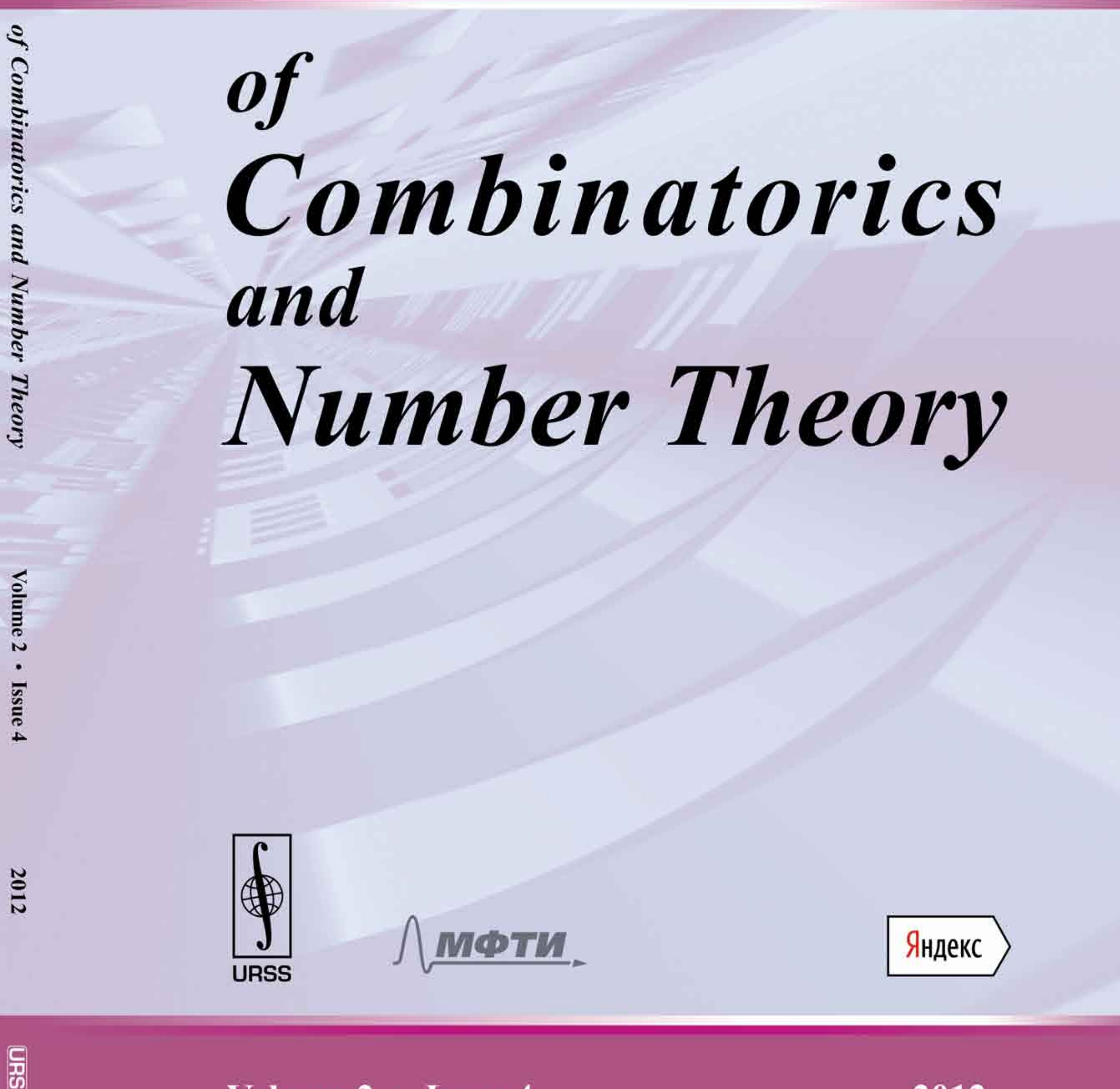
Reprint from

ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory



Volume 2 • Issue 4 2012

Moscow Journal of Combinatorics and Number Theory Volume 2 • Issue 4 2012

Moscow Journal of Combinatorics and Number Theory. 2012. Vol. 2. Iss. 4. 88 p.

The journal was founded in 2010.

Published by the Moscow Institute of Physics and Technology
with the support of Yandex.

Издание настоящего выпуска журнала осуществляется
при финансовой поддержке ООО «ЯНДЕКС».

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

Website of our journal
`http://mjcnt.phystech.edu`

E-mail
`mjcnt@phystech.edu`

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141700

URSS Publishers
56, Nakhimovsky Prospekt,
Moscow,
Russia,
117335

Издательство «УРСС»
Нахимовский пр-т, 56
Москва,
Российская Федерация,
117335

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77–41900.

Формат 70×100/16. Печ. л. 5,5. Зак. № ВУ-17.

Отпечатано в ООО «ЛЕНАНД».
117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.

ISSN 2220–5438

13920 ID 171734
9 785453 000500

Moscow Journal
of Combinatorics
and Number Theory
2012, vol. 2, iss. 4,
pp. 63–86, [pp. 357–380]

Choosability in simple hypergraphs

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Abstract: The work deals with a combinatorial problem concerning list colorings of simple hypergraphs. A hypergraph $H = (V, E)$ is said to be r-choosable if, for every family of color lists $L = \{L(v) : v \in V\}$, satisfying $|L(v)| = r$ for every $v \in V$, there exists a choice function f such that $f(v) \in L(v)$ for every $v \in V$, and no edge of H is monochromatic under f. We prove that if H is an n-uniform non-r-choosable simple hypergraph, $n \geqslant 3$, $r \geqslant 2$, then its maximum vertex degree $\Delta(H)$ satisfies the inequality

$$\Delta(H) \geqslant c\, r^{n-1} n^{-1/3},$$

where $c > 0$ is an absolute constant.

Keywords: list colorings, hypergraph colorings, sparse hypergraphs, random recoloring method

AMS Subject classification: 05C15, 05C65, 05D40

Received: 21.10.2012

1. Introduction and background of the problem

This work deals with an extremal combinatorial problem concerning list colorings of simple hypergraphs. First of all, we recall some definitions.

A hypergraph $H = (V, E)$ is said to be *simple* if every two of its distinct edges do not share more than one common vertex, i. e., formally,

$$\text{for any } \; e,\; f \in E,\; e \neq f: \;\; |e \cap f| \leqslant 1.$$

Let $\Delta(H)$ denote the maximum vertex degree of a hypergraph H.

A coloring f of the vertex set V is called *proper* for hypergraph $H=(V,E)$, if there is no monochromatic edges in E under f. A hypergraph is said to be r-colorable if there is a proper coloring with r colors for it. The chromatic number $\chi(H)$ of a hypergraph H is the minimum r such that H is r-colorable.

In 1973 P. Erdős and L. Lovász (see [3]) established the following quantitative relation between the chromatic number and the maximum vertex degree in n-uniform hypergraphs.

THEOREM 1 (P. Erdős, L. Lovász, [3]). *Let $n \geqslant 3$ and $r \geqslant 2$ be integers. If H is an n-uniform non-r-colorable hypergraph, then*

$$\Delta(H) \geqslant \frac{1}{4} r^{n-1} n^{-1}. \tag{1}$$

This classical bound of Erdős and Lovász has been improved in a series of papers. In 2000 J. Radhakrishnan and A. Srinivasan (see [9]) proved that, for any n-uniform non-2-colorable hypergraph H, the following inequality holds:

$$\Delta(H) \geqslant 0.17 \frac{2^n}{\sqrt{n \ln n}}. \tag{2}$$

Their result was recently extended by A. V. Kostochka, M. Kumbhat and V. Rödl (see [7]) to an arbitrary number of colors. They showed that if

$$r = r(n) = o\big(\sqrt{\ln \ln n}\big),$$

then

$$\Delta(H) > e^{-4r^2} \left(\frac{n}{\ln n}\right)^{\frac{\lfloor \log_2 r \rfloor}{\lfloor \log_2 r \rfloor + 1}} \frac{r^n}{n}, \tag{3}$$

for any n-uniform non-r-colorable hypergraph H. However, by using the same proof technique as in [7] one can obtain good lower bounds for the maximum vertex degree even when the chromatic number is large in comparison with uniformity n.

THEOREM 2. *Let $h \geqslant 1$ be an integer. Then there exist $n_0 = n_0(h)$ and $c(h) > 0$ such that, for any $n \geqslant n_0$ and $r \geqslant 2^h$, the following statement holds. If H is an n-uniform non-r-colorable hypergraph, then*

$$\Delta(H) \geqslant c(h)\, r^{n-1} \left(n (\ln n)^h\right)^{-1/(h+1)}. \tag{4}$$

The last known result concerning estimating maximum vertex degree in uniform hypergraphs with large chromatic number was obtained by the author of this paper (see [10]): for all $n, r \geqslant 3$ and arbitrary n-uniform non-r-colorable hypergraph H, one has

$$\Delta(H) > \frac{1}{8}\, r^{n-1} n^{-1/2}. \tag{5}$$

It is easy to see, that bound (5) improves (3) and (4) in the case $r = 3$. But for $r > 3$ inequality (4) gives a better bound (taking $h = 2$):

$$\Delta(H) \geqslant \mathrm{const} \cdot r^{n-1} n^{-1/3} (\ln n)^{-2/3}.$$

The first improvement of Theorem 1 for simple hypergraphs was obtained in 1990 by Z. Szabó (see [13]) in connection with estimating famous Van der Waerden function.

THEOREM 3 (Z. Szabó, [13]). *For any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that, for any $n > n_0$, the following statement holds. If H is an n-uniform non-2-colorable simple hypergraph, then*

$$\Delta(H) \geqslant 2^{n-1} n^{-\varepsilon}.$$

Theorem 3 was extended to an arbitrary number of colors and to a wider class of partial Steiner systems by A. V. Kostochka and M. Kumbhat in 2009 (see [6]). In particular case of simple hypergraphs, they showed that, for any $\varepsilon > 0$ and $r \geqslant 2$,

$$\Delta(H) \geqslant r^{n-1} n^{-\varepsilon}, \tag{6}$$

where H is an n-uniform non-r-colorable simple hypergraph and $n > n_0(\varepsilon, r)$ is sufficiently large. Since ε can be taken arbitrary small, we can replace it in the right-hand part of (6) by some infinitesimal function. The calculations from [6] give the following bound:

$$\Delta(H) \geqslant r^{n-1} n^{-\varepsilon(n,r)}, \tag{7}$$

where

$$\varepsilon(n, r) = \Theta\left(\sqrt[4]{\frac{\ln r}{\ln n}}\right).$$

The result (7) of Kostochka and Kumbhat was slightly improved by the present author (see [11]) as follows.

THEOREM 4 (D. A. Shabanov, [11]). *There exists an integer* $n_0 > 0$ *such that, for any* $n > n_0$ *and* $r \leqslant n^{1/9}$, *the following statement holds. If* H *is an* n*-uniform non-*r*-colorable simple hypergraph, then*

$$\Delta(H) \geqslant r^{n-1}\, n^{-3/t(n,r)}, \tag{8}$$

where

$$t(n,r) = \left\lfloor \sqrt{\min\left(\frac{\ln n}{\ln r}, \frac{\ln n}{2\ln\bigl((4/3)\ln n\bigr)}\right)} \right\rfloor.$$

Finally, A. Frieze and D. Mubayi (see [5]) proved that if H is an n-uniform non-r-colorable simple hypergraph, then

$$\Delta(H) \geqslant c(n)\, r^{n-1} \ln r, \tag{9}$$

where $c(n) > 0$ does not depend on r. Their bound is sharp up to some factor depending on n, since in 2010 A. V. Kostochka and V. Rödl showed (see [8]) that, for any $n, r \geqslant 2$, there exists an n-uniform non-r-colorable simple hypergraph H satisfying

$$\Delta(H) \leqslant \lceil n\, r^{n-1} \ln r \rceil.$$

Unfortunately, the function $c(n)$ from (9) is very small, e. g. it follows from the proof in [5] that

$$c(n) = O(n^{1-2n}).$$

So, the bound (9) improves even the classical result (1) of Erdős and Lovász only if r is very large in comparison with n: $\ln r = \Omega(n^{2n-2})$.

In the current paper we study the quantitative relation between the maximum vertex degree and the choice number in simple uniform hypergraphs. Recall that a hypergraph H is said to be r*-choosable* if, for every family of sets

$$L = \{L(v) :\ v \in V\}$$

(L is called *list assignment*), such that $|L(v)| = r$ for all $v \in V$, there is a proper coloring from the lists (for every $v \in V$ we should use a color from $L(v)$). The *choice number* (or *list chromatic number*) of a hypergraph H, denoted by $ch(H)$, is the least r such that H is r-choosable. The choice numbers of graphs were independently introduced by V. G. Vizing (see [14]) and by P. Erdős, A. Rubin and H. Taylor (see [4]).

It is clear that $\chi(H) \leqslant ch(H)$, so, if a hypergraph is not r-choosable, then it is also not r-colorable. However, not all the proofs of the results (1)–(9) work in the case of list colorings.

If H is an n-uniform non-r-choosable hypergraph, then one easily establish the lower bound (1):

$$\Delta(H) \geqslant \frac{1}{4}\, r^{n-1} n^{-1}.$$

The proof of the estimate (2) given by Radhakrishnan and Srinivasan also works for list colorings. In the paper [10] it was shown that the bound (5) holds for non-r-choosable hypergraphs. Hence, if H is an n-uniform non-r-choosable hypergraph, then

$$\Delta(H) > \frac{1}{8}\, r^{n-1} n^{-1/2}. \tag{10}$$

But the proofs of inequalities (3) and (4) cannot be generalized to the case of an arbitrary r-uniform list assignment. So, (10) remains the best general lower bound for the maximum vertex degree in hypergraphs with high choice number.

The results of Kostochka — Kumbhat (7) and Shabanov (8) can be extended for non-r-choosable hypergraphs without any problems, but the proof of the bound (9) by Frieze and Mubayi does not work for list colorings, as the authors wrote themselves in the final comment of [5]. Thus, for class of simple hypergraphs we have the following picture.

Suppose H is an n-uniform non-r-choosable hypergraph. If $r \leqslant n^{1/36}$, then the best lower bound for $\Delta(H)$ is provided by the list version of Theorem 4:

$$\Delta(H) \geqslant r^{n-1}\, n^{-3/t(n,r)}, \tag{11}$$

where

$$t(n,r) = \left\lfloor \sqrt{\min\left(\frac{\ln n}{\ln r}, \frac{\ln n}{2\ln\bigl((4/3)\ln n\bigr)}\right)} \right\rfloor.$$

But for $r > n^{1/36}$ this relation becomes asymptotically weaker than the inequality (10). So, in the area $r > n^{1/36}$ the best asymptotic lower bound for $\Delta(H)$ is (10), which holds for any n-uniform hypergraphs with choice number greater than r.

The main result of the paper improves the inequality (10) for simple hypergraphs.

THEOREM 5. *Let $n \geqslant 3$ and $r \geqslant 2$ be integers. If $H = (V, E)$ is an n-uniform non-r-colorable simple hypergraph, then*

$$\Delta(H) \geqslant \frac{1}{11} r^{n-1} n^{-1/3}. \tag{12}$$

It is easy to see that (12) improves (10) for all $n \geqslant 7$ and any r. It is also asymptotically better than (11) if $r > n^{1/81}$.

Theorem 5 is an immediate corollary of the following theorem, which, in fact, gives a lower bound for the maximum edge degree in n-uniform simple hypergraphs with high choice number. Recall that, for any edge $f \in E$ of a hypergraph $H = (V, E)$, the number of other edges intersecting f is called its *degree* in H. We denote by $D(H)$ the maximum edge degree of the hypergraph H.

THEOREM 6. *Let $n \geqslant 3$ and $r \geqslant 2$ be integers. If $H = (V, E)$ is an n-uniform simple hypergraph satisfying the condition*

$$D(H) \leqslant \frac{1}{11} r^{n-1} n^{2/3}, \tag{13}$$

then H is r-choosable.

In the next section we shall prove Theorem 6.

2. Proof of Theorem 6

Let $L = \{L(v),\ v \in V\}$ be an arbitrary r-uniform list assignment for the vertex set V. We have to show the existence of a proper coloring from the lists for hypergraph H. To prove this we shall construct some random coloring from the lists L and estimate the probability that this coloring is not proper for H.

For any edge $f \in E$, let us use the following notation:

$$L(f) = \bigcap_{v \in f} L(v).$$

It is clear that $|L(f)| \leqslant r$ for every edge f, and f can be monochromatic under some coloring from the lists, only when all its vertices are colored with some $u \in L(f)$.

2.1. Randomized algorithm for hypergraph coloring

The construction of a random coloring is based on the method of *random recoloring*. This method was proposed by J. Beck (see [2]) and developed by J. Spencer (see [12]), Radhakrishnan and Srinivasan (see [9]) for two-colorings. In our work we follow the ideas Radhakrishnan and Srinivasan of constructing random colorings for simple hypergraphs and related systems.

We describe a randomized algorithm for random coloring of the vertex set V. This algorithm consists of two stages.

First Stage. Main coloring. During the first stage we randomly and uniformly color the vertex set: any vertex $v \in V$ is independently from other vertices is colored with any color $u \in L(v)$ with equal probability $1/r$.

Let χ_0 denote the obtained random coloring (*main coloring*). Let $\mathcal{M}(\chi_0)$ denote the set of monochromatic edges of H in the coloring χ_0. The main coloring χ_0 can also contain almost monochromatic edges. An edge $f \in E$ is said to be *almost monochromatic* in χ_0 if there is a color $u \in L(f)$ and a vertex $v \in f$ such that the vertex v is the only vertex in f which is not colored with u in χ_0. In this case, the color u is called *dominating* in f and the vertex v is called *dangerous of the color* u.

Second Stage. Recoloring process. During the second stage we want to recolor some vertices of the edges from $\mathcal{M}(\chi_0)$ but we shall forbid the almost monochromatic edges to become completely monochromatic.

Suppose $\{X_v,\ v \in V\}$ are independent nonnegative equally distributed random variables with some continuous distribution. Our recoloring process has continuous time and starts at the time $t = 0$. For any vertex $v \in V$, we do the following.

1. The vertex v can be recolored only at the time moment X_v.
2. At the time X_v we check whether there is a monochromatic edge f, containing v, such that f is monochromatic in the main coloring χ_0 and none of the vertices of f has changed its color up to the time X_v.
3. If such edge f exists, then we choose randomly and uniformly a color $a \in L(v)$ not coinciding with the color of v in the main coloring χ_0.
4. If v is not dangerous of the color a, then we recolor v with a.
5. Otherwise, we do not change the color of v in the process.

Let χ_t, $t > 0$ denote the random coloring obtained during the recoloring process up to the time t. We want to estimate the probability of the event

$$\mathcal{F}(t) = \bigl\{H \text{ has at least one monochromatic edge in } \chi_t\bigr\}$$

and show that, for some $t > 0$, it is strictly less than 1. Now we shall give a formal construction of the random coloring χ_t.

2.2. Formal construction of random coloring

Consider, on some probability space, the following set of random variables.

1. $\{\xi_v,\ v \in V\}$ — independent random variables, for any v the random variable ξ_v has uniform distribution on the set $L(v)$.
2. $\{X_v,\ v \in V\}$ — independent exponential random variables, i. e.

$$\Pr(X_v > t) = e^{-t}, \quad t > 0.$$

3. $\{\eta_v,\ v \in V\}$ — independent (of each other and of $\{X_v,\ v \in V\}$) random variables with the following conditional distribution:

$$\Pr(\eta_v = a|\xi_v = u) = \frac{1}{r-1} \quad \text{for any} \quad a \neq u,\ a, u \in L(v),\ v \in V.$$

The random vector $\xi = \{\xi_v,\ v \in V\}$ can be interpreted as an uniform random coloring from the lists L (we assign the color ξ_v to the vertex v). This is our main coloring χ_0 from the algorithm. We shall give a formal construction for the coloring χ_t.

Let $f \in E$ be an edge of H. For every $u \in L(f)$, let $\mathcal{M}(f, u)$ and $\mathcal{AM}(f, u)$ denote the following events:

$$\mathcal{M}(f, u) = \bigcap_{j \in f}\{\xi_j = u\}, \quad \mathcal{AM}(f, u) = \Bigl\{\sum_{j \in f} I\{\xi_j \neq u\} = 1\Bigr\}. \qquad (14)$$

It is clear that $\mathcal{M}(f, u)$ is the event that f is monochromatic of a color u in ξ, and $\mathcal{AM}(f, u)$ is the event that f is almost monochromatic with dominating color u in ξ.

For every vertex $v \in V$ and every color $a \in L(v)$, let $\mathcal{D}(v, a)$ and $\mathcal{D}(v)$ denote the following events:

$$\begin{aligned} \mathcal{D}(v, a) &= \bigcup_{\substack{f \in E:\\ v \in f,\, a \in L(f)}} \big(\mathcal{AM}(f, a) \cap \{\xi_v \neq a, \eta_v = a\}\big), \\ \mathcal{D}(v) &= \bigcup_{a \in L(v)} \mathcal{D}(v, a). \end{aligned} \tag{15}$$

It is easy to see that the event $\mathcal{D}(v, a)$ means that v is dangerous of the color a and $\eta_v = a$. Note that due to our algorithm if the event $\mathcal{D}(v)$ occurs then the vertex v is not recolored at the time X_v.

Finally, for every vertex v, let $\mathcal{R}(v)$ denote the following event:

$$\mathcal{R}(v) = \bigcup_{f \in E:\, v \in f} \bigcap_{u \in L(f)} \left(\mathcal{M}(f, u) \cap \left\{ \sum_{j \in f} I\{X_j < X_v, \mathcal{D}(j)\} = n - 1 \right\}\right). \tag{16}$$

Here $I\{A\}$ is an indicator of the event A. Our algorithm implies that we should try to recolor the vertex v if the event $\mathcal{R}(v)$ occurs, since it is contained in some monochromatic edge f, in which none of the vertices has changed its color up to the time X_v.

Further we denote a random coloring

$$\zeta(t) = \{\zeta_v(t),\ v \in V\}, \quad t > 0,$$

as follows

$$\zeta_v(t) = \xi_v I\left\{\overline{\mathcal{R}(v)} \cup \mathcal{D}(v) \cup \{X_v > t\}\right\} + \eta_v I\left\{\mathcal{R}(v) \cap \overline{\mathcal{D}(v)} \cap \{X_v \leqslant t\}\right\}. \tag{17}$$

It is clear that, for any $v \in V$, the random variable $\zeta_v(t)$ takes values only from the set $L(v)$. So, the random vector $\zeta(t) = \{\zeta_v(t),\ v \in V\}$ gives a random coloring χ_t from the algorithm.

Remark 1. It should be noted that one can use an arbitrary continuous distribution for the random variables X_v, $v \in V$ instead of exponential. In fact, all we need is a random ordering of the vertex set. Using continuous time helps to simplify the calculations.

Our main aim is to estimate the probability of the event that the coloring $\zeta(t)$ is not a proper list coloring for the hypergraph H. Let $\mathcal{F}(t)$ denote this event:

$$\mathcal{F}(t) = \bigcup_{f\in E} \bigcup_{u\in L(f)} \bigcap_{j\in f} \{\zeta_j(t) = u\}. \tag{18}$$

We shall separate the event $\mathcal{F}(t)$ into some “local” parts.

How can a monochromatic edge $f \in E$ appear in the coloring $\zeta(t)$? We have three possibilities.

1. The edge f is monochromatic of some color u in the main coloring ξ, it contains at least two vertices, for which we have already tried to make a recoloring, and f is monochromatic of the same color u in $\zeta(t)$. We denote this event by $\mathcal{A}(f,t)$. Formally

$$\mathcal{A}(f,t) = \bigcup_{u\in L(f)} \left(\bigcap_{j\in f}\{\xi_j = u, \zeta_j(t) = u\}\right) \cap \left\{\sum_{j\in f} I\{X_j < t\} \geqslant 2\right\}. \tag{19}$$

2. The edge f is monochromatic of some color u in the main coloring ξ, it contains at most one vertex, for which we have already tried to make a recoloring, and f is monochromatic of the same color u in $\zeta(t)$. We denote this event by $\mathcal{B}(f,t)$. Formally,

$$\mathcal{B}(f,t) = \bigcup_{u\in L(f)} \left(\bigcap_{j\in f}\{\xi_j = u,\ \zeta_j(t) = u\}\right) \cap \left\{\sum_{j\in f} I\{X_j < t\} \leqslant 1\right\}. \tag{20}$$

3. The edge f is not monochromatic of some color u in the main coloring ξ, but it is monochromatic of this color u in the coloring $\zeta(t)$. We denote this event by $\mathcal{C}(f,t)$. Formally,

$$\mathcal{C}(f,t) = \bigcup_{u\in L(f)} \left(\bigcap_{j\in f}\{\zeta_j(t) = u\} \cap \overline{\mathcal{M}(f,u)}\right). \tag{21}$$

It follows from (19), (20) and (21) that

$$\mathcal{F}(t) = \bigcup_{f\in E} \left(\mathcal{A}(f,t) \cup \mathcal{B}(f,t) \cup \mathcal{C}(f,t)\right). \tag{22}$$

Further, we shall consider these three events separately.

2.3. First event

Suppose that the event $\mathcal{A}(f,t)$ occurs and f is monochromatic of a color $u \in L(f)$. The event $\mathcal{A}(f,t)$ implies that for at least two vertices $v \in f$, we have $X_v < t$. Suppose w and w' are two vertices of f such that

$$X_w < X_{w'} < X_v \quad \text{for all} \quad v \in f \setminus \{w, w'\}.$$

Since f is monochromatic in the main coloring ξ, the event $\mathcal{R}(w)$ also occurs (see (16)). Moreover, the equality $\zeta_w(t) = \xi_w = u$ implies that the vertex w at the time X_w does not change its color and, consequently, the event $\mathcal{R}(w')$ occurs (we have already tried to recolor w'). Thus, the equalities $\zeta_w(t) = \xi_w = u$ and $\zeta_{w'}(t) = \xi_{w'} = u$ can hold only

- if both vertices w and w' are dangerous of some colors, say, $a \in L(w)$ and $a' \in L(w')$, AND, moreover,
- $\eta_w = a$ and $\eta_{w'} = a'$,

i. e. when the events $\mathcal{D}(w,a)$ and $\mathcal{D}(w',a')$ hold. The event $\mathcal{D}(w,a)$ implies that there is an edge f' such that $w \in f'$ and f' is almost monochromatic with the dominating color a in the main coloring ξ. Respectively, the event $\mathcal{D}(w',a')$ implies that there is an edge f'' such that $w' \in f''$ and f'' is almost monochromatic with the dominating color a' in ξ. Since H is simple, the vertices w and w' are uniquely defined by the edges f' and f'': $\{w\} = f \cap f'$, $\{w'\} = f \cap f''$.

For any three edges f, f', f'' satisfying the conditions $|f \cap f'| = 1$, $|f \cap f''| = 1$, $f \cap f' \cap f'' = \varnothing$, let $\mathcal{A}(f,f',f'')$ denote the following event:

$$\mathcal{A}(f,f',f'') = \bigcup_{u \in L(f)} \bigcup_{\substack{a,a' \neq u: \\ a \in L(f'),\, a' \in L(f'')}} \Big(\mathcal{M}(f,u) \cap \mathcal{AM}(f',a) \cap \mathcal{AM}(f'',a') \cap$$

$$\cap \ \{\eta_w = a,\ \eta_{w'} = a'\} \cap \bigcap_{j \in f \setminus \{w,w'\}} \{X_w < X_{w'} < X_j\} \Big), \tag{23}$$

where $\{w\} = f \cap f'$ and $\{w'\} = f \cap f''$. Thus, due to the above discussion we get the following relation:

$$\mathcal{A}(f,t) \subset \bigcup_{\substack{f' \in E: \\ |f \cap f'| = 1}} \bigcup_{\substack{f'' \in E:\, |f \cap f''| = 1 \\ f \cap f' \cap f'' = \varnothing}} \mathcal{A}(f,f',f''). \tag{24}$$

Let us also introduce the event $\mathcal{A}$:

$$\mathcal{A}=\bigcup_{f\in E}\ \bigcup_{u\in L(f)}\ \bigcup_{\substack{v,v'\in f:\\ v\neq v'}}\Big(\mathcal{M}(f,u)\cap\bigcap_{j\in f\setminus\{v,v'\}}\{X_v<X_{v'}<X_j\}\cap\bigcap_{j\in f}\{\zeta_j(X_{v'})=u\}\Big). \tag{25}$$

This event states that during the recoloring process there is an edge f, which is monochromatic in the main coloring ξ, and after the consideration of its first two vertices v and v' this edge remains monochromatic of the same color. It is easy to see from (23) and (25) that the event $\mathcal{A}$ is also covered by the union of the events $\mathcal{A}(f,f',f'')$:

$$\mathcal{A}\subset\bigcup_{f\in E}\ \bigcup_{\substack{f'\in E:\\ |f\cap f'|=1}}\ \bigcup_{\substack{f''\in E:\,|f\cap f''|=1\\ f\cap f'\cap f''=\varnothing}}\mathcal{A}(f,f',f''). \tag{26}$$

The event $\mathcal{A}$ will help us to consider the third event $\mathcal{C}(f,t)$.

Now let us estimate the probability of the event $\mathcal{A}(f,f',f'')$ depending on the cardinality of the set $f'\cap f''$. Since all the random variables ξ_j, X_j are mutually independent, in the case $|f'\cap f''|=0$ we have

$$\mathsf{Pr}\big(\mathcal{A}(f,f',f'')\big)=\sum_{u\in L(f)}\ \sum_{\substack{a\in L(f'):\\ a\neq u}}\ \sum_{\substack{a'\in L(f''):\\ a'\neq u}}\Bigg[\Big(\prod_{j\in f}\mathsf{Pr}(\xi_j=u)\Big)\Big(\prod_{j\in f'\setminus f}\mathsf{Pr}(\xi_j=a)\Big)\times$$

$$\times\Big(\prod_{j\in f''\setminus f}\mathsf{Pr}(\xi_j=a')\Big)\cdot\mathsf{Pr}(\eta_w=a|\xi_w=u)\cdot\mathsf{Pr}(\eta_{w'}=a'|\xi_{w'}=u)\times$$

$$\times\,\mathsf{Pr}\Big(\bigcap_{j\in f\setminus\{w,w'\}}\{X_w<X_{w'}<X_j\}\Big)\Bigg]\leqslant$$

$\big($since $|f'\setminus f|=|f''\setminus f|=n-1$ and $\big|L(f')\setminus\{u\}\big|\leqslant r-1$, $\big|L(f'')\setminus\{u\}\big|\leqslant r-1\big)$

$$\leqslant r(r-1)^2r^{2-3n}\frac{1}{(r-1)^2}\cdot\frac{(n-2)!}{n!}=r^{3-3n}\cdot\frac{1}{n(n-1)}. \tag{27}$$

Finally, if $|f'\cap f''|=1$ (recall that H is simple), then the unique common vertex w'' of the edges f' and f'' does not belong to the edge f and, moreover, the colors a and a' should coincide (see (23)) since $a=\xi_{w''}=a'$. Otherwise the probability is equal to zero. So, in this case we have

$$\mathsf{Pr}\big(\mathcal{A}(f,f',f'')\big)=\sum_{u\in L(f)}\sum_{\substack{a\in L(f'):\\ a\neq u}}\Bigg[\bigg(\prod_{j\in f}\mathsf{Pr}(\xi_j=u)\bigg)\bigg(\prod_{j\in f'\setminus f}\mathsf{Pr}(\xi_j=a)\bigg)\times$$

$$\times\bigg(\prod_{j\in f''\setminus(f\cup f')}\mathsf{Pr}(\xi_j=a)\bigg)\cdot\mathsf{Pr}(\eta_w=a|\xi_w=u)\cdot\mathsf{Pr}(\eta_{w'}=a|\xi_{w'}=u)\times$$

$$\times\,\mathsf{Pr}\bigg(\bigcap_{j\in f\setminus\{w,w'\}}\{X_w<X_{w'}<X_j\}\bigg)\Bigg]\leqslant$$

$\big($since $|f'\setminus f|=n-1$ and $\big|f''\setminus(f\cup f')\big|=n-2\big)$

$$\leqslant r(r-1)r^{3-3n}\frac{1}{(r-1)^2}\cdot\frac{(n-2)!}{n!}\leqslant 2\cdot r^{3-3n}\cdot\frac{1}{n(n-1)}. \tag{28}$$

From the obtained bounds (27) and (28) we see that, for any three edges f,f',f'' satisfying the conditions

$$|f\cap f'|=1,\quad |f\cap f''|=1,\quad f\cap f'\cap f''=\varnothing,$$

the following estimate for the probability of the event $\mathcal{A}(f,f',f'')$ holds:

$$\mathsf{Pr}\big(\mathcal{A}(f,f',f'')\big)\leqslant 2\cdot r^{3-3n}\cdot\frac{1}{n(n-1)}. \tag{29}$$

2.4. Second event

For any edge $f\in E$, let us consider the following event

$$\mathcal{Q}(f,t)=\bigcup_{u\in L(f)}\bigg(\bigcap_{j\in f}\{\xi_j=u\}\bigg)\cap\bigg\{\sum_{j\in f}I\{X_j<t\}\leqslant 1\bigg\}. \tag{30}$$

Then it is easy to see from (20) that

$$\mathcal{B}(f,t)\subset\mathcal{Q}(f,t). \tag{31}$$

Let us find the probability of the event $\mathcal{Q}(f,t)$. Since the random variable

$$\sum_{j\in f} I\{X_j<t\}$$

is independent of the random coloring ξ and it has a binomial distribution with parameters $(n,1-e^{-t})$, we have

$$\begin{aligned}\mathsf{Pr}\big(\mathcal{Q}(f,t)\big) = \Big(\sum_{u\in L(f)}\prod_{j\in f}\mathsf{Pr}(\xi_j=u)\Big)\cdot\mathsf{Pr}\Big(\sum_{j\in f} I\{X_j<t\}\leqslant 1\Big) \leqslant \\ \leqslant r^{1-n}\big(n(1-e^{-t})e^{-t(n-1)}+e^{-tn}\big). \qquad (32)\end{aligned}$$

2.5. Third event

Let the event $\mathcal{C}(f,t)$ occur. Recall that

$$\mathcal{C}(f,t)=\bigcup_{u\in L(f)}\Big(\bigcap_{j\in f}\{\zeta_j(t)=u\}\cap\overline{\mathcal{M}(f,u)}\Big).$$

Since the edge f is not monochromatic of a color u in the main coloring ξ and becomes monochromatic of u in $\zeta(t)$, f is not an almost monochromatic edge with dominating color u. Thus, there are at least two vertices in f, which are colored with colors not coinciding with u in the main coloring, and all such vertices should be recolored with u up to the time t during the recoloring process.

Suppose that $v\in f$ is the last vertex, that was recolored with u during the recoloring process and $v'\in f$ is the penultimate such vertex. Since v is recolored at time X_v, there is an edge f' such that

- f' is monochromatic of some color $a\neq u$ in the main coloring,
- at time X_v this edge is still monochromatic of color a,
- moreover, if the event $\overline{\mathcal{A}}$ (see (25)) holds, then at most one vertex of $f'\setminus\{v\}$ has been considered up to the time X_v, i. e.

$$\sum_{j\in f'\setminus\{v\}} I\{X_j<X_v\}\leqslant 1.$$

The above discussion also gives an edge f'' for the vertex v' with the same properties. Note that due to the simplicity of hypergraph H the vertices v and v'

are uniquely defined by the edges f and f'. Thus, we get the following relation

$$\mathcal{C}(f,t)\cap\overline{\mathcal{A}}\subset \bigcup_{u\in L(f)}\ \bigcup_{\substack{f'\in E:\\ f'\cap f\neq\varnothing}}\ \bigcup_{\substack{f''\in E:\, f''\cap f\neq\varnothing,\\ f\cap f'\cap f''=\varnothing}}\ \bigcup_{\substack{a\in L(f'):\\ a\neq u}}\ \bigcup_{\substack{a'\in L(f''):\\ a'\neq u}} \Bigg(\bigcap_{j\in f\setminus(f'\cup f'')} \{\zeta_j(X_{v'})=u\}\cap$$

$$\cap\mathcal{M}(f',a)\cap\mathcal{M}(f'',a')\cap\{\eta_v=u,\ \eta_{v'}=u\}\cap\{X_{v'}<X_v\}\cap$$

$$\cap\Big\{\sum_{j\in f'\setminus\{v\}} I\{X_j<X_v\}\leqslant 1\Big\}\cap\Big\{\sum_{j\in f''\setminus\{v'\}} I\{X_j<X_{v'}\}\leqslant 1\Big\}\Bigg), \tag{33}$$

where $v=f\cap f'$ and $v'=f\cap f''$.

For any vertex $j\in f\setminus(f'\cup f'')$, the equality $\zeta_j(X_{v'})=u$ can hold in two cases:

- either $\xi_j=u$,
- or $\xi_j\neq u$, but $X_j<X_{v'}$ (j should be recolored before v') and $\eta_j=u$.

Hence,

$$\{\zeta_j(X_{v'})=u\}\subset\{\xi_j=u\}\sqcup\big\{\xi_j\neq u,\ X_j<X_v,\ \eta_j=u\big\}. \tag{34}$$

Let us denote by $\mathcal{C}(f,f',f'')$, where the edges f, f' an f'' satisfy the conditions $|f\cap f'|=|f\cap f''|=1$ and $f\cap f'\cap f''=\varnothing$, the following event

$$\mathcal{C}(f,f',f'')=$$

$$=\bigcup_{u\in L(f)}\ \bigcup_{\substack{a\in L(f'):\\ a\neq u}}\ \bigcup_{\substack{a'\in L(f''):\\ a'\neq u}} \Bigg(\bigcap_{j\in f\setminus(f'\cup f'')} \big\{\{\xi_j=u\}\sqcup\{\xi_j\neq u,\ X_j<X_{v'},\ \eta_j=u\}\big\}\cap$$

$$\cap\,\mathcal{M}(f',a)\cap\mathcal{M}(f'',a')\cap\{\eta_v=u,\ \eta_{v'}=u\}\cap\{X_{v'}<X_v\}\cap$$

$$\cap\Big\{\sum_{j\in f'\setminus\{v\}} I\{X_j<X_v\}\leqslant 1\Big\}\cap\Big\{\sum_{j\in f''\setminus\{v'\}} I\{X_j<X_{v'}\}\leqslant 1\Big\}\Bigg), \tag{35}$$

where $v=f\cap f'$ and $v'=f\cap f''$.

Using (33), (34) and (35) one has the following relation:

$$\mathcal{C}(f,t)\cap\overline{\mathcal{A}}\subset \bigcup_{\substack{f'\in E:\\ f'\cap f\neq\varnothing}}\ \bigcup_{\substack{f''\in E:\, f''\cap f\neq\varnothing,\\ f\cap f'\cap f''=\varnothing}} \mathcal{C}(f,f',f''). \tag{36}$$

Estimate for the probability of the event $\mathcal{C}(f,f',f'')$ depends on the cardinality of the set of $f'\cap f''$. If $|f'\cap f''|=0$ then

$$\mathrm{Pr}\bigl(\mathcal{C}(f,f',f'')\bigr)=\int\limits_0^\infty\int\limits_0^\infty \mathrm{Pr}\,\bigl(\mathcal{C}(f,f',f'')|X_v=x,X_{v'}=y\bigr)\,e^{-x}e^{-y}\,dx\,dy=$$

$$=\int\limits_0^\infty e^{-x}\int\limits_0^x \mathrm{Pr}\Biggl[\bigcup_{u\in L(f)}\ \bigcup_{\substack{a\in L(f'):\\ a\neq u}}\ \bigcup_{\substack{a'\in L(f''):\\ a'\neq u}}\Biggl(\bigcap_{j\in f\setminus(f'\cup f'')}\{\{\xi_j=u\}\sqcup$$

$$\sqcup\{\xi_j\neq u,\ X_j<y,\ \eta_j=u\}\}\cap\mathcal{M}(f',a)\cap\mathcal{M}(f'',a')\cap\{\eta_v=u,\eta_{v'}=u\}\cap$$

$$\cap\Biggl\{\sum_{j\in f'\setminus\{v\}} I\{X_j<x\}\leqslant 1\Biggr\}\cap\Biggl\{\sum_{j\in f''\setminus\{v'\}} I\{X_j<y\}\leqslant 1\Biggr\}\Biggr)\Biggr]e^{-y}\,dy\,dx\leqslant$$

$$\leqslant\int\limits_0^\infty e^{-x}\int\limits_0^x r(r-1)^2\biggl(\frac{1}{r}+\frac{r-1}{r}(1-e^{-y})\frac{1}{(r-1)}\biggr)^{n-2}r^{-2n}\cdot\frac{1}{(r-1)^2}\times$$

$$\times\bigl(e^{-x(n-1)}+(n-1)(1-e^{-x})e^{-x(n-2)}\bigr)\times$$

$$\times\bigl(e^{-y(n-1)}+(n-1)(1-e^{-y})e^{-y(n-2)}\bigr)e^{-y}\,dy\,dx\leqslant$$

(since $1-e^{-x}\leqslant x$ and $e^{-x}<1$)

$$\leqslant r^{3-3n}\int\limits_0^\infty e^{-x}\int\limits_0^x(1+y)^{n-2}e^{-x(n-2)}(1+(n-1)x)e^{-y(n-2)}(1+(n-1)y)\,dy\,dx\leqslant$$

(since $(1+y)^{n-2}<e^{y(n-2)}$)

$$\leqslant r^{3-3n}\int\limits_0^\infty e^{-x(n-1)}(1+(n-1)x)\int\limits_0^x(1+(n-1)y)\,dy\,dx\leqslant$$

$$\leqslant r^{3-3n}\int\limits_0^\infty xe^{-x(n-1)}(1+(n-1)x)\biggl(1+(n-1)\frac{x}{2}\biggr)\,dx=r^{3-3n}\cdot\frac{7}{(n-1)^2}.\qquad(37)$$

If $|f' \cap f''| = 1$, $v'' = f' \cap f''$, then $\xi_{v''} = a = a'$, i. e. the colors a and a' should coincide. Moreover, $v'' \notin f$. So, in this case we have

$$\Pr\bigl(\mathcal{C}(f,f',f'')\bigr) = \int_0^\infty \int_0^\infty \Pr\bigl(\mathcal{C}(f,f',f'') | X_v = x, X_{v'} = y\bigr) e^{-x} e^{-y}\, dx\, dy =$$

$$= \int_0^\infty e^{-x} \int_0^x \Pr\Biggl[\bigcup_{u \in L(f)} \bigcup_{\substack{a \in L(f'): \\ a \neq u}} \Biggl(\bigcap_{j \in f \setminus (f' \cup f'')} \bigl\{ \{\xi_j = u\} \sqcup \{\xi_j \neq u,\, X_j < y,\, \eta_j = u\} \bigr\} \cap$$

$$\cap\, \mathcal{M}(f',a) \cap \mathcal{M}(f'',a) \cap \{\eta_v = u, \eta_{v'} = u\} \cap$$

$$\cap \Biggl\{ \sum_{j \in f' \setminus \{v\}} I\{X_j < x\} \leqslant 1 \Biggr\} \cap \Biggl\{ \sum_{j \in f'' \setminus \{v'\}} I\{X_j < y\} \leqslant 1 \Biggr\} \Biggr) \Biggr] e^{-y}\, dy\, dx \leqslant$$

(since $\displaystyle\sum_{j \in f'' \setminus \{v'\}} I\{X_j < y\} \geqslant \sum_{j \in f'' \setminus \{v', v''\}} I\{X_j < y\}$)

$$\leqslant \int_0^\infty e^{-x} \int_0^x r(r-1) \left(\frac{1}{r} + \frac{r-1}{r} (1 - e^{-y}) \frac{1}{(r-1)} \right)^{n-2} r^{1-2n} \cdot \frac{1}{(r-1)^2} \times$$

$$\times \left(e^{-x(n-1)} + (n-1)(1 - e^{-x}) e^{-x(n-2)} \right) \times$$

$$\times \left(e^{-y(n-2)} + (n-2)(1 - e^{-y}) e^{-y(n-3)} \right) e^{-y}\, dy\, dx \leqslant$$

(since $1 - e^{-x} \leqslant x$)

$$\leqslant r^{3-3n} \cdot \left(\frac{r}{r-1} \right) \int_0^\infty e^{-x} \int_0^x (1+y)^{n-2} e^{-x(n-2)} \times$$

$$\times (1 + (n-1)x) e^{-y(n-3)} (1 + (n-2)y) e^{-y}\, dy\, dx \leqslant$$

(since $y \leqslant x$, $(1+y)^{n-2} < e^{y(n-2)}$ and $r/(r-1) \leqslant 2$)

$$\leqslant 2 \cdot r^{3-3n} \int_0^\infty e^{-x(n-1)} (1 + (n-1)x) \int_0^x (1 + (n-1)y)\, dy\, dx =$$

$$=2\cdot r^{3-3n}\int\limits_0^\infty xe^{-x(n-1)}(1+(n-1)x)\left(1+(n-1)\frac{x}{2}\right)dx=r^{3-3n}\cdot\frac{14}{(n-1)^2}. \tag{38}$$

Thus, from the estimates (2.5) and (38) we obtain the final upper bound for the probability of the event $\mathcal{C}(f,f',f'')$, holding for any three edges f, f' an f'' such that

$$|f\cap f'|=|f\cap f''|=1 \quad \text{and} \quad f\cap f'\cap f''=\varnothing,$$

$$\Pr\big(\mathcal{C}(f,f',f'')\big)\leqslant r^{3-3n}\cdot\frac{14}{(n-1)^2}. \tag{39}$$

Now we are ready to estimate the probability of the event $\mathcal{F}(t)$.

2.6. Application of Local Lemma

Remember that by the definitions (18), (19), (20) and (21) of the events $\mathcal{F}(t)$, $\mathcal{A}(f,t)$, $\mathcal{B}(f,t)$ and $\mathcal{C}(f,t)$ we have the equality (22):

$$\mathcal{F}(t)=\bigcup_{f\in E}\big(\mathcal{A}(f,t)\cup\mathcal{B}(f,t)\cup\mathcal{C}(f,t)\big).$$

So, it follows from the relations (24), (26), (31), (36) that

$$\begin{aligned}\mathcal{F}(t)&\subset\bigcup_{f\in E}\big(\mathcal{A}(f,t)\cup\mathcal{B}(f,t)\cup\mathcal{C}(f,t)\big)\subset\\&\subset\mathcal{A}\cup\bigcup_{f\in E}\Big(\mathcal{A}(f,t)\cup\mathcal{B}(f,t)\cup\big(\mathcal{C}(f,t)\cap\overline{\mathcal{A}}\big)\Big)\subset\\&\subset\bigcup_{f\in E}\left(\mathcal{Q}(f,t)\cup\bigcup_{\substack{f'\in E:\\|f\cap f'|=1}}\ \bigcup_{\substack{f''\in E:\,|f\cap f''|=1,\\ f\cap f'\cap f''=\varnothing}}\big(\mathcal{A}(f,f',f'')\cup\mathcal{C}(f,f',f'')\big)\right).\end{aligned} \tag{40}$$

Further, we shall use a classical claim called Local Lemma. This statement was first proved in the paper of P. Erdős and L. Lovász [3]. We shall formulate it in a special case.

THEOREM 7. *Let events* $\mathcal{B}_1,\ldots,\mathcal{B}_M$ *be given on some probability space. Let* $S_1,\ldots,S_M$ *be subsets of* $\mathcal{R}_M=\{1,\ldots,M\}$ *such that for any* $i=1,\ldots,M$, *the event* $\mathcal{B}_i$ *is* *independent of the algebra generated by the events*

$$\{\mathcal{B}_j : j \in \mathcal{R}_M \backslash (S_i \cup \{i\})\}.$$

If for any $i = 1, \dots, M$ *the following inequality hold*

$$\sum_{j \in S_i \cup \{i\}} \mathrm{Pr}(\mathcal{B}_j) \leqslant \frac{1}{4}, \tag{41}$$

then

$$\mathrm{Pr}\Big(\bigcap_{j=1}^{M} \overline{\mathcal{B}_j}\Big) \geqslant \prod_{j=1}^{M} \big(1 - 2 \cdot \mathrm{Pr}(\mathcal{B}_j)\big) > 0.$$

The proof of the Local Lemma can be found, for example, in the book [1].

Consider the system $\Psi(t)$ consisting of all the events $\mathcal{Q}(f,t)$, $f \in E$ and all the events

$$\mathcal{A}(f,f',f''), \quad \mathcal{C}(f,f',f''), \quad f,f',f'' \in E$$

satisfying

$$|f \cap f'| = 1, \quad |f \cap f''| = 1, \quad |f \cap f' \cap f''| = 0.$$

Due to (40) the following inequality holds

$$\mathrm{Pr}(\mathcal{F}(t)) \leqslant \mathrm{Pr}\Big(\bigcup_{\mathcal{Q} \in \Psi(t)} \mathcal{Q}\Big) = 1 - \mathrm{Pr}\Big(\bigcap_{\mathcal{Q} \in \Psi(t)} \overline{\mathcal{Q}}\Big). \tag{42}$$

We shall show that the probability of

$$\bigcap_{\mathcal{Q} \in \Psi(t)} \overline{\mathcal{Q}}$$

is greater than zero. Due to the Local Lemma (see Theorem 7), it is sufficient to find, for every $\mathcal{Q} \in \Psi(t)$, a system of events $\Psi_{\mathcal{Q}} \subset \Psi(t)$ such that $\mathcal{Q}$ and the algebra generated by $\{\mathcal{J} \in \Psi(t) \backslash (\Psi_{\mathcal{Q}} \cup \{\mathcal{Q}\})\}$ are independent, and, moreover, such that

$$\sum_{\mathcal{J} \in \Psi_{\mathcal{Q}} \cup \mathcal{Q}} \mathrm{Pr}(\mathcal{J}) \leqslant \frac{1}{4}. \tag{43}$$

For any $\mathcal{Q} \in \Psi$, we define *the domain* $D(\mathcal{Q})$ of the event $\mathcal{Q}$ as follows:

$$D(\mathcal{Q}) = \begin{cases} f, & \text{if } \mathcal{Q} = \mathcal{Q}(f,t) \text{ for some } f \in E; \\ f \cup f' \cup f'', & \text{if } \mathcal{Q} = \mathcal{A}(f,f',f'') \text{ or } \mathcal{C}(f,f',f''). \end{cases}$$

By the definitions (23) (30), (35) the event $\mathcal{Q}$ belongs to the algebra generated by the random variables $\{\xi_j, \eta_j, X_j : j \in D(\mathcal{Q})\}$. Then this event is independent of the algebra generated by the random variables

$$\{\xi_j, \eta_j, X_j : j \in V \setminus D(\mathcal{Q})\}.$$

Let us consider, for any $\mathcal{Q} \in \Psi(t)$ the system $\Psi_{\mathcal{Q}}$ consisting of all the events $\mathcal{Q}(g,t)$ such that $g \cap D(\mathcal{Q}) \neq \varnothing$ and all the events $\mathcal{A}(g,g',g'')$, $\mathcal{C}(g,g',g'')$ such that

$$D(\mathcal{Q}) \cap (g \cup g' \cup g'') \neq \varnothing.$$

In other words,

$$\Psi_{\mathcal{Q}} = \left\{ \widetilde{\mathcal{Q}} :\ \widetilde{\mathcal{Q}} \in \Psi(t), D(\widetilde{\mathcal{Q}}) \cap D(\mathcal{Q}) \neq \varnothing \right\}.$$

Thus, the event $\mathcal{Q}$ is independent of the algebra generated by $\{\mathcal{J} \in \Psi(t) \setminus \Psi_{\mathcal{Q}}\}$, if we choose $\Psi_{\mathcal{Q}}$ in this way. Moreover, $\mathcal{Q} \in \Psi_{\mathcal{Q}}$. We have to check the inequality (43). By the choice of the set $\Psi_{\mathcal{Q}}$ we get the relation

$$\sum_{\mathcal{J} \in \Psi_{\mathcal{Q}}} \mathrm{Pr}(\mathcal{J}) \leqslant \sum_{g \in E:\, g \cap D(\mathcal{Q}) \neq \varnothing} \mathrm{Pr}(\mathcal{Q}(g,t)) + \sum_{\substack{g,g',g'' \in E:\, |g \cap g'|=1, \\ |g \cap g''|=1,\, |g' \cap g'' \cap g''|=0, \\ D(\mathcal{Q}) \cap (g \cup g' \cup g'') \neq \varnothing}} \mathrm{Pr}\big(\mathcal{A}(g,g',g'')\big) + \\ + \sum_{\substack{g,g',g'' \in E:\, |g \cap g'|=1, \\ |g \cap g''|=1,\, |g' \cap g'' \cap g''|=0, \\ D(\mathcal{Q}) \cap (g \cup g' \cup g'') \neq \varnothing}} \mathrm{Pr}(\mathcal{C}(g,g',g'')). \tag{44}$$

Let us denote by $s_1(\mathcal{Q})$, $s_2(\mathcal{Q})$ and $s_3(\mathcal{Q})$ the number of summands in the first sum, the second sum and the third sum in the right-hand side of (44) respectively. It is clear that

$$s_2(\mathcal{Q}) = s_3(\mathcal{Q}).$$

Using these notations from the relation (44) and the estimates (29), (32), (39), we get the inequality

$$\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}} \mathrm{Pr}(\mathcal{J}) \leqslant s_1(\mathcal{Q})r^{1-n}\bigl(n(1-e^{-t})e^{-t(n-1)}+e^{-tn}\bigr)+$$
$$+s_2(\mathcal{Q})r^{3-3n}\cdot\frac{2}{n(n-1)}+s_3(\mathcal{Q})r^{3-3n}\cdot\frac{14}{(n-1)^2}. \tag{45}$$

Now we shall consider two cases depending on $\mathcal{Q}$. Recall that $D(H)$ denote the maximum edge degree of the hypergraph H.

1. $\mathcal{Q}=\mathcal{Q}(f,t)$ for some $f\in E$. There exist at most $D(H)$ other edges g intersecting an arbitrary edge of E. So,

$$s_1(\mathcal{Q})\leqslant D(H)+1. \tag{46}$$

Now let us consider $s_2(\mathcal{Q})$. We have to estimate the number of edge triples (g,g',g'') satisfying the conditions:

$$|g\cap g'|=1,\quad |g\cap g''|=1,\quad |g'\cap g''\cap g''|=0,\quad f\cap(g\cup g'\cup g'')\neq\varnothing.$$

If $g\cap f\neq\varnothing$, then g can be chosen by at most $D(H)+1$ ways, g' — by at most $D(H)$ ways (g' should intersect with g and not coincide with it) and, finally, g'' — by at most $D(H)-1$ ways (g'' should intersect with g and not coincide with it and with g'). Thus, there are at most $D^3(H)$ such configurations. The same argument applied to the situations $g'\cap f\neq\varnothing$ and $g''\cap f\neq\varnothing$ also gives at most $D^3(H)$ configurations in every case. Hence,

$$s_2(\mathcal{Q})\leqslant 3D^3(H). \tag{47}$$

2. $\mathcal{Q}=\mathcal{A}(f,f',f'')$ or $\mathcal{C}(f,f',f'')$ for some $f,f',f''\in E$, satisfying

$$|f\cap f'|=1,\quad |f\cap f''|=1,\quad |f\cap f'\cap f''|=0.$$

Let us estimate the number of edges g such that $g\cap(f\cup f'\cup f'')\neq\varnothing$. By the definition of $D(H)$ there are at most $D(H)+1$ edges intersecting f. Thus, there are at most $D(H)-1$ edges g intersecting f' and having empty intersection with f. The same holds for f''. So, we have the following upper bound for $s_1(Q)$:

$$s_1(Q) \leqslant 3D(H)-1. \tag{48}$$

Since in the second case our event depends on three edges f, f', f'' it is easy to see from the previously obtained inequality (47) that

$$s_2(Q) \leqslant 9D^3(H). \tag{49}$$

It is clear from (46), (47), (48) and (49) that the maximal values of the upper bounds for $s_1(Q)$ and $s_2(Q)$ are in the second case. So, to prove (43) it is sufficient to establish (due to (45)) the following inequality:

$$\begin{gathered}(3D(H)-1)r^{1-n}\left(n(1-e^{-t})e^{-t(n-1)}+e^{-tn}\right)+\\ +9D^3(H)r^{3-3n}\cdot\frac{2}{n(n-1)}+9D^3(H)r^{3-3n}\cdot\frac{14}{(n-1)^2}\leqslant\frac{1}{4}.\end{gathered} \tag{50}$$

From the initial condition (13) we have

$$D(H) \leqslant \frac{1}{11} r^{n-1} n^{2/3}. \tag{51}$$

Thus,

$$9D^3(H)r^{3-3n}\cdot\frac{2}{n(n-1)}+9D^3(H)r^{3-3n}\cdot\frac{14}{(n-1)^2}\leqslant$$

(using (51) and the condition $n \geqslant 3$)

$$\leqslant \frac{9}{11^3}\left(\frac{2n}{n-1}+\frac{14n^2}{(n-1)^2}\right) \leqslant \frac{9}{11^3}\left(3+\frac{63}{2}\right)=\frac{310,5}{1331}<\frac{1}{4}.$$

Hence, we can choose a very large $t>0$ satisfying the inequality

$$3(D(H)-1)r^{1-n}\left(n(1-e^{-t})e^{-t(n-1)}+e^{-tn}\right)+\frac{310,5}{1331}<\frac{1}{4}.$$

Such choice of the parameter t implies the required inequality (50). Thus, the condition (43) holds and Local Lemma can be applied to the system of events $\Psi(t)$.

Local Lemma states that

$$\Pr\Big(\bigcap_{Q\in\Psi(t)} \overline{Q}\Big) > 0.$$

By (42) we have shown that for some t the probability of the event $\mathcal{F}(t)$ is less than 1, and, consequently, there is a proper coloring from the lists L for H, i. e. H is r-choosable. Theorem 6 is proved.

Remark 2. The constant 1/11 is right-hand side of (13) can be replaced by a greater number for large n. For example, our proof shows that one can take $1/(8.5) - O(1/n)$ instead of it.

Acknowledgements. This work was supported by Russian Foundation of Fundamental Research (grant № 12–01–00683-a), by the program "Leading Scientific Schools" (grant № NSh-2519.2012.1) and by the grant of the President of Russian Federation MK-1122.2012.1.

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