mentioned that Gonek also deduced [7] the universality of $\zeta(s)$ from that of $\log\zeta(s)$. However, in the case of $\log\zeta(s)$, the approximated function $f(s)$ is not necessarily non-vanishing on the disc $|s| \leqslant r$ (on the set K in Gonek's version). Using of $\log\zeta(s)$ in place of $\zeta(s)$ often is more convenient because the investigation of products is replaced by that of sums.

Theorem 9. ([7], [11]) *Suppose that* $0 < r < \frac{1}{4}$. *Let* $f(s)$ *be a continuous function on the disc* $|s| \leqslant r$ *which is analytic in the interior of this disc. Then, for every* $\varepsilon > 0$, *there exists a real number* $\tau = \tau(\varepsilon)$ *such that*

$$\max_{|s|\leqslant r} \left|\log\zeta\left(s + \frac{3}{4} + i\tau\right) - f(s)\right| < \varepsilon.$$

Clearly, Theorem 9 implies Theorem 1. However, a version of Theorem 2 for $\log\zeta(s)$ is not known. As it was noted in [17], such a version does not follows directly from Theorem 2.

Also, an analogue of Theorem 2 is true [1], [12] for the derivative $\zeta'(s)$. In this case, as in Theorem 9, the approximated function can have zeros on K. Therefore, for brevity, denote by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K. Then we have the following statement.

Theorem 10 ([1], [12]). *Let* $K \in \mathcal{K}$ *and* $f \in H(K)$. *Then, for every* ε,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] :\ \sup_{s\in K} \left|\zeta'(s+i\tau) - f(s)\right| < \varepsilon\right\} > 0.$$

Theorems 9 and 10 show that the function $F(\zeta(s))$ with $F(g) = \log g$ and $F(g) = g'$ preserves the universality property of $\zeta(s)$. Thus, a problem arises to indicate other functions F such that $F(\zeta(s))$ could be universal in the above sense. This problem has been considered in [15] and [17].

The first class of universal functions $F(\zeta(s))$ is inspired by Theorem 10. We use an analogue of the classical Lipschitz condition in the space of analytic functions. We say that the operator $F :\ H(D) \to H(D)$ belongs to the class $Lip(\alpha)$ if it satisfies the hypotheses:

1° For every polynomial $p = p(s)$ and every $K \in \mathcal{K}$, there exists an element $g \in F^{-1}\{p\} \subset H(D)$ such that $g(s) \neq 0$ on K;

2° For every $K \in \mathcal{K}$, there exists positive constants c and α, and a set $K_1 \in \mathcal{K}$, such that

$$\sup_{s \in K} |F(g_1(s)) - F(g_2(s))| \leqslant c \sup_{s \in K_1} |g_1(s) - g_2(s)|^\alpha$$

for all $g_1, g_2 \in H(D)$.

The universality of $F(\zeta(s))$ with $F \in Lip(\alpha)$ is given in the following theorem.

THEOREM 11 ([17]). *Suppose that $F \in Lip(\alpha)$, $K \in \mathcal{K}$ and $f \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |F(\zeta(s + i\tau)) - f(s)| < \varepsilon \right\} > 0.$$

It follows easily from the Cauchy integral formula that $F: \ H(D) \to H(D)$ given by $F(g) = g^{(n)}$, $g \in H(D)$, $n \in \mathbb{N}$, is an element of the $Lip(1)$. Thus, all derivatives $\zeta^{(n)}(s)$, $n \in \mathbb{N}$, are universal, and Theorem 11 is a generalization of Theorem 10.

Theorem 11 is deduced directly from Theorem 2 by using the Mergelyan theorem.

The probabilistic approach described in Introduction for the proof of Theorem 2 leads to other classes of universal functions $F(\zeta(s))$. We recall that

$$S = \{g \in H(D) : \ g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

THEOREM 12 ([15]). *Suppose that $F: \ H(D) \to H(D)$ is a continuous operator such that, for every open set $G \in H(D)$, the set $(F^{-1}G) \cap S$ is non-empty. Let K and $f(s)$ be the same as in Theorem 11. Then the same assertion as in Theorem 11 is true.*

Theorem 12 can be made more convenient. The hypothesis on the intersection $(F^{-1}G) \cap S$ can be replaced by a simple one involving polynomials.

THEOREM 13 ([17]). *Suppose that $F: \ H(D) \to H(D)$ is a continuous operator such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S$ is non-empty. Let K and $f(s)$ be the same as in Theorem 11. Then the same assertion as in Theorem 11 is true.*

Theorem 13 is possible because the approximation in the space $H(D)$ reduces to that on compact sets $K \in \mathcal{K}$. Really, for an arbitrary open region $G \in \mathbb{C}$, we can use in the space $H(G)$ the metric

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s\in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s\in K_m} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(G),$$

where $\{K_m : m \in \mathbb{N}\}$ is a sequence of compact subsets of the region G such that

$$G = \bigcup_{m=1}^{\infty} K_m,$$

$K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and if $K \subset G$ is a compact subset, then $K \subset K_m$ for some m. Therefore, it is not difficult to see that the approximation in $H(G)$ reduces to that on the sets K_m with large enough m. Obviously, in the case of the space $H(D)$, we can choose the sets $K_m \in \mathcal{K}$. This remark, the Mergelyan theorem and Theorem 12 leads to Theorem 13.

The hypothesis that $(F^{-1}\{p\}) \cap S \neq \varnothing$ is related to the non-vanishing of the function from the set $F^{-1}\{p\}$. Therefore, it is more convenient to replace the region D by a bounded one. Let $V > 0$ be an arbitrary number,

$$D_V = \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, \ |t| < V \right\}$$

and

$$S_V = \{ g \in H(D_V) : \ g(s) \neq 0 \text{ or } g(s) \equiv 0 \}.$$

Then we have the following analogue of Theorem 13.

THEOREM 14 ([15]). *Let K and $f(s)$ be the same as in Theorem* 11, *and that $V > 0$ be such that $K \subset D_V$. Suppose that $F : H(D_V) \to H(D_V)$ is a continuous operator such that, for each polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \cap S_V$ is non-empty. Then the same assertion as in Theorem* 11 *is true.*

It is easily seen that, for some operators $F : H(D_V) \to H(D_V)$, for each polynomial $p = p(s)$, there exists a polynomial $q = q(s)$, $q \in F^{-1}\{p\}$ and $q(s) \neq 0$ for $s \in D_V$. For example, this is true for the operator given by

$$F(g) = c_1 g' + \cdots + c_r g^{(r)}, \quad g \in H(D_V), \quad c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\}.$$

Therefore, the function $F(\zeta(s))$ is universal in the sense of Theorem 11.

Now we will approximate analytic functions from some subsets of $H(D)$ by shifts $F(\zeta(s+i\tau))$. This allows to extend the class of universal functions.

For $a_1, \dots, a_r \in \mathbb{C}$ and $F:\ H(D) \to H(D)$, define

$$H_{F(0);a_1,\dots,a_r}(D) = \Big\{ g \in H(D):\ (g(s) - a_j)^{-1} \in H(D),\ j = 1, \dots, r \Big\} \cup \{F(0)\}.$$

THEOREM 15 ([17]). *Suppose that $F:\ H(D) \to H(D)$ is a continuous operator such that $F(S) \supset H_{F(0);a_1,\dots,a_r}(D)$. If $r = 1$, let $K \in \mathcal{K}$, and $f \in H(K)$ and $f(s) \neq a_1$ on K. If $r \geqslant 2$, let $K \subset D$ be an arbitrary compact subset, and $f \in H_{F(0);a_1,\dots,a_r}(D)$. Then the same assertion as in Theorem 11 is true.*

We give an example in Theorem 15. Let $r = 2$ and $a_1 = -1$, $a_2 = 1$. Then Theorem 15 shows that the functions from the class $H_{0;-1,1}(D)$ can be approximated uniformly on compact subsets of D by shifts $\sin(\zeta(s+i\tau))$. A similar assertion also holds for the functions $\cos(\zeta(s))$, $\sinh(\zeta(s))$ and $\cosh(\zeta(s))$. For example, to show the inclusion $F(S) \supset H_{0;-1,1}(D)$ with $F(g) = \sin g$, $g \in H(D)$, we have to consider the equation

$$\frac{e^{ig(s)} - e^{-ig(s)}}{2i} = f(s)$$

which implies that $g(s) \in S$ if $f(s) \in H(D)$ does not take the values -1 and 1 on D.

The case $r \geqslant 2$ of Theorem 15 can be generalized for all functions $f \in F(S)$. Thus, we have the following statement.

THEOREM 16 ([17]). *Suppose that $F:\ H(D) \to H(D)$ is a continuous operator. Let $K \subset D$ be an arbitrary compact subset, and $f \in F(S)$. Then the same assertion as in Theorem 11 is true.*

In Theorem 16 and in the case $r \geqslant 2$ of Theorem 15, we do not require that the set $K \in \mathcal{K}$. In these cases, the condition $f \in F(S)$ implies that the function $f(s)$ is an element of the support of the measure P_ζ which was defined in Introduction, thus, we do not need to use the Mergelyan theorem.

Let $F:\ H(D) \to H(D)$ be given by $F(g) = gg'$, $g \in H(D)$. Then we have the equation

$$\Big(g^2(s)\Big)' = \frac{1}{2} f(s).$$

Hence, for some $s_0 \in D$,

$$g^2(s) = \frac{1}{2}\int\limits_{s_0}^{s} f(z)\mathrm{d}z \stackrel{def}{=} f_1(s),$$

and $g(s) = \sqrt{f_1(s)}$ if $f_1(s) \neq 0$ on D. Then, $f(s) \in F(S)$ and, by Theorem 16, can be approximated by shifts $\zeta(s+i\tau)\zeta'(s+i\tau)$.

Now let $M(D)$ denote the space of meromorphic functions on D equipped with the topology of uniform convergence on compacta. The space $H(D)$ is a subspace of $M(D)$. Theorem 16 can be rewritten for operators with values in $M(D)$.

Theorem 17 ([19]). *Suppose that $F: H(D) \to M(D)$ is a continuous operator. Let $K \subset D$ be an arbitrary compact subset, and $f \in F(S) \cap H(D)$. Then the same assertion as in Theorem* 11 *is true.*

For example, if $f(s)$ is a non-vanishing analytic function on D, then $f(s)$ can be approximated uniformly on compact subsets $K \subset D$ by shifts $\zeta^{-1}(s+i\tau)$. Moreover, any function $f \in H(D)$ uniformly on $K \subset D$ is approximated by $\frac{\zeta'(s+i\tau)}{\zeta(s+i\tau)}$.

The results of Section 3 remain valid with suitable changes for other zeta-functions: Hurwitz zeta-function [18], zeta-functions attached to cusp forms [21], periodic Hurwitz zeta-functions [20], Lerch zeta-functions [16].

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