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# On groups with Perfect Order Subsets

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**Abstract:** A finite group $G$ is said to have *Perfect Order Subsets* if for every $d$, the number of elements of $G$ of order $d$ (if there are any) divides $|G|$. Answering a question of Finch and Jones from 2002, we prove that if $G$ is Abelian, then such a group has order divisible by 3 except in the case $G = \mathbb{Z}/2^k\mathbb{Z}$. We also place additional restrictions on the order of such groups.

**Keywords:** Abelian groups, primes, Pratt trees

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## 1. Introduction

Consider the multiplicative function

$$f(n) = \prod_{p^a \| n} (p^a - 1).$$

A finite group $G$ is said to have *Perfect Order Subsets* if for every $d$, the number of elements of $G$ of order $d$ (if there are any) divides $|G|$. This notion was introduced in the paper [1] by C. Finch and L. Jones. In the case of finite Abelian groups, the authors reduced the problem of which groups have this property to the case of groups of the form

$$G = \prod_{i=1}^{k} \left( \frac{\mathbb{Z}}{p_i \mathbb{Z}} \right)^{a_i},$$

where $p_i$ are primes and $a_i \geqslant 1$. For these groups, it follows from results in [1] that $G$ has *Perfect Order Subsets* if and only if $f(n)|n$ where $n = |G|$. Only 11 examples of such $n$ are known, given below, and only one of these is divisible by the square of an odd prime.

$$
\begin{aligned}
&2\\
&2\cdot 3\\
&2^2\cdot 3\\
&2^3\cdot 3\cdot 7\\
&2^4\cdot 3\cdot 5\\
&2^5\cdot 3\cdot 5\cdot 31\\
&2^8\cdot 3\cdot 5\cdot 17\\
&2^{16}\cdot 3\cdot 5\cdot 17\cdot 257\\
&2^{17}\cdot 3\cdot 5\cdot 17\cdot 257\cdot 131\,071\\
&2^{32}\cdot 3\cdot 5\cdot 17\cdot 257\cdot 65\,537\\
&2^{11}\cdot 3\cdot 5\cdot 11^2\cdot 23\cdot 89
\end{aligned}
$$

The authors of [1] asked several basic questions about such groups (see also [6]). One of which asks if $|G|$ is not a power of 2, must 3 divide $|G|$? We prove that this is the case for Abelian groups.

THEOREM 1. *If $f(n)|n$ and $n > 2$, then $3|n$.*

We also show that $f(n)|n$ implies that $n/f(n)$ is bounded. Note that the divergence of

$$\prod_p \left(1 - \frac{1}{p}\right)^{-1}$$

implies that $n/f(n)$ is unbounded for general $n$. On the other hand, all of the known examples of $n$ such that $n > 6$ and $f(n)|n$ (given in [1]) satisfy $n = 2f(n)$.

THEOREM 2. *For any $n \in \mathbb{N}$, if $f(n)|n$, then $n/f(n) \leqslant 85$.*

The most important property of numbers $n$ with $f(n)|n$ is given by the following easy proposition.

PROPOSITION 1. *If $f(n)|n$, then for every prime $p|n$, every prime divisor of $p - 1$ also divides $n$.*

By Proposition 1, knowing that $3 \nmid n$ allows us to exclude many possible prime factors of $n$. Inductively, define a set $\mathscr{P}$ or primes as follows: (i) $2 \in \mathscr{P}$, (ii) $3 \notin \mathscr{P}$, (iii) for every prime $p \geqslant 5$, $p \in \mathscr{P}$ if and only if all prime factors of $p-1$ are in $\mathscr{P}$. Thus,

$$\mathscr{P} = \{2, 5, 11, 17, 23, 41, 47, 83, 89,\\ 101, 137, 167, 179, 251, 257, 353, 359, 401, 461, 503, \ldots\}. \qquad (1)$$

By Proposition 1, every prime dividing $n$ must come from $\mathscr{P}$. The set $\mathscr{P}$ has a alternative interpretation as the set of all primes whose *Pratt tree* (see [3]) does not contain a node labelled 3.

Our proof of Theorem 1 is primarily based on the lower bound in the following estimate:

THEOREM 3. *We have*

$$0.2512 \leqslant \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p}\right) \leqslant 0.2793.$$

Since $\mathscr{P}$ omits all primes $p \equiv 1 \pmod 3$, and hence omits all primes $q$ such that $q-1$ has a prime factor which is $1 \pmod 3$, standard application of sieve methods (e. g., Theorem 4.2 of [5]) yields the upper bound

$$\mathscr{P}(x) := \#\{p \leqslant x : p \in \mathscr{P}\} \ll \frac{x}{\log x} \prod_{\substack{p \leqslant x \\ p \equiv 1 \pmod 3}} \left(1 - \frac{1}{p}\right) \ll \frac{x}{(\log x)^{3/2}}.$$

From this one obtains immediately from partial summation that the product in Theorem 3 converges. Obtaining good numerical bounds requires more work.

## 2. Number theory tools

Our first result is an estimate of Rosser and Schoenfeld [7].

LEMMA 1. *For any* $x > 1$,

$$\left(1 + \frac{1}{\log^2 x}\right)^{-1} \leqslant e^{\gamma}(\log x) \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \leqslant \left(1 + \frac{1}{\log^2 x}\right).$$

The following general sieve estimate is Theorem 1 of [2].

LEMMA 2. *Let $S$ be a set of primes containing 2, and put*

$$H(t)=\prod_{\substack{p\leqslant t\\ p\in S}}\left(1-\frac{1}{p}\right).$$

*Then $N(x)$, the number of primes $p\leqslant x$ such that all the prime factors of $p-1$ are in $S$, satisfies*

$$N(x)\leqslant\frac{x}{(1+1/\log x)I(x)},\qquad I(x)=\int_1^{\sqrt{x}}\frac{\log t}{t}H(t)\,dt.$$

LEMMA 3. *Let $S$ be any set of primes with the property that for all $p\in S$ and prime $q|(p-1)$, $q\in S$. For any $x\geqslant 2$,*

$$\prod_{p\in S}\left(1-\frac{1}{p}\right)\geqslant\prod_{\substack{p\leqslant x\\ p\in S}}\left(1-\frac{1}{p}\right)-\frac{8}{\log x}.$$

PROOF. We may assume $S$ is nonempty, so $2\in S$. Write

$$S(x)=\#\{p\in S:p\leqslant x\}.$$

For $j\geqslant 0$, let $y_j=x^{2^j}$ and

$$H_j=\prod_{\substack{p\leqslant y_j\\ p\in S}}\left(1-\frac{1}{p}\right).$$

Without loss of generality, suppose

$$H_0>\frac{8}{\log x},$$

so in particular $x\geqslant e^{16}$. We derive by induction lower estimates for $H_j$. By Lemma 2, when $y_{j-1}\leqslant t\leqslant y_j$, we have

$$S(t)\leqslant\frac{8t}{(1+1/\log t)H_{j-1}\log^2 t}.$$

By partial summation and the inequality

$$\log\left(1-\frac{1}{t}\right)\geqslant-\frac{1}{t-1},$$

$$\log\left(\frac{H_j}{H_{j-1}}\right) = \sum_{\substack{y_{j-1}<p\leqslant y_j \\ p\in S}} \log\left(1-\frac{1}{p}\right) =$$

$$= S(y_j)\log\left(1-\frac{1}{y_j}\right) - S(y_{j-1})\log\left(1-\frac{1}{y_{j-1}}\right) - \int_{y_{j-1}}^{y_j} \frac{S(u)}{u^2-u}\,du \geqslant$$

$$\geqslant -\frac{S(y_j)}{y_j-1} - \frac{y_j}{y_j-1}\int_{y_{j-1}}^{y_j} \frac{S(u)}{u^2}\,du \geqslant$$

$$\geqslant -\frac{8}{H_{j-1}}\left(\frac{y_j}{y_j-1}\right)\left(\frac{1}{\log^2 y_j + \log y_j} + \int_{y_{j-1}}^{y_j} \frac{du}{u(\log^2 u + \log u)}\right).$$

Using the relation $y_j = y_{j-1}^2$, we find that the integral above equals

$$\log\left(1+\frac{1}{\log y_j + 1}\right).$$

Now

$$\log(1+\varepsilon) \leqslant \varepsilon - \frac{1}{3}\varepsilon^2 \quad \text{for} \quad \varepsilon = \frac{1}{\log y_j + 1} \leqslant \frac{1}{3}.$$

Thus,

$$\log\left(\frac{H_j}{H_{j-1}}\right) \geqslant -\frac{8}{H_{j-1}}\left(\frac{y_j}{y_j-1}\right)\left(\frac{1}{\log^2 y_j + \log y_j} + \frac{1}{\log y_j + 1} - \frac{1}{3(\log y_j + 1)^2}\right) =$$

$$= -\frac{8}{H_{j-1}\log y_j}\left(\frac{y_j}{y_j-1}\right)\left(1 - \frac{\log y_j}{3(\log y_j + 1)^2}\right).$$

Since $y_j \geqslant y_0 \geqslant e^{16}$, the right side above is $\geqslant -8/(H_{j-1}\log y_j)$. Therefore,

$$H_j \geqslant H_{j-1}\exp\left\{-\frac{8}{H_{j-1}\log y_j}\right\} \geqslant H_{j-1} - \frac{8}{\log y_j} = H_{j-1} - \frac{8\cdot 2^{-j}}{\log x}.$$

Iterating this inequality concludes the proof. □

## 3. Proof of Theorem 1

We first describe how to deduce Theorem 1 from Theorem 3. Observe that

$$\frac{f(n)}{n} = \prod_{p^a \| n} \left(1 - \frac{1}{p^a}\right). \tag{2}$$

Suppose that $f(n)|n$ and $3 \nmid n$. If $2^9|n$, then (2) and Theorem 3 imply

$$\frac{f(n)}{n} \geqslant \frac{511}{512} \prod_{\substack{p \geqslant 5 \\ p \in \mathscr{P}}} \left(1 - \frac{1}{p}\right) \geqslant \frac{511}{512}(0.5024) > \frac{1}{2}.$$

Hence, $f(n) \nmid n$. Thus, $2^k \| n$, where $1 \leqslant k \leqslant 8$. If $k = 1$, then $4 \nmid f(n)$, which means that $n = 2p^a$ for some odd prime $p$. But then (2) and $p \geqslant 5$ imply

$$2 < \frac{n}{f(n)} = \frac{2}{1 - 1/p^a} < 3,$$

so that $f(n) \nmid n$. If $k$ is even, then $3|(2^k - 1)|f(n)|n$, a contradiction. Finally, if $k \in \{3, 5, 7\}$, then $n$ has at most 7 odd prime factors, hence

$$\frac{f(n)}{n} \geqslant \frac{7}{8} \prod_{\substack{5 \leqslant p \leqslant 83 \\ p \in \mathscr{P}}} \left(1 - \frac{1}{p}\right) > \frac{1}{2},$$

so $f(n) \nmid n$. Therefore, $f(n)|n$ implies $3|n$.

Proof of Theorem 3. The proof has two parts. The first is a computer calculation of all of the elements of $\mathscr{P}$ which are less than

$$x_0 = 2^{44} \approx 1.76 \times 10^{13},$$

consisting of 39 479 071 primes. This computation took about 120 hours on the first authors' desktop computer. Rather than compute the elements of $\mathscr{P}$ one by one, the algorithm sieved a large interval of integers $(A, B]$ (size about $10^8$), both sieving out the residue classes 0 (mod $p$) for primes $\leqslant \sqrt{B}$, but also sieving the residue classes 1 (mod $p$) for primes $p \in \mathscr{P}$, $p \leqslant B/2$. Stopping the computation at a power of 2 was convenient for the second part of the proof — using the results of the computation to estimate $\mathscr{P}(x)$ for $x > x_0$.

LEMMA 4. *Let* $x_0 = 2^{44}$. *Then*

$$\prod_{\substack{p\in\mathscr{P}\\ p\leqslant x_0}}\left(1-\frac{1}{p}\right) = 0.27923438887\ldots$$

*Furthermore, with* $s = 0.6$ *we have*

$$\mathscr{P}(x) \leqslant \begin{cases} \alpha x^s + 2 & (2^9 \leqslant x \leqslant x_0), \quad \alpha = 0.445836183,\\ \alpha' x^s + 2 & (x \leqslant x_0), \qquad\quad \alpha' = 0.501761301. \end{cases}$$

It appears that $\mathscr{P}(x) \approx x^{5/8}$. Recently K. Ford [4] proved that $\mathscr{P}(x) \ll x^{1-c}$ for some $c > 0$.

Note that if $p \in \mathscr{P}$, then $p - 1 \equiv 1 \pmod 3$. As all prime factors of $p - 1$ are in $\mathscr{P}$ and hence congruent to 2 modulo 3, we have

$$1 \equiv p - 1 \equiv 2^{\Omega(p-1)} \pmod 3,$$

hence $\Omega(p-1)$ is even.

A second computer program was used to generate even numbers which are products of primes in $\mathscr{P}$. Specifically, let

$$\mathscr{N}^- = \left\{n : 2|n, P^+(n) \leqslant x_0, \Omega(n) \text{ odd}, p|n \implies p \in \mathscr{P}\right\} = \{2, 8, 20, 32, 44, \ldots\},$$
$$\mathscr{N}^+ = \left\{n : 2|n, P^+(n) \leqslant x_0, \Omega(n) \text{ even}, p|n \implies p \in \mathscr{P}\right\} = \{4, 10, 16, 22, 34, \ldots\},$$

and, setting $\delta = 1/10$, let

$$h_j^- = \sum_{\substack{n\in\mathscr{N}^-\\ n<2^{j\delta}}} \frac{1}{n^s}.$$

If $n \in \mathscr{N}^\pm$ and the odd part of $n$ is given, then the parity of the exponent of 2 in the prime factorization of $n$ is fixed. Thus,

$$\sum_{\substack{n\in\mathscr{N}^\pm\\ P^+(n)<2^{j\delta}}} \frac{1}{n^s} \leqslant g_j := \frac{2^{-s}}{1-4^{-s}} \prod_{\substack{p\in\mathscr{P}\\ 2<p<2^{j\delta}}} (1-p^{-s})^{-1}. \tag{3}$$

The elements of $\mathscr{N}^-$ were computed exactly up to $2^{36}$. Our next task is to use this data to obtain crude upper bounds on $\mathscr{P}(x)$ in the range $x_0 < x \leqslant 2^{72}$:

LEMMA 5. *Let* $\delta = 1/10$ *and* $s = 0.6$. *For every integer* $j$ *satisfying* $44 < j\delta \leqslant 72$, *we have*

$$\mathscr{P}(x) \leqslant C_j x^s \qquad (2^{(j-1)\delta} < x \leqslant 2^{j\delta}),$$

*where*

$$C_j = \frac{72}{2^{(j-1)\delta s}} + \min_{\max(9, j\delta-44) \leqslant t\delta \leqslant 44} \left[\alpha'\left(g_t - h^-_{\min(t,j-1-t)}\right) + \alpha\left(h^-_{j-t} - h^-_{j-1-44/\delta}\right)\right] + \\ + \sum_{i=1+44/\delta}^{j-1} C_i\left(h^-_{j+1-i} - h^-_{j-i}\right).$$

*Moreover, the sequence* $(C_j)$ *is increasing.*

PROOF. We proceed by induction on $j$. Suppose $j\delta > 44$ and the given bounds have been proved for $x_0 < x \leqslant 2^{(j-1)\delta}$. Let $\max(9, j\delta - 44) \leqslant t\delta \leqslant 44$ and put $y = 2^{t\delta}$. Suppose that $2^{(j-1)\delta} < x \leqslant 2^{j\delta}$. Suppose that $p \in \mathscr{P}$ with $p \leqslant x$, let $q = P^+(p-1)$ and $p - 1 = qn$. Then

$$P^+(n) \leqslant \min\left(q, \frac{x}{q}\right) \leqslant x_0,$$

so $n \in \mathscr{N}^-$. We have (i) $q \leqslant 5$, (ii) $q > 5$ and $n \geqslant x/y$, (iii) $q > 5$ and $x/x_0 \leqslant n < x/y$, or (iv) $q > 5$ and $n < x/x_0$. In case (i), $p - 1$ is a power of two (there are exactly 4 such $p$) or $p - 1 = 2^a 5^b$ with $a \geqslant 1$, $b \geqslant 1$ (there are 68 such primes $p \leqslant 2^{72}$). Now let $\mathscr{P}^*(x) = \mathscr{P}(x) - 2$. Using (3), the number of primes counted in case (ii) is at most

$$\sum_{\substack{n \in \mathscr{N}^- \\ x/y \leqslant n < x \\ P^+(n) \leqslant y}} \mathscr{P}^*\left(\frac{x}{n}\right) \leqslant \alpha' x^s \sum_{\substack{n \in \mathscr{N}^- \\ x/y \leqslant n < x \\ P^+(n) \leqslant y}} \frac{1}{n^s} \leqslant \\ \leqslant \alpha' x^s \left(\sum_{\substack{n \in \mathscr{N}^- \\ P^+(n) \leqslant y}} \frac{1}{n^s} - \sum_{\substack{n \in \mathscr{N}^- \\ n < \min(y, x/y)}} \frac{1}{n^s}\right) \leqslant \alpha' x^s \left(g_t - h^-_{\min(t,j-1-t)}\right).$$

In case (iii), $q \leqslant x_0$, hence the number of such $p$ is bounded above by

$$\sum_{\substack{n\in\mathcal{N}^- \\ x/x_0\leqslant n<x/y}} \mathscr{P}^*\left(\frac{x}{n}\right) \leqslant \alpha x^s \sum_{\substack{n\in\mathcal{N}^- \\ x/x_0\leqslant n<x/y}} \frac{1}{n^s} \leqslant \alpha x^s \bigl(h^-_{j-t} - h^-_{j-1-44/\delta}\bigr).$$

In the final case, we use the induction hypothesis, in particular the supposition that $C_{j-1} > C_{j-2} > \cdots$. Thus, the number of primes counted in case (iv) is at most

$$\sum_{\substack{n\in\mathcal{N}^- \\ n<x/x_0}} \mathscr{P}^*\left(\frac{x}{n}\right) \leqslant \sum_{\substack{n\in\mathcal{N}^- \\ n<x/x_0}} C_i \left(\frac{x}{n}\right)^s, \qquad i = \left\lceil \frac{\log x/n}{\delta \log 2} \right\rceil.$$

Therefore,

$$\sum_{\substack{n\in\mathcal{N}^- \\ n<x/x_0}} \mathscr{P}^*\left(\frac{x}{n}\right) \leqslant \sum_{i=44/\delta+1}^{j-1} x^s C_i \sum_{\substack{n\in\mathcal{N}^- \\ 2^{(j-i)\delta}\leqslant n<2^{(j-i+1)\delta}}} \frac{1}{n^s} \leqslant$$

$$\leqslant x^s \sum_{i=44/\delta+1}^{j-1} C_i \bigl(h^-_{j+1-i} - h^-_{j-i}\bigr).$$

Combining the estimates in cases (i)–(iv) proves the given assertion in the range

$$2^{(j-1)\delta} < x \leqslant 2^{j\delta}.$$

The monotonicity of the sequence $(C_j)$ follows by direct calculation. □

We now develop bounds on $\mathscr{P}(x)$ for $x > 2^{72}$. Let

$$N^- = \sum_{n\in\mathcal{N}^-} \frac{1}{n}, \qquad N^+ = \sum_{n\in\mathcal{N}+} \frac{1}{n}.$$

By direct application of the computed elements of $\mathscr{P}$ which are $\leqslant x_0$, we obtain

$$N^+ + N^- = \frac{1}{2} \prod_{\substack{p\leqslant x_0 \\ p\in\mathscr{P}}} \left(1 - \frac{1}{p}\right)^{-1} = 1.790610\ldots$$

and

$$N^+ - N^- = \sum_{n\in\mathcal{N}^+\cup\mathcal{N}^-} \frac{(-1)^{\Omega(n)}}{n} = -\frac{1}{2}\prod_{\substack{p\leqslant x_0\\ p\in\mathscr{P}}}\left(1+\frac{1}{p}\right)^{-1} = -0.1968977\ldots .$$

Thus,

$$N^- \leqslant 0.993755, \qquad N^+ \leqslant 0.796857. \tag{4}$$

Primarily due to the fact that $N^-$ is so close to 1, our bounds from now on take the shape

$$\mathscr{P}(x) \leqslant K_i x \qquad (2^{i-1} < x \leqslant 2^i). \tag{5}$$

First, using the values of $C_j$ from Lemma 5, we obtain (5) for $45 \leqslant i \leqslant 72$, where

$$K_i = \max_{(i-1)/\delta < j \leqslant i/\delta} C_j (2^{(j-1)\delta})^{s-1}.$$

For convenience, define

$$K_i^* = \max(K_{45}, \ldots, K_i).$$

Lemma 6. *For $i \geqslant 73$, we have* (5), *where*

$$K_i = (2^{i-1})^{s-1} g_{44/\delta} + \frac{1}{x_0} + \frac{K_{i-1}}{2} + \frac{K_{i-3}}{8} + \left(N^- - \frac{5}{8}\right) K_{i-4}^* +$$
$$+ \sum_{2\leqslant k\leqslant (i-2)/44} \frac{(K_{i-44(k-1)}^*)^k}{k!} N_k \big(1 + (i-44k)\log 2\big)^{k-1},$$

*where*

$$N_k = \begin{cases} N^+ & k \text{ even}, \\ N^- & k \text{ odd}. \end{cases}$$

Proof. Again, we use induction on $i$. Suppose that $2^{i-1} < x \leqslant 2^i$. If $p \in \mathscr{P}$, then $p \equiv 2 \pmod 3$. Thus, if $P^+(p-1) \leqslant x_0$ then $p-1 \in \mathcal{N}^+$. Hence, the number of $p \leqslant x$ with $p \in \mathscr{P}$ and $P^+(p-1) \leqslant x_0$ is at most

$$\sum_{\substack{n\leqslant x-1\\ n\in\mathcal{N}^+}} \left(\frac{x}{n}\right)^s \leqslant x^s g_{44/\delta}.$$

The number of $p-1$ divisible by the square of a prime $> x_0$ is trivially at most

$$\sum_{q>x_0} \frac{x}{q^2} \leqslant \frac{x}{x_0}.$$

If $P^+(p-1) > x_0$ and $p-1$ is not divisible by the square of any prime $> x_0$, let $k$ be the number of prime factors of $p-1$ which are $> x_0$. Using the fact that the smallest 3 elements of $\mathscr{N}^-$ are $2, 8, 20$, the number of $p$ with $k=1$ is at most

$$\sum_{\substack{n\in\mathscr{N}^- \\ n<x/x_0}} \mathscr{P}\left(\frac{x}{n}\right) \leqslant \mathscr{P}\left(\frac{x}{2}\right) + \mathscr{P}\left(\frac{x}{8}\right) + \sum_{\substack{n\in\mathscr{N}^- \\ 20\leqslant n\leqslant x/x_0}} \mathscr{P}\left(\frac{x}{n}\right) \leqslant$$
$$\leqslant \frac{x}{2}K_{i-1} + \frac{x}{8}K_{i-3} + \left(N^- - \frac{5}{8}\right)K^*_{i-4}.$$

Now suppose $k \geqslant 2$ and put $\mathscr{N}_k = \mathscr{N}^-$ if $k$ is odd and $\mathscr{N}_k = \mathscr{N}^+$ if $k$ is even. Observe that $i \geqslant 44k$. As there are $k!$ ways to order the prime factors of $p-1$ which are $> x_0$, the number of $p \leqslant x$ corresponding to this value of $k$ is at most

$$\frac{1}{k!} \sum_{\substack{n\in\mathscr{N}_k \\ n<x/x_0^k}} \sum_{\substack{x_0<q_1\leqslant x/(nx_0^{k-1}) \\ q_1\in\mathscr{P}}} \cdots \sum_{\substack{x_0<q_{k-1}\leqslant x/(nx_0^{k-1}) \\ q_{k-1}\in\mathscr{P}}} \mathscr{P}\left(\frac{x}{nq_1\cdots q_{k-1}}\right) \leqslant$$
$$\leqslant \frac{K^*_{i-44(k-1)}}{k!} x \sum_{n\in\mathscr{N}_k} \frac{1}{n}\Bigg( \sum_{\substack{x_0<q\leqslant x/x_0^{k-1} \\ q\in\mathscr{P}}} \frac{1}{q}\Bigg)^{k-1} \leqslant$$
$$\leqslant \frac{K^*_{i-44(k-1)}}{k!} xN_k \left( \frac{\mathscr{P}(x/x_0^{k-1})}{x/x_0^{k-1}} + \int_{x_0}^{x/x_0^{k-1}} \frac{\mathscr{P}(u)}{u^2}\, du \right)^{k-1} \leqslant$$
$$\leqslant \left(K^*_{i-44(k-1)}\right)^k \frac{x}{k!} N_k \left(1 + \log\left(\frac{x}{x_0^k}\right)\right)^{k-1}. \qquad \square$$

Heuristically, the terms in the sum corresponding to $k=1$ dominate the others. These terms total at most $K^*_{i-1}N^- < K^*_{i-1}$, which means that the sequence $(K_i)$ changes very slowly with $i$. In fact, $K_i \leqslant 0.0001407$ for $45 \leqslant i \leqslant 640$. Using computed values of $K_i$ for $i \leqslant 640$, we obtain, with $x_1 = 2^{640}$,

$$\begin{aligned}\prod_{\substack{p\leqslant x_1\\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right) &\geqslant \prod_{\substack{p\leqslant x_0\\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right)\exp\left\{-\sum_{p>x_0}\frac{1}{p^2}-\sum_{\substack{x_0<p\leqslant x_1\\ p\in\mathscr{P}}}\frac{1}{p}\right\}\geqslant\\ &\geqslant \prod_{\substack{p\leqslant x_0\\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right)\exp\left\{-\frac{1}{x_0}-\frac{\mathscr{P}(x_1)}{x_1}+\frac{\mathscr{P}(x_0)}{x_0}-\int_{x_0}^{x_1}\frac{\mathscr{P}(u)}{u^2}\,du\right\}\geqslant\\ &\geqslant \prod_{\substack{p\leqslant x_0\\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right)\exp\left\{\frac{39\,479\,070}{x_0}-K_{640}-\sum_{i=45}^{640}K_i\log 2\right\}\geqslant\\ &\geqslant 0.2693. \qquad (6)\end{aligned}$$

To finish the proof of Theorem 3, take $S = \mathscr{P}$ and $x = x_1 = 2^{640}$ in Lemma 3, and use (6). $\square$

## 4. Proof of Theorem 2

PROPOSITION 2. *Suppose $f(n)|n$ and $n/f(n) \geqslant 5$. Then $\omega(n) \geqslant 46$ and $2^{45}|n$.*

PROOF. If $2\|n$ and $n > 2$, then $n = 2p^b$ for a prime $p$, so $(p^b - 1)|(2p^b)$ and hence $(p^b - 1)|2$. This implies $p = 3$ and $n = 6$. If $2^2|n$ and $2^6 \nmid n$, then $n$ has at most 6 odd prime factors and

$$\frac{n}{f(n)} \leqslant \frac{4}{3}\prod_{3\leqslant p\leqslant 13}\frac{p}{p-1} < 4.$$

Now assume $2^6|n$. If $\omega(n) \leqslant 45$, then

$$\frac{f(n)}{n} \geqslant \frac{63}{64}\prod_{3\leqslant p\leqslant 200}\left(1-\frac{1}{p}\right) > \frac{1}{5}.$$

Hence, $\omega(n) \geqslant 46$, and thus $n$ has at least 45 odd prime factors. This implies that $2^{45}|f(n)|n$. $\square$

We first prove the following result about primes dividing $n$ to a small power.

THEOREM 4. *If* $f(n)|n$ *and* $Q = \{p|n : p^{40} \nmid n\}$, *then*

$$\prod_{q\in Q}\left(1-\frac{1}{q}\right)^{-1} \leqslant 85.32.$$

PROOF. By Proposition 2, we may assume $2^{45}|n$, so that $2 \notin Q$. Let $t_0$ be the smallest prime that

$$\prod_{\substack{p\leqslant t_0\\ p\in Q}}\left(1-\frac{1}{p}\right)^{-1} \geqslant 16.016e^{\gamma}. \tag{7}$$

If no such $t_0$ exists, then the theorem follows, since $16.016e^{\gamma} < 30$. Next, Lemma 1 implies

$$\frac{1}{32.032e^{\gamma}} \geqslant \prod_{p\leqslant t_0}\left(1-\frac{1}{p}\right) \geqslant \left(1+\frac{1}{\log^2 t_0}\right)^{-1}\frac{e^{-\gamma}}{\log t_0},$$

which implies that $t_0 \geqslant e^{32}$.

Let $S = \{p : p|n\}$ and $S(x) = \#\{p \leqslant x : p \in S\}$. For any prime $q$ with $q^b \| n$, there are at most $b$ primes $p|n$ with $p \equiv 1 \pmod q$. Hence, by Lemma 2, for $x \geqslant t_0$ we have

$$\begin{aligned} S(x) \leqslant S(\sqrt{x}) + \sum_{\substack{q\in Q\\ q\leqslant\sqrt{x}}} \sum_{\substack{p|n\\ p\equiv 1 \pmod q}} 1 + \\ + \#\left\{\sqrt{x} < p \leqslant x : \forall\, q \leqslant \sqrt{x} \text{ with } q\in Q,\ q\nmid(p-1)\right\} \leqslant \\ \leqslant 40S(\sqrt{x}) + \#\left\{\sqrt{x} < p \leqslant x : \forall\, q \leqslant \sqrt{x} \text{ with } q\in Q,\ q\nmid(p-1)\right\} \leqslant \\ \leqslant 20\sqrt{x} + \frac{x}{(1+1/\log x)I(x)}, \end{aligned}$$

where

$$I(x) = \int_1^{\sqrt{x}} H(t)\frac{\log t}{t}\,dt, \qquad H(t) = \prod_{\substack{p\leqslant t\\ p^{40}|n}}\left(1-\frac{1}{p}\right).$$

By Lemma 1 and (7),

$$H(t) \geqslant \prod_{p \leqslant \max(t_0, t)} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \leqslant t_0 \\ p \in Q}} \left(1 - \frac{1}{p}\right)^{-1} \geqslant$$

$$\geqslant \frac{16.016}{\log \max(t, t_0)} \left(1 + \frac{1}{\log^2 t_0}\right)^{-1} \geqslant \frac{16}{\log \max(t, t_0)}.$$

Hence,

$$I(x) \geqslant \begin{cases} \dfrac{2 \log^2 x}{\log t_0} & (x \leqslant t_0^2), \\ 8 \log\left(\dfrac{x}{t_0}\right) & (x > t_0^2). \end{cases}$$

Since

$$\sqrt{x} \leqslant \frac{x}{8000 \log^2 x}$$

for $x \geqslant t_0$, we obtain

$$S(x) \leqslant \begin{cases} \dfrac{x \log t_0}{2 \log^2 x} & (t_0 \leqslant x \leqslant t_0^2), \\ \dfrac{x}{8 \log(x/t_0)} & (x > t_0^2). \end{cases} \tag{8}$$

Note that by (7), $S(t_0) \geqslant 1$. By (8) and partial summation, if

$$t = t_0^{C+1} \geqslant t_0^2$$

then

$$\prod_{\substack{p \in S \\ t_0 < p \leqslant t}} \left(1 - \frac{1}{p}\right) \geqslant \exp\left\{- \sum_{\substack{p \in S \\ t_0 < p \leqslant t}} \frac{1}{p} - \sum_{p > t_0} \frac{1}{p^2}\right\} \geqslant$$

$$\geqslant \exp\left\{-\frac{1}{t_0} + \frac{S(t_0)}{t_0} - \frac{S(t)}{t} - \int_{t_0}^{t} \frac{S(u)}{u^2}\, du\right\} \geqslant$$

$$\geqslant \exp\left\{-\frac{1}{8C \log t_0} - \frac{1}{4} - \frac{1}{8} \log C\right\}.$$

Applying Lemma 3 gives

$$\prod_{p\in S}\left(1-\frac{1}{p}\right) \geqslant \prod_{\substack{p\in S\\ p\leqslant t}}\left(1-\frac{1}{p}\right) - \frac{8}{\log t}$$

$$\geqslant \prod_{\substack{p\in S\\ p\leqslant t_0}}\left(1-\frac{1}{p}\right)\cdot \exp\left\{-\frac{1}{8C\log t_0}-\frac{1}{4}-\frac{1}{8}\log C\right\} - \frac{8}{(C+1)\log t_0}.$$

By Lemma 1, we obtain the bound

$$\prod_{\substack{p\in Q\\ p>t_0}}\left(1-\frac{1}{p}\right) \geqslant \prod_{\substack{p\in S\\ p>t_0}}\left(1-\frac{1}{p}\right)$$

$$\geqslant \exp\left\{-\frac{1}{8C\log t_0}-\frac{1}{4}-\frac{1}{8}\log C\right\} - \frac{8}{(C+1)\log t_0}\prod_{p\leqslant t_0}\left(1-\frac{1}{p}\right)^{-1} \geqslant$$

$$\geqslant \exp\left\{-\frac{1}{8C\log t_0}-\frac{1}{4}-\frac{1}{8}\log C\right\} - \frac{8e^{\gamma}(1+1/\log^2 t_0)}{C+1} \geqslant$$

$$\geqslant \exp\left\{-\frac{1}{256C}-\frac{1}{4}-\frac{1}{8}\log C\right\} - \frac{8e^{\gamma}(1+1/1024)}{C+1}.$$

Taking $C = 296$ produces a lower bound for the above product of 0.33437. Therefore,

$$\prod_{p\in Q}\left(1-\frac{1}{p}\right) \geqslant \frac{1}{16.016e^{\gamma}}\left(1-\frac{1}{t_0}\right)0.33437 \geqslant \frac{1}{85.32}$$

and the proof of Theorem 4 is complete. □

Proof of Theorem 2. By Theorem 4,

$$\frac{n}{f(n)} = \prod_{p^a\|n}\frac{1}{1-p^{-a}} \leqslant \prod_{p\in Q}\frac{1}{1-p^{-1}}\prod_{p}\frac{1}{1-p^{-40}} \leqslant 85.4. \quad \square$$

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