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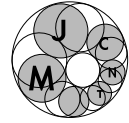


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Counterexamples to Borsuk's conjecture on spheres of small radii

Andrey Kupavskii and Andrey Raigorodskii (Moscow)

Abstract: In this work, the classical Borsuk conjecture is discussed, which states that any set of diameter 1 in the Euclidean space \mathbb{R}^d can be divided into $d + 1$ parts of smaller diameter. During the last two decades, many counterexamples to the conjecture have been proposed in high dimensions. However, all of them are sets of diameter 1 that lie on spheres whose radii are close to the value $1/\sqrt{2}$. The main result of this paper is as follows: *for any $r > 1/2$, there exists a d_0 such that for all $d \geq d_0$, a counterexample to Borsuk's conjecture can be found on a sphere $S_r^{d-1} \subset \mathbb{R}^d$.*

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1. Introduction

This paper is devoted to the classical Borsuk partition problem. In 1933 K. Borsuk (see [5]) posed the following question: *is it true that any set $\Omega \subset \mathbb{R}^d$ having diameter 1 can be divided into some parts $\Omega_1, \dots, \Omega_{d+1}$ whose diameters are strictly smaller than 1?* Here by the *diameter* of a set Ω we mean the quantity

$$\text{diam } \Omega = \sup_{\mathbf{x}, \mathbf{y} \in \Omega} |\mathbf{x} - \mathbf{y}|,$$

where, in turn, $|\mathbf{x} - \mathbf{y}|$ denotes the standard Euclidean distance between vectors.

Let us use some additional notation. So by $f(\Omega)$ we denote the value

$$f(\Omega) = \min \{ f : \Omega = \Omega_1 \cup \dots \cup \Omega_f, \forall i \text{ diam } \Omega_i < \text{diam } \Omega \}.$$

Furthermore,

$$f(d) = \max_{\Omega \subset \mathbb{R}^d, \text{diam } \Omega = 1} f(\Omega),$$

i. e., $f(d)$ is the minimum number of parts of smaller diameter, into which an arbitrary set of diameter 1 in \mathbb{R}^d can be divided. In these terms, Borsuk's question is as follows: *is it true that always $f(d) = d + 1$?* The positive answer on this question is usually called "Borsuk's conjecture".

The history of Borsuk's conjecture is somewhat dramatic. Almost all the specialists in the field of combinatorial geometry strongly believed that the conjecture should be true. Multiple results supporting it have been proved. For example, if Ω has smooth boundary, then one certainly obtains $f(\Omega) \leq d + 1$. This result is due to H. Hadwiger (see [10]), and it seems to show the evidence of the conjecture. However, in 1993 J. Kahn and G. Kalai published a breakthrough paper, where they constructed a *finite* set of points in a very high dimension d that could not be decomposed into $d + 1$ subsets of smaller diameter (see [12]).

Now, Borsuk's conjecture is shown to be true for $d \leq 3$ and false for $d \geq 298$ (see [16]). Also, we know that

$$(1.2255 \dots + o(1))^{\sqrt{d}} \leq f(d) \leq (1.224 \dots + o(1))^d.$$

Here the lower bound was found in 1999 by A. M. Raigorodskii (see [13]) and the upper bound is due to O. Schramm (see [20]).

The colorful history of Borsuk's conjecture is exhibited in numerous books and survey papers. We refer the reader to [4, 7, 9, 14–16].

2. Formulation of the problem and statements of the results

A careful analysis of all known counterexamples to Borsuk's conjecture shows that they are always finite sets of points in \mathbb{R}^d lying on spheres whose radii are close to $1/\sqrt{2}$. This is quite natural, since, by Jung's theorem (see [11]), any set in \mathbb{R}^d

having diameter 1 can be covered by a ball of radius

$$\sqrt{\frac{d}{2d+2}} \sim \frac{1}{\sqrt{2}},$$

and the intuition is that in order to get a counterexample, we have to take a set with as big covering ball as possible. The main result of our work is the following theorem which completely breaks such intuition.

THEOREM 1. *Let $S_r^{d-1} \subset \mathbb{R}^d$ be the sphere of radius r with centre at the origin. For any $r > 1/2$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, one can find a set $\Omega \subset S_r^{d-1}$ which has diameter 1 and does not admit a partition into $d + 1$ parts of smaller diameter.*

Actually, we shall prove an even stronger result. In order to formulate it, let us introduce the quantity

$$f_r(d) = \max_{\Omega \subset S_r^{d-1}, \text{diam } \Omega=1} f(\Omega).$$

In these terms, Theorem 1 says that for any $r > 1/2$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, $f_r(d) > d + 1$. Moreover, one has

THEOREM 2. *For any $r > 1/2$, there exist numbers $k = k(r) \in \mathbb{N}$, $c = c(r) > 1$ and a function $\delta = \delta(d) = o(1)$ such that*

$$f_r(d) \geq (c + \delta)^{\sqrt[k]{d}}.$$

Theorem 2 means that if r is fixed and exceeds $1/2$, then the order of magnitude of the value $f_r(d)$ is at least $e^{g(d)}$, where $g(d)$ is a fixed positive power of d . Thus not only the quantity $f_r(d)$ is greater than $d + 1$ starting from some d_0 , but also it is *substantially* greater than the conjectured value. This fact allows us to use some optimization and to prove eventually the following theorem.

THEOREM 3. *Let $r = r(d) = 1/2 + \varphi(d)$, where*

$$\varphi = o(1) \quad \text{and} \quad \varphi(d) \geq c \frac{\ln \ln d}{\ln d}$$

for all d and a large enough $c > 0$. Then, there exists a d_0 such that for $d \geq d_0$, $f_{r(d)}(d) > d + 1$.

In other words, we can disprove Borsuk's conjecture by constructing sets of diameter 1 that lie on spheres as radii tending to $1/2$ with $d \rightarrow \infty$. Here, in the case of

$$\varphi(d) = \Theta\left(\frac{\ln \ln d}{\ln d}\right),$$

there is already no room to spare in the bound $f_r(d) > d+1$. So using our approach, one only may discuss the value of a constant c in Theorem 3, which is of course not significant.

Now, a natural question arises: perhaps $f_r(d) \leq d+1$, provided

$$r = r(d) = \frac{1}{2} + O\left(\frac{\ln \ln d}{\ln d}\right)?$$

Unfortunately, we only can prove

THEOREM 4. *Let $r = r(d) = 1/2 + \varphi(d)$, where $\varphi = O(1/d)$. Then, $f_r(d) \leq d+1$.*

Thus, we get a gap between the two functions $1/d$ and $\ln(\ln d)/(\ln d)$. To reduce this gap would be of a great interest.

The structure of the rest of our paper will be as follows. In Section 3, we shall prove both Theorem 1 and Theorem 2. Section 4 will be devoted to the proof of Theorem 3. In Section 5, we shall briefly discuss Theorem 4 which is in fact very simple. In Section 6, we shall show that in some sense, the constructions from Sections 3 and 4 are best possible. In Section 7, we shall say a few words about possible extensions and strengthenings of our main results.

3. Proofs of Theorems 1 and 2

Since Theorem 1 is an immediate consequence of Theorem 2, we just prove the second result.

Let us fix an arbitrary $r > 1/2$. Without loss of generality, we also assume that $r < 1/\sqrt{2}$. Let d be large enough (during the proof, we will see what it means). To make clear our further exposition, we subdivide it into four parts. So in Subsection 3.1, we construct a d' -dimensional set Ω' with $d' < d$; in Subsection 3.2, we show that Ω' can be transformed into some $\Omega \subset S_r^{d-1}$; in Subsection 3.3, we

prove that

$$f_r(d) \geq f(\Omega) \geq f(\Omega') \geq (c + \delta)^{\sqrt[2k]{d}}$$

with some appropriate $c > 1$ and $\delta = o(1)$; in Subsection 3.4, we prove a key lemma which is formulated in Subsection 3.3.

3.1. Construction

Take

$$k = \min \left\{ k' \in \mathbb{N} : r^2 > \frac{2k' + 1}{8k'} \right\}.$$

This value is correctly defined, since

$$r > \frac{1}{2} \quad \text{and} \quad \frac{2k' + 1}{8k'} \rightarrow \frac{1}{4} \quad \text{as} \quad k' \rightarrow \infty.$$

Put $n = \max\{m : m \equiv 0 \pmod{4}, m^{2k} < d\}$. Clearly

$$\sqrt[2k]{d} - 5 \leq n < \sqrt[2k]{d}. \tag{1}$$

Consider the function

$$u(a_0) = \frac{1 + 2k \left(\frac{a_0}{2}\right)^{2k-1}}{2 + 4k \left(\frac{a_0}{2}\right)^{2k-1} + (4k - 2) \left(\frac{a_0}{2}\right)^{2k}}. \tag{2}$$

Obviously,

$$u(0) = \frac{1}{2}, \quad u(2) = \frac{2k + 1}{8k} < r^2 < \frac{1}{2},$$

and u is monotone decreasing on the interval $(0, 2)$. Therefore, the equation

$$\frac{1 + 2k \left(\frac{a_0}{2}\right)^{2k-1}}{2 + 4k \left(\frac{a_0}{2}\right)^{2k-1} + (4k - 2) \left(\frac{a_0}{2}\right)^{2k}} = r^2$$

has a unique solution $a_0 \in (0, 2)$.

Take

$$a = \min \left\{ a' : a' \geq \frac{a_0 n}{2}, \frac{a'}{4} + \frac{n}{4} \text{ is a prime number} \right\}.$$

It is known (see [3]) that between x and $x + O(x^{0.525})$, there is certainly a prime number. Thus, the quantity

$$p = \frac{a}{4} + \frac{n}{4} \tag{3}$$

is asymptotically equal to $p_0 n$, where

$$p_0 = \frac{a_0}{8} + \frac{1}{4} \in \left(\frac{1}{4}, \frac{1}{2} \right).$$

Consider the set

$$\Sigma = \left\{ \mathbf{x} = (x_1, \dots, x_n) : \forall i \ x_i \in \{-1, 1\}, x_1 = 1, x_1 + \dots + x_n = 0 \right\}.$$

Let $\mathcal{A} = \{a_1, \dots, a_w\}$ be the set of all possible $2k$ -character words over the alphabet $X = \{1, \dots, n\}$, $w = n^{2k}$. Assume that $a_i = \{\nu(i, 1), \dots, \nu(i, 2k)\}$, $i = 1, \dots, w$. Fix an $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma$. Consider

$$\mathbf{x}^{*2k} = \left(x_{\nu(1,1)} \cdot \dots \cdot x_{\nu(1,2k)}, \dots, x_{\nu(w,1)} \cdot \dots \cdot x_{\nu(w,2k)}, \sqrt{2ka^{2k-1}}x_1, \dots, \sqrt{2ka^{2k-1}}x_n \right).$$

Clearly the number of coordinates in any vector \mathbf{x}^{*2k} equals $w + n$. Put

$$\Omega' = \{ \mathbf{x}^{*2k} : \mathbf{x} \in \Sigma \}.$$

One can readily see that Ω' lies in $\mathbb{R}^{d'}$ with $d' = w < d$. The point is that for any i , $x_i = x_1^{2k-1} \cdot x_i$, since $x_1 = 1$. The construction is complete.

3.2. Transforming Ω' into an $\Omega \subset S_r^{d-1}$

First, let us calculate the diameter of Ω' . For the scalar product of any two vectors $\mathbf{x}^{*2k}, \mathbf{y}^{*2k} \in \Omega'$, we have the relation

$$(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}) = \sum_{i=1}^w x_{\nu(i,1)} \cdot \dots \cdot x_{\nu(i,2k)} \cdot y_{\nu(i,1)} \cdot \dots \cdot y_{\nu(i,2k)} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}) =$$

$$\begin{aligned}
&= \sum_{i_1=1}^n \dots \sum_{i_{2k}=1}^n x_{i_1} \cdot \dots \cdot x_{i_{2k}} \cdot y_{i_1} \cdot \dots \cdot y_{i_{2k}} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}) = \\
&= \left(\sum_{i_1=1}^n x_{i_1} y_{i_1} \right) \cdot \dots \cdot \left(\sum_{i_{2k}=1}^n x_{i_{2k}} y_{i_{2k}} \right) + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}) = \\
&= (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Obviously the minimum of the scalar product $(\mathbf{x}^{*2k}, \mathbf{y}^{*2k})$ is attained on those and only those pairs of vectors $\mathbf{x}, \mathbf{y} \in \Sigma$ whose scalar product equals $-a$. Such pairs of vectors exist for large enough values of d . Indeed, by Construction, $a \sim a_0 n/2$ and $a_0 < 2$. So for large n , $-a > -n$. Moreover, $a \equiv 0 \pmod{4}$, and it is easy to see that for every two vectors $\mathbf{x}, \mathbf{y} \in \Sigma$, one necessarily has $(\mathbf{x}, \mathbf{y}) \in (-n, n]$ and $(\mathbf{x}, \mathbf{y}) \equiv 0 \pmod{4}$.

Thus, we get

$$\text{diam}^2 \Omega' = 2(\mathbf{x}^{*2k}, \mathbf{x}^{*2k}) - 2(a^{2k} - 2ka^{2k}) = 2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}.$$

At the same time, Ω' lies on the sphere $S_\rho^{d'-1}$, where

$$\rho^2 = (\mathbf{x}^{*2k}, \mathbf{x}^{*2k}) = n^{2k} + 2ka^{2k-1}n.$$

Compressing Ω' so that a new set Ω'' has diameter 1, we see that $\Omega'' \subset S_{r'}^{d'-1}$ with

$$(r')^2 = \frac{n^{2k} + 2ka^{2k-1}n}{2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}}.$$

Since, again by Construction, $a \geq a_0 n/2$ and u defined by (2) is monotone decreasing, we get the inequalities

$$\begin{aligned}
(r')^2 &= \frac{n^{2k} + 2ka^{2k-1}n}{2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}} \leq \\
&\leq \frac{n^{2k} + 2k \left(\frac{a_0 n}{2} \right)^{2k-1} n}{2n^{2k} + 4k \left(\frac{a_0 n}{2} \right)^{2k-1} n + (4k-2) \left(\frac{a_0 n}{2} \right)^{2k}} = u(a_0) = r^2.
\end{aligned}$$

Now, it remains to interpret $S_{r'}^{d'-1}$ as an intersection of the sphere S_r^{d-1} and a plane of dimension d' . This can be done, since $d' < d$ and $r' \leq r$. Let Ω be an image of Ω'' under such interpretation.

3.3. Lower bound for $f(\Omega)$

It is clear that

$$f_r(d) \geq f(\Omega) = f(\Omega'') = f(\Omega').$$

So it remains to show that $f(\Omega') \geq (c + \delta)^{\sqrt[2k]{d}}$ for appropriate c and δ . First, assume that

$$f(\Omega') < f = \frac{|\Omega'|}{\sum_{i=0}^{p-1} C_n^i}.$$

Then, Ω' can be represented as

$$\Omega' = \Omega'_1 \cup \dots \cup \Omega'_f \quad \forall i \quad \text{diam } \Omega'_i < \text{diam } \Omega'. \tag{4}$$

Obviously the correspondence $\mathbf{x} \rightarrow \mathbf{x}^{*2k}$ is a bijection between Σ and Ω' . So

$$|\Omega'| = |\Sigma| = C_{n-1}^{n/2-1}$$

and, moreover, the partition (4) induces a partition

$$\Sigma = \Sigma_1 \cup \dots \cup \Sigma_f.$$

By the choice of f and by pigeon-hole principle, there exists a part Σ_i with

$$|\Sigma_i| > \sum_{i=0}^{p-1} C_n^i.$$

In the next subsection, we shall prove the following lemma.

LEMMA 1. *If $Q \subset \Sigma$ is such that*

$$|Q| > \sum_{i=0}^{p-1} C_n^i,$$

then there exist $\mathbf{x}, \mathbf{y} \in Q$ with $(\mathbf{x}, \mathbf{y}) = -a$.

Of course Lemma 1 is applied only for $n \gg 1$. By this lemma, Σ_i contains two different vectors \mathbf{x}, \mathbf{y} with scalar product equal to $-a$. It means (cf. Subsection 3.2) that

$$\text{diam } \Omega'_i = |\mathbf{x}^{*2k} - \mathbf{y}^{*2k}| = \text{diam } \Omega',$$

which contradicts the properties of partition (4).

Thus, we have shown that

$$f(\Omega') \geq \frac{|\Omega'|}{p-1} = \frac{C_{n-1}^{n/2-1}}{\sum_{i=0}^{p-1} C_n^i}.$$

Let us recall that

$$p \sim p_0 n, \quad p_0 \in \left(\frac{1}{4}, \frac{1}{2} \right).$$

In this case, standard analytical tools like Stirling's formula and Chernoff's inequality entail the following asymptotic relations:

$$C_{n-1}^{n/2-1} = (2 + o(1))^n, \quad \sum_{i=0}^{p-1} C_n^i = (c' + o(1))^n, \quad c' < 2.$$

Taking the necessary ratio, we get

$$f(\Omega') \geq (c + \delta')^n, \quad c > 1, \quad \delta' = o(1).$$

Finally, by inequalities (1) we have

$$f_r(d) \geq f(\Omega') \geq (c + \delta')^n \geq (c + \delta')^{2k\sqrt{d}-5} = (c + \delta)^{2k\sqrt{d}}, \quad \delta = o(1).$$

Theorem 2 is proved.

3.4. Proof of Lemma 1

We shall use the now classical linear algebra method in combinatorics (see [1, 2, 17]). Consider an arbitrary $Q = \{\mathbf{x}_1, \dots, \mathbf{x}_s\} \subset \Sigma$ such that for every two different i, j , one has $(\mathbf{x}_i, \mathbf{x}_j) \neq -a$. We need to show that

$$s \leq \sum_{i=0}^{p-1} C_n^i.$$

To each vector $\mathbf{x} \in \Sigma$ we assign a polynomial $P_{\mathbf{x}} \in \mathbb{Z}/p\mathbb{Z}[y_1, \dots, y_n]$. Namely,

$$P_{\mathbf{x}}(\mathbf{y}) = \prod_{i=0, i \not\equiv -a \pmod{p}}^{p-1} (i - (\mathbf{x}, \mathbf{y})), \quad \mathbf{y} = (y_1, \dots, y_n).$$

Here the product is taken over all residues i modulo p except for $i \equiv -a \pmod{p}$. So the degree of any $P_{\mathbf{x}}$ does not exceed $p - 1$. The most important property of such polynomials is as follows.

PROPERTY. *For every $\mathbf{x}, \mathbf{y} \in \Sigma$, the congruence $(\mathbf{x}, \mathbf{y}) \equiv -a \pmod{p}$ is equivalent to the congruence $P_{\mathbf{x}}(\mathbf{y}) \not\equiv 0 \pmod{p}$.*

Property is evident, and we shall just use it. However, before doing so, we transform any $P_{\mathbf{x}}$ into some $P'_{\mathbf{x}}$. More precisely, we represent every polynomial $P_{\mathbf{x}}$ as a linear combination of monomials. Of course each monomial has the form

$$y_{i_1}^{\alpha_1} \cdot \dots \cdot y_{i_q}^{\alpha_q}, \quad q \leq p - 1.$$

If α_ν is even, we remove y_{i_ν} from the monomial. If it is odd, we replace it by 1. Clearly the new monomial is a product of pairwise different variables. However, the new polynomials $P'_{\mathbf{x}}$, $\mathbf{x} \in \Sigma$, also satisfy Property. The point is that in Property, only variables whose values are ± 1 are considered.

By the just-given construction,

$$\dim (\text{linear span } \{P'_{\mathbf{x}}\}_{\mathbf{x} \in \Sigma}) \leq \sum_{i=0}^{p-1} C_n^i.$$

If we succeed in showing that the vectors $\mathbf{x}_1, \dots, \mathbf{x}_s$ from the set Q correspond to the set $P'_{\mathbf{x}_1}, \dots, P'_{\mathbf{x}_s}$ of linearly independent polynomials (over the field $\mathbb{Z}/p\mathbb{Z}$), then Lemma 1 is proved.

So let us assume that

$$c_1 P'_{\mathbf{x}_1}(\mathbf{y}) + \dots + c_s P'_{\mathbf{x}_s}(\mathbf{y}) \equiv 0 \pmod{p} \quad \forall \mathbf{y} \in \Sigma. \quad (5)$$

Take $\mathbf{y} = \mathbf{x}_i$ with an arbitrary i . On the one hand, $(\mathbf{x}_i, \mathbf{x}_i) = n$. By Construction, $n - 4p = -a$ (see (3)). Therefore,

$$(\mathbf{x}_i, \mathbf{x}_i) \equiv -a \pmod{p},$$

i. e., by Property, $P'_{x_i}(x_i) \not\equiv 0 \pmod p$. On the other hand, if $j \neq i$, then

$$(x_i, x_j) < n \quad \text{and} \quad (x_i, x_j) \neq -a.$$

Moreover, since

$$a \equiv 0 \pmod 4, \quad n \equiv 0 \pmod 4, \quad (x, y) \equiv 0 \pmod 4 \quad \forall x, y \in \Sigma,$$

we see that

$$(x_i, x_j) \notin \{n - p, n - 2p, n - 3p, n - 5p, n - 6p, n - 7p\}.$$

Finally, $-a < 0$, and so $n - 8p < -n$, which means that $(x_i, x_j) \not\equiv a \pmod p$ and that, by Property,

$$P'_{x_j}(x_i) \equiv 0 \pmod p.$$

By relation (5), we get $c_i \equiv 0 \pmod p$ (here the primality of p is essential). Since i was arbitrary, we obtain the linear independence of our polynomials, and Lemma is proved.

4. Proof of Theorem 3

Take $r = 1/2 + \varphi$, where

$$\varphi(d) = \frac{6 \ln \ln d}{\ln d}.$$

We shall prove that for large enough d , $f_r(d) > d + 2$. Then, the assertion of Theorem 3 will follow, since for any

$$r' > r, \quad S_{r'}^d \supset S_r^{d-1}.$$

Of course, further exposition will be very close to the one in Section 3. However, there will be some technical subtleties. First of all, we rewrite Subsection 3.1.

Take $k = \lceil 1/\varphi \rceil$. Then, for large values of d ,

$$\frac{2k + 1}{8k} = \frac{1}{4} + \frac{1}{8k} \leq \frac{1}{4} + \frac{\varphi}{8} < \frac{1}{4} + \varphi + \varphi^2 = r^2.$$

In other similar inequalities below, we shall not write “for large values of d ” anymore; we shall just assume that d is big enough. As in Subsection 3.1, put

$$n = \max\{m : m \equiv 0 \pmod{4}, m^{2k} < d\}.$$

Now, inequality (1) means that

$$n \sim d^{\varphi/2} = \exp\left\{(\ln d) \cdot \frac{3 \ln \ln d}{\ln d}\right\} = \ln^3 d.$$

Let $a_0 = 2 - \varphi/2$. We want to show that $u(a_0) < r^2$, provided u is the same function as in (2). A standard computation is below. First,

$$\begin{aligned} u(a_0) &= \frac{1 + 2k \left(\frac{a_0}{2}\right)^{2k-1}}{2 + 4k \left(\frac{a_0}{2}\right)^{2k-1} + (4k-2) \left(\frac{a_0}{2}\right)^{2k}} < \\ &< \frac{2k \left(\frac{a_0}{2}\right)^{2k-1}}{(8k-2) \left(\frac{a_0}{2}\right)^{2k}} \cdot \frac{1 + \frac{1}{2k(a_0/2)^{2k-1}}}{1 + \frac{2}{(8k-2)(a_0/2)^{2k}}}. \end{aligned}$$

Further,

$$\frac{2k}{8k-2} = \frac{1}{4} \cdot \frac{1}{1 - \frac{1}{4k}}.$$

Since

$$\frac{1}{1-x} \leq 1 + 2x,$$

we have

$$\frac{2k}{8k-2} \leq \frac{1}{4} \cdot \left(1 + \frac{\varphi}{2}\right).$$

Now,

$$\frac{(a_0/2)^{2k-1}}{(a_0/2)^{2k}} = \frac{a_0}{2} = 1 - \frac{\varphi}{4}.$$

Finally,

$$\frac{1 + \frac{1}{2k(a_0/2)^{2k-1}}}{1 + \frac{1}{(8k-2)(a_0/2)^{2k}}} \leq \left(1 + \frac{1}{2k(a_0/2)^{2k-1}}\right) \cdot \left(1 - \frac{1}{(8k-2)(a_0/2)^{2k}}\right). \quad (6)$$

Clearly

$$\left(\frac{a_0}{2}\right)^{2k} \sim \left(\frac{a_0}{2}\right)^{2k-1} \sim \left(1 - \frac{\varphi}{4}\right)^{2/\varphi} \sim e^{-1/2}.$$

Consequently, the right-hand side of (6) can be bounded from above by

$$\left(1 + \frac{1.1\varphi}{2e^{-1/2}}\right) \cdot \left(1 - \frac{0.9\varphi}{8e^{-1/2}}\right) \leq 1 + \frac{3.6\varphi}{8e^{-1/2}}.$$

Thus,

$$\begin{aligned} u(a_0) &\leq \frac{1}{4} \cdot \left(1 + \frac{\varphi}{2}\right) \cdot \left(1 - \frac{\varphi}{4}\right) \cdot \left(1 + \frac{3.6\varphi}{8e^{-1/2}}\right) \leq \\ &\leq \frac{1}{4} \cdot \left(1 + \frac{\varphi}{4}\right) \cdot \left(1 + \frac{3.6\varphi}{8e^{-1/2}}\right) \leq \frac{1}{4} + \frac{\varphi}{4} < \frac{1}{4} + \varphi + \varphi^2 = r^2. \end{aligned}$$

Again, as in Construction, we take

$$a = \min \left\{ a' : a' \geq \frac{a_0 n}{2}, \frac{a'}{4} + \frac{n}{4} \text{ is a prime number} \right\}.$$

By the already cited results from [3], $a < n$ and

$$p = \frac{a}{4} + \frac{n}{4} \leq \frac{n}{2} - \frac{\varphi n}{16} + O(n^{0.525}).$$

On the one hand,

$$\varphi n = \Theta((\ln \ln d)(\ln d)^2).$$

On the other hand,

$$n^{0.525} = (\ln d)^{3 \cdot 0.525} = (\ln d)^{1.575} = o((\ln \ln d)(\ln d)^2).$$

Hence,

$$p \leq \frac{n}{2} - \frac{\varphi n}{20}.$$

The sets Σ and Ω' are just the same as in Subsection 3.1. The only change in Subsection 3.2 is in replacing the last equality by the inequality in the estimate of $(r')^2$ by r^2 .

Reproducing Subsection 3.3 word by word, we obtain the bound

$$f_r(d) \geq \frac{C_{n-1}^{n/2-1}}{\sum_{i=0}^{p-1} C_n^i} = \frac{\frac{1}{2} C_n^{n/2}}{\sum_{i=0}^{p-1} C_n^i}.$$

It remains to establish the estimate

$$\frac{\frac{1}{2} C_n^{n/2}}{\sum_{i=0}^{p-1} C_n^i} > d + 2.$$

Clearly

$$\sum_{i=0}^{p-1} C_n^i < n C_n^p \leq n C_n^{n/2-x}, \quad x = \left\lfloor \frac{\varphi n}{20} \right\rfloor.$$

We have

$$\begin{aligned} \frac{C_n^{n/2}}{C_n^{n/2-x}} &= \frac{\left(\frac{n}{2}+1\right) \cdots \left(\frac{n}{2}+x\right)}{\left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}-1\right) \cdots \left(\frac{n}{2}-x+1\right)} = \frac{\left(1+\frac{2}{n}\right) \cdots \left(1+\frac{2x}{n}\right)}{\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{2(x-1)}{n}\right)} = \\ &= \exp \left\{ \frac{x(x+1)}{2} - \frac{x(x-1)}{2} + O\left(\frac{x^3}{n^2}\right) \right\} = \exp \left\{ \frac{2x^2}{n} + O\left(\frac{x^3}{n^2}\right) \right\} = \\ &= \exp \left\{ (1+o(1)) \frac{\varphi^2 n}{200} \right\} = \exp \left\{ \Theta \left(\frac{(\ln \ln d)^2}{(\ln d)^2} \cdot (\ln d)^3 \right) \right\} = \\ &= \exp \left\{ \Theta \left((\ln \ln d)^2 \cdot (\ln d) \right) \right\}. \end{aligned}$$

Thus,

$$\frac{\frac{1}{2}C_n^{n/2}}{\sum_{i=0}^{p-1} C_n^i} = d^{\Theta((\ln \ln d)^2)} > d + 2,$$

and the proof is complete.

5. Proof of Theorem 4

Take $S = S_r^{d-1}$ with an arbitrary r . Inscribe into S a d -dimensional regular simplex Δ . Consider its faces $\Delta_1, \dots, \Delta_{d+1}$. If O is the origin, then denote by S_i the set $\text{conv}\{\Delta_i, O\} \cap S$. Of course

$$S = S_1 \cup \dots \cup S_{d+1}.$$

Moreover, it is well-known that (see [4])

$$\text{diam } S_i = 2r \left(1 - \Theta \left(\frac{1}{d} \right) \right).$$

If

$$r = r(d) \leq \frac{1}{2} + \frac{c}{d}$$

with an appropriate $c > 0$, then we get $\text{diam } S_i < 1$, and we are done.

In principle, it's possible to obtain rather good upper bounds for $f_r(d)$ in the general case. The simplest way to do that is to use old results of C. A. Rogers (see [19]): *any sphere of radius $r > 1/2$ in \mathbb{R}^d can be covered by $(2r + o(1))^d$ spherical caps of diameter 1*. This result gives already the estimate

$$f_r(d) \leq (2r + o(1))^d$$

when r is fixed. More subtle estimates can be discovered by using an approach which appeared in the paper [6]. For example, Rogers tells us that for $r > \sqrt{3/8}$, we can only show that

$$f_r(d) \leq (c + o(1))^d$$

with $c > \sqrt{3/2}$. However, J. Bourgain and J. Lindenstrauss, the authors of [6], provide us with a universal bound

$$f_r(d) \leq \left(\sqrt{\frac{3}{2}} + o(1) \right)^d.$$

Their ideas might be carefully applied even in some cases when $r \rightarrow 1/2$, but, in this paper, we do not focus on that kind of results.

6. Improving Construction from Subsection 3.1 is hard

Let us discuss some key properties of Construction from § 3.1 and a possibility of improving it. Indeed, one of the most important steps in Construction was in assigning, to each n -dimensional vector $\mathbf{x} \in \Sigma$, a d' -dimensional vector \mathbf{x}^{*2k} , so that eventually we got a set $\Omega' \subset \mathbb{R}^{d'}$. The correspondence between vectors \mathbf{x} from Σ and their images \mathbf{x}^{*2k} was organized in such a way that

$$(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}) = (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).$$

In other words, the scalar product of any two elements of the set Ω' was a polynomial depending on the scalar product of the preimages of those elements. Denote this polynomial by h_a^* . Thus,

$$h_a^*(t) = t^{2k} + 2ka^{2k-1}t.$$

In order to prove Theorems 1–3, we essentially used the following properties of h_a^* (see § 3.2): first of all, h_a^* , considered as function on the interval $[-n, n]$, attains its minimum at the point $-a \in (-n, 0)$ and so

$$\left. \frac{dh_a^*}{dt} \right|_{t=-a} = 0;$$

second, the quantity

$$\frac{h_a^*(n)}{2h_a^*(n) - 2h_a^*(-a)} = \frac{n^{2k} + 2ka^{2k-1}n}{2n^{2k} + 4ka^{2k-1}n + (4k - 2)a^{2k}}$$

is a close approximation to the radius of a sphere, on which a counterexample to Borsuk's conjecture lies. Therefore, the results of Theorems 1–3 would be improved, provided we could replace h_a^* by another polynomial h_a having the same first property and a smaller value of the expression

$$\frac{h_a(n)}{2h_a(n) - 2h_a(-a)}.$$

Moreover, we may also vary the value $a \in (0, n)$ by taking $a \sim a'n$ with $a' \in (0, 1)$ as close to 1 as necessary.

Let Δ_a^m be the set of all polynomials

$$h_a(t) = u_m t^m + \dots + u_1 t + u_0$$

of degree m with non-negative coefficients and having the property that h_a , considered as function on the interval $[-n, n]$, attains its minimum at point $-a \in [-n, 0)$ and

$$\left. \frac{dh_a}{dt} \right|_{t=-a} = 0$$

(the last condition is non-trivial only for $a = n$).

PROPOSITION 1. *If m is even, then*

$$\min_{a \in (0, n]} \min_{h_a \in \Delta_a^m} \frac{h_a(n)}{2h_a(n) - 2h_a(-a)} = \min_{h \in \Delta_n^m} \frac{h(n)}{2h(n) - 2h(-n)} = \frac{h^*(n)}{2h^*(n) - 2h^*(-n)},$$

where $h^*(t) = t^m + mn^{m-1}t$. *If m is odd, then*

$$\min_{h \in \Delta_n^m} \frac{h(n)}{2h(n) - 2h(-n)} \geq \min_{h \in \Delta_n^{m-1}} \frac{h(n)}{2h(n) - 2h(-n)}.$$

Before proving Proposition, let us briefly comment on it. Actually, Proposition tells us that, in § 3.1 and § 3.2, everything was done in an optimum way: it's better to take polynomials of even degree, and, among them, h_a^* is asymptotically best possible (when $a \rightarrow n$, $h_a^* \rightarrow t^{2k} + 2kn^{2k-1}t$). It is worth noting that, in principle, for any $h \in \Delta_a^m$, one can transform Σ into such Ω' that the scalar product of any two vectors from Ω' is equal to the value of h at the scalar product of the preimages of those two vectors (here it is important to assume that any $h \in \Delta_a^m$ has

non-negative coefficients). However, according to Proposition, this fact is already not quite useful for our purposes.

PROOF OF PROPOSITION 1. First, we note that the quantity $h_a(-a)$ should be non-positive in order to minimize the ratio

$$\frac{h_a(n)}{2h_a(n) - 2h_a(-a)} = \frac{1}{2 - \frac{2h_a(-a)}{h_a(n)}}, \quad (7)$$

since $h_a(n) > 0$ for any h_a . Polynomials $h_a \in \Delta_a^m$ such that $h_a(-a) \leq 0$ do really exist, so we may assume that $h_a(-a) \leq 0$. Under this assumption, our minimization is equivalent to the maximization of the expression

$$\frac{|h_a(-a)|}{h_a(n)}$$

over the set $\tilde{\Delta}_a^m \subset \Delta_a^m$ containing those and only those polynomials h_a , for which $h_a(-a) \leq 0$.

Furthermore, we may suppose that $u_0 = 0$, since, for $u_0 > 0$, ratio (7) is definitely greater.

Now, let us prove that

$$\min_{a \in (0, n]} \min_{h_a \in \tilde{\Delta}_a^m} \frac{h_a(n)}{2h_a(n) - 2h_a(-a)} = \min_{h \in \tilde{\Delta}_n^m} \frac{h(n)}{2h(n) - 2h(-n)} \quad (8)$$

or, which is the same, that

$$\max_{a \in (0, n]} \max_{h_a \in \tilde{\Delta}_a^m} \frac{|h_a(-a)|}{h_a(n)} = \max_{h \in \tilde{\Delta}_n^m} \frac{|h(-n)|}{h(n)}. \quad (9)$$

Indeed, all the coefficients of h_a are non-negative, and so $h_a(a) \leq h_a(n)$. Therefore, for any $a \in (0, n]$, we have

$$\max_{h_a \in \tilde{\Delta}_a^m} \frac{|h_a(-a)|}{h_a(n)} \leq \max_{h_a \in \tilde{\Delta}_a^m} \frac{|h_a(-a)|}{h_a(a)} = \max_{h \in \tilde{\Delta}_n^m} \frac{|h(-n)|}{h(n)},$$

which completes the proof of (8) and (9).

It remains to show that

$$\max_{h \in \tilde{\Delta}_n^m} \frac{|h(-n)|}{h(n)} \leq \frac{|h^*(-n)|}{h^*(n)} = \frac{m-1}{m+1}.$$

Take an arbitrary polynomial $h \in \tilde{\Delta}_n^m$. We represent it in the form

$$h = h_1 + h_2 = \sum_i c_{\alpha_i} t^{\alpha_i} + \sum_i c_{\beta_i} t^{\beta_i}, \quad c_{\alpha_i} > 0, \quad c_{\beta_i} > 0,$$

where, in h_1 , only odd degrees of t are taken, and, in h_2 , only even degrees are present. In this notation, we have

$$\frac{|h(-n)|}{h(n)} = \frac{\left| \sum_i c_{\alpha_i} n^{\alpha_i} - \sum_i c_{\beta_i} n^{\beta_i} \right|}{\sum_i c_{\alpha_i} n^{\alpha_i} + \sum_i c_{\beta_i} n^{\beta_i}}.$$

Hence, we are led to check the inequality

$$\sum_i c_{\alpha_i} n^{\alpha_i} \leq m \sum_i c_{\beta_i} n^{\beta_i}. \quad (10)$$

However, we know that $h'(-n) = 0$, i. e.,

$$\sum_i \alpha_i c_{\alpha_i} n^{\alpha_i-1} = \sum_i \beta_i c_{\beta_i} n^{\beta_i-1}.$$

Then, we get the following series of inequalities:

$$\sum_i c_{\alpha_i} n^{\alpha_i} \leq n \sum_i \alpha_i c_{\alpha_i} n^{\alpha_i-1} = n \sum_i \beta_i c_{\beta_i} n^{\beta_i-1} \leq m \sum_i c_{\beta_i} n^{\beta_i}.$$

Thus, (10) is true, and the proof of the first part of Proposition is complete.

The second part would follow from the inequality

$$\max_{h \in \tilde{\Delta}_n^m} \frac{|h(-n)|}{h(n)} \leq \frac{m-2}{m},$$

which is tantamount to

$$\sum_i c_{\alpha_i} n^{\alpha_i} \leq (m-1) \sum_i c_{\beta_i} n^{\beta_i}. \quad (11)$$

Eventually, (11) is provided by the following series of estimates:

$$\sum_i c_{\alpha_i} n^{\alpha_i} \leq n \sum_i \alpha_i c_{\alpha_i} n^{\alpha_i-1} = n \sum_i \beta_i c_{\beta_i} n^{\beta_i-1} \leq (m-1) \sum_i c_{\beta_i} n^{\beta_i}.$$

Here we get the factor $m-1$ instead of m , since m is odd and so $\beta_i \leq m-1$ for any i .

Proposition 1 is proved. \square

7. Improving Construction from Subsection 3.1 is still possible

The only place in Construction, which may be further improved, is in the form of the set Σ . In principle, we could take an arbitrary subset Σ of the integer lattice \mathbb{Z}^n . In order to be able to apply the whole machinery we used to prove our results we must assume that for any $\mathbf{x} \in \Sigma$, the value of (\mathbf{x}, \mathbf{x}) is the same. For example,

$$\Sigma = \left\{ \mathbf{x} = (x_1, \dots, x_n) : \forall i \ x_i \in \{-1, 0, 1\}, x_1^2 + \dots + x_n^2 = k \right\}$$

with some k could be of help. In the papers [18], [8] and in the book [17], an analogous approach was developed in order to get optimal bounds for the chromatic numbers of spaces and for the standard Borsuk number. However, the difference between the results obtained from $(-1, 1)$ -constructions and the more general ones turned out to appear only in the expressions of the form $c + \delta$ (cf. Theorem 2). Although sometimes this is important as well, we do not think it is of interest to thoroughly investigate here the corresponding bounds.

Another possible refinement is in calculating more carefully the dimension of the set Ω' . Actually, it is not n^{2k} ; it does not exceed

$$C_n^{2k} \leq \frac{n^{2k}}{(2k)!}.$$

For fixed values of $r > 1/2$, this is very important, but for sequences $r_d \sim 1/2$, this fact does not give anything.

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