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Disjoint edges in separated hypergraphs

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Abstract: Let $X_1, X_2, \ldots, X_r$ be pairwise disjoint sets. A family $\mathcal{F}$ of k-subsets of $X_1 \cup \ldots \cup X_r$ is called *separated* if $|F \cap X_j| \leqslant 1$ holds for all $F \in \mathcal{F}$ and $1 \leqslant j \leqslant r$.The maximum number of pairwise disjoint members of $\mathcal{F}$ is called the *matching number*.
The present paper provides best possible bounds on the maximum size of separated families with given matching number.

Keywords: finite family of sets, hypergraph, matching number, pairwise disjoint edges

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1. Introduction

For a finite set X of n vertices and a family

$$\mathcal{F} \subset \binom{X}{k}$$

define the matching number $\nu(\mathcal{F})$ as the maximum number of pairwise disjoint edges in $\mathcal{F}$.

A classical problem in extremal set theory is to determine the maximum number of edges in a family

$$\mathcal{F} \subset \binom{X}{k}$$

with given matching number. Let us fix a positive integer s and define two families $\mathcal{A}_1(n)$ and $\mathcal{A}_k$ of matching number s.

$$\mathcal{A}_1(n) = \left\{ A \in \binom{X}{k} : A \cap T \neq 0 \right\}$$

where $T \in \binom{X}{s}$ is fixed, $n \geqslant sk$.

$$\mathcal{A}_k = \binom{[(s+1)k-1]}{k},$$

here and in the sequel $[\ell] = \{1, 2, \ldots, \ell\}$.

Let us recall the following longstanding conjecture of Erdős [3].

Conjecture 1 [3]. If

$$n \geqslant (s+1)k - 1 \quad \text{and} \quad \mathcal{F} \subset \binom{X}{k}$$

satisfies $\nu(\mathcal{F}) = s$ then

$$|\mathcal{F}| \leqslant \max\big\{|\mathcal{A}_1(n)|, |\mathcal{A}_k|\big\} \quad \text{holds.} \tag{1}$$

For $n > n_0(k, s)$ Erdős [3] proved (1). For $k = 1$ it is trivial, for $k = 2$ it is an old result of Erdős and Gallai [4] (for a different proof see [1]), for $k = 3$ it was proved recently (cf. [8] and [12]). However the general case is still wide open (cf. [2], [7], [9]). Let us also mention that the case $s = 1$ is the famous Erdős—Ko—Rado Theorem [5].

In the present paper we consider the analogous problem for *separated* families.

Definition 1. *Let $X_1, X_2, \ldots, X_r$ be pairwise disjoint sets. A family $\mathcal{F}$ of k-subsets of $X_1 \cup \ldots \cup X_r$ is called separated if $|F \cap X_j| \leqslant 1$ holds for all $F \in \mathcal{F}$ and $1 \leqslant j \leqslant r$.*

First we are to prove the following.

Theorem 1. *Let $\mathcal{F}$ be a separated family of k-subsets with $\nu(\mathcal{F}) = s$, where $1 \leqslant s < n$ is fixed. Suppose moreover that $|X_1| = \ldots = |X_r| = n$. Then*

$$|\mathcal{F}| \leqslant s\binom{r-1}{k-1}n^{k-1} \quad \textit{holds.} \tag{2}$$

The bound (2) is best possible. In order to see this take an s-subset D of X_i for some fixed i, $1 \leqslant i \leqslant r$ and define

$$\mathcal{F}_D = \left\{ F \in \binom{X_1 \cup \ldots \cup X_r}{k} : F \cap D \neq 0,\ |F \cap X_j| \leqslant 1,\ 1 \leqslant j \leqslant r \right\}.$$

Then we extend this result to the non-equipartite case, showing that $\mathcal{F}_D$ still provides the best contstruction (cf. Theorem 3).

2. The case $r = k$

This case is rather simple. Let us solve it in a more general form. Instead of

$$|X_1| = \ldots = |X_k|,$$

we only assume $|X_i| = n_i$, $i = 1, \ldots, k$, $s < n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$. We choose further an integer q with $s < q \leqslant n_1$.

Theorem 2. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_q$ be separated families of k-element sets such that there are no $1 \leqslant i_0 < i_1 < 1 < \ldots < i_s \leqslant q$ and pairwise disjoint edges $F_{i_j} \in \mathcal{F}_{i_j}$, $0 \leqslant j \leqslant s$.*

Then

$$|\mathcal{F}_1| + \ldots + |\mathcal{F}_q| \leqslant s \cdot n_1 \cdot n_2 \cdot \ldots \cdot n_k \quad \textit{holds.} \tag{3}$$

Proof. Consider permutations $\pi(i) = (P_i(1), \ldots, P_i(n_i))$ of the elements of X_i, $i = 1, \ldots, k$ and define $\mathcal{P} = \{P_1, \ldots, P_{n_1}\}$ where

$$P_j = \{P_1(j), P_2(j), \ldots, P_k(j)\}.$$

We define a bipartite graph G with partite sets

$$A = \{1, \ldots, n_1\} \quad \text{and} \quad B = \{1, \ldots, q\} \quad \text{and} \quad (a, b) \in G$$

iff $P_a \in \mathcal{F}_b$ holds.

Claim 1. $\nu(G) \leqslant s$

Proof of the Claim 1. Indeed, if (a_t, b_t), $0 \leqslant t \leqslant s$ were independent edges then $P_{a_0}, \ldots, P_{a_s}$ are pairwise disjoint edges, each belonging to a different $\mathcal{F}_b$. □

CLAIM 2.

$$\sum_{1\leqslant b\leqslant q} |\mathcal{F}_b \cap \mathcal{P}| \leqslant sn_1 \quad \text{holds.} \tag{4}$$

PROOF OF THE CLAIM 2. Using Claim 1 and the König-Hall Theorem (cf. [12]), we can find an s-element set $C = A \cup B$ such that every edge of G has at least one endpoint in C. Consequently, the number of edges of G, which is exactly the LHS of (4) is at most $|C|$ times the maximum degree, i. e. $s \cdot n_1$. □

Intuitively the proof of the theorem is complete, because we got an upper bound for the LHS of (4) which is s/q times the trivial bound qn_1.

Averaging over all permutations should prove (3).

To make it precise one has to check that for all $1 \leqslant b \leqslant q$, $F = (x_1, \dots, x_k) \in \mathcal{F}_b$ and all $1 \leqslant a \leqslant n_1$ the proportion of permutations $\pi(i)$ with $P_i(a) = x_a$ is $1/n_i$. Thus the expected size of $|\mathcal{P} \cap \mathcal{F}_b|$ is $|\mathcal{P}||\mathcal{F}_b|/n_1 \cdot \ldots \cdot n_k = |\mathcal{F}|/n_2 \cdot \ldots \cdot n_k$.

Using (4) we infer

$$\sum_{1\leqslant b\leqslant q} |\mathcal{F}_b|/n_2 \cdot \ldots \cdot n_k \leqslant sn_1, \quad \text{i. e.,} \quad \sum_{1\leqslant b\leqslant q} |\mathcal{F}_b| \leqslant sn_1 \cdot \ldots \cdot n_k \quad \text{as desired.} \quad \square$$

COROLLARY 1. (2) *holds for* $k = r$.

PROOF. Let $n_1 = \ldots = n_k = n$, $q = n$ and $\mathcal{F}_1 = \mathcal{F}_2 = \ldots = \mathcal{F}_n = \mathcal{F}$. $\nu(\mathcal{F}) = s$ implies that $\mathcal{F}_1, \dots, \mathcal{F}_n$ satisfy the assumption Theorem 2. Thus $n|\mathcal{F}| = |\mathcal{F}_1| + \ldots + |\mathcal{F}_n| \leqslant sn^k$, that is $|\mathcal{F}| \leqslant s \cdot n^{k-1}$ as desired. □

Remark. From the proof it is clear that in case of equality either $C \subset A$ or $q = n$ and $C \subset B$ hold for *all* partitions. This can be used to prove *uniqueness* of the optimal families in (2) for $s \geqslant 3$ and also in Theorem 2.

3. The general case

Our main tool is Katona's Cyclic Permutation Method (cf. [10]).

Let $m > k$ be fixed and consider the m-set $X = \{x_1, \dots, x_m\}$ and $B = B(k)$ the family of cyclic arcs of length k, that is, $B = \{B_1, \dots, B_m\}$, where

$$B_i = \{x_i, x_{i+1}, \dots, x_{i+k-1}\}$$

(for $i + j > m$, $x_{i+j} \overset{\text{def}}{=} x_{i+j-m}$).

LEMMA 1. *If* $m \geqslant s$ $\quad \mathcal{A} \subset B$ *and* $\nu(\mathcal{A}) \leqslant s$, *then*

$$|\mathcal{A}| \leqslant sk \quad \textit{holds.} \tag{6}$$

PROOF. Let us first consider the case $m = (s+1)k$.

Let us partition $\mathcal{B}$ into $\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_{k-1}$ by defining $\mathcal{B}_j = \{B_i : i \equiv j \ (\mathrm{mod}\ k)\}$. It should be clear that each $\mathcal{B}_j$ is a partition of X into $s+1$ pairwise disjoint k-sets. Thus

$$|\mathcal{A} \cap \mathcal{B}_j| \leqslant s \quad \text{for} \quad 0 \leqslant j < k.$$

Summing for j gives

$$|\mathcal{A}| = \sum_{0 \leqslant j < k} |\mathcal{A} \cap \mathcal{B}_j| \leqslant sk, \quad \text{as desired.}$$

Let us now apply induction on m.

Suppose that the statement is true for m and consider it for $m+1$.

Since $m \geqslant (s+1)k$, $\nu(\mathcal{B}) > s$. Consequently, $\mathcal{A} \subsetneqq \mathcal{B}$. WLOG assume that $B_{m+1} \notin \mathcal{A}$ holds.

Let $\mathcal{B}^{(m)}$ denote the set of cyclic arcs for m, $\mathcal{B}^{(m)} = \{B_i^{(m)} : i = 1, \ldots, m\}$. Define the map $\varphi : (\mathcal{B} - \{B_{m+1}\}) \to B^{(m)}$ by $\varphi(B_i) = B_i^{(m)}$. Note that for the B_i with $x_{m+1} \notin B_i$, the map φ is the identity. For those with $x_{m+1} \in B_i$, x_{m+1} is replaced by $x_{i+k-1-m}$. For B_{m+1} we would have

$$\varphi(B_{m+1}) = \varphi(B_1) = B_1^{(m)}$$

but fortunately $B_{m+1} \notin \mathcal{A}$.

It should be immediate that $\nu(\varphi(\mathcal{A})) \leqslant \nu(\mathcal{A}) \leqslant s$. Thus by the induction hypothesis $|\mathcal{A}| = |\varphi(\mathcal{A})| \leqslant sk$, concluding the proof. □

PROOF OF THEOREM 1. Let $\pi_i = (x_1^{(i)}, \ldots, x_n^{(i)})$ be a random ordering of the elements of X_i, $i = 1, \ldots, r$. Let also $\varrho = (i_1, \ldots, i_r)$ be a cyclic ordering of $[r]$. We use these to define a cyclic ordering π of the elements of $X_1 \cup \ldots \cup X_r$ that is:

$$\pi = \left(x_1^{(i_1)}, x_1^{(i_2)}, \ldots, x_1^{(i_r)}, x_2^{(i_1)}, x_2^{(i_2)}, \ldots, x_n^{(i_{r-1})}, x_n^{(i_r)}\right).$$

We apply the Lemma for $m = nr$, $\mathcal{A} = \mathcal{F} \cap \mathcal{B}$. We infer $|\mathcal{A}| \leqslant sk$.

For a set $F \in \mathcal{F}$ define $T(F) = \{i : F \cap X_i \neq 0\}$. Then

$$T(F) \in \binom{[r]}{k}.$$

There are $k!(r-k)!$ choices of the cyclic permutation ϱ, such that $T|F|$ occurs as an arc (of length k) in it. Then in π there are n places of arcs of length k with the same $T(F)$. By randomness, for each of them the probability, that the corresponding set is exactly F is n^{-k}. Altogether there are n places making the probability for F occurring as an arc of π to n^{-k+1}. On the other hand, this probability is 0 for all ϱ where $T(F)$ is not an arc.

Choosing ϱ randomly over all $(r-1)!$ choices with uniform distribution gives the probability

$$\frac{k!(r-k)!}{(r-1)!} n^{-k+1} = \frac{r \cdot n}{\binom{r}{k} \cdot n^k}.$$

Consequently, the expectation of

$$|\mathcal{F} \cap \mathcal{B}| \quad \text{is} \quad |\mathcal{F}| \cdot \frac{r \cdot n}{\binom{r}{k} n^k}.$$

Since we proved $|\mathcal{F} \cap \mathcal{B}| \leqslant sk$, we deduce

$$|\mathcal{F}| \leqslant \frac{s \cdot k}{r \cdot n} \binom{r}{k} n^k = s \binom{r-1}{k-1} \cdot n^{k-1} \quad \text{as desired.} \qquad \square$$

4. The proof of main result

Using the shifting technique we extend Theorem 2 to the case $|X_i| = n_i$, $i = 1, \ldots, r$.

Set

$$\mathcal{H} = \left\{H \subset X_1 \cup \ldots \cup X_r : |H| = k, \ |H \cap X_i| \leqslant 1, \ 1 \leqslant i \leqslant r\right\}.$$

Suppose that $n_1 \leqslant n_i$ for $2 \leqslant i \leqslant r$. For

$$D \in \binom{X_1}{s}$$

define

$$\mathcal{F}_D = \{F \in H : F \cap D \neq \varnothing\}.$$

Obviously, $\nu(\mathcal{F}_D) = s$ holds for $|X_1| \geqslant s$.

THEOREM 3. *Suppose that* $\mathcal{F} \subset \mathcal{H}$ *satisfies* $\nu(\mathcal{F}) = s$, $s < n_1$. *Then* $|\mathcal{F}| \leqslant |\mathcal{F}_D|$ *holds.*

PROOF. Since the case $n_1 = n_2 = \ldots = n_r$ is exactly Theorem 1, we may assume that $n_r > n_1$ holds and apply induction on $n_1 + \ldots + n_r$. Let us apply the shifting operation (for basic facts concerning shifting cf. [6]). Let us order the elements of X_r, say, $X_r = (y_1, \ldots, y_{n_r})$. For $1 \leqslant i < j \leqslant n_r$ define

$$S_{ij}(F) = \begin{cases} \overline{F} = (F - \{y_j\}) \cup \{y_i\} & \text{if} \quad y_i \in F, \quad y_i \notin \mathcal{F}, \quad \overline{F} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

Set also

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) : F \in \mathcal{F}\}.$$

Then $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$ and $\nu(S_{ij}(\mathcal{F})) \leqslant \nu(\mathcal{F})$ hold.

Repeated application of the operation S_{ij} for all $1 \leqslant i < j \leqslant n_r$will eventually produce a family $\mathcal{F}^*$ that is *stable*. That is $S_{ij}(\mathcal{F}^*) = \mathcal{F}^*$ for all $1 \leqslant i < j \leqslant n_r$.

For an arbitrary element x and a family $\mathcal{T}$ define

$$\mathcal{T}(\overline{x}) = \{T \in \mathcal{T} : x \notin T\}, \quad \mathcal{T}(x) = \{T - \{x\} : x \in T \in \mathcal{T}\}.$$

Obviously, $|\mathcal{T}| = |\mathcal{T}(\overline{x})| + |\mathcal{T}(x)|$ and $\nu(\mathcal{T}(\overline{x})) \leqslant \nu(\mathcal{T})$ hold.

CLAIM 3. $\nu\big(\mathcal{F}^*(y_{n_r})\big) \leqslant s$.

We could just refer the reader to [6], but we prefer to show the easy proof.

Suppose for contradiction that $F_1, \ldots, F_{s+1}$ are pairwise disjoint members of $\mathcal{F}^*(y_{n_r})$. Define $G_i = F_i \cup \{y_i\}$ for $i - 1, \ldots, s+1$. By stability $G_1, \ldots, G_{s+1}$ are pairwise disjoint members of $\mathcal{F}^*$, a contradiction proving the claim.

Since both $\mathcal{F}^*(\overline{y}_{n_r})$ and $\mathcal{F}^*(y_{n_r})$ are families on

$$X_i \cup \ldots \cup X_{n_r} - \{y_{n_r}\},$$

we may apply the induction hypothesis to deduce

$$\left|\mathcal{F}^*(\overline{y}_{n_r})\right| \leqslant \left|\mathcal{F}_D(\overline{y}_{n_r})\right| \quad \text{and} \quad \left|\mathcal{F}^*(y_{n_r})\right| \leqslant \left|\mathcal{F}_D(y_{n_r})\right|.$$

Adding these two inequalities $|\mathcal{F}| = |\mathcal{F}^*| \leqslant |\mathcal{F}_D|$ follows. □

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