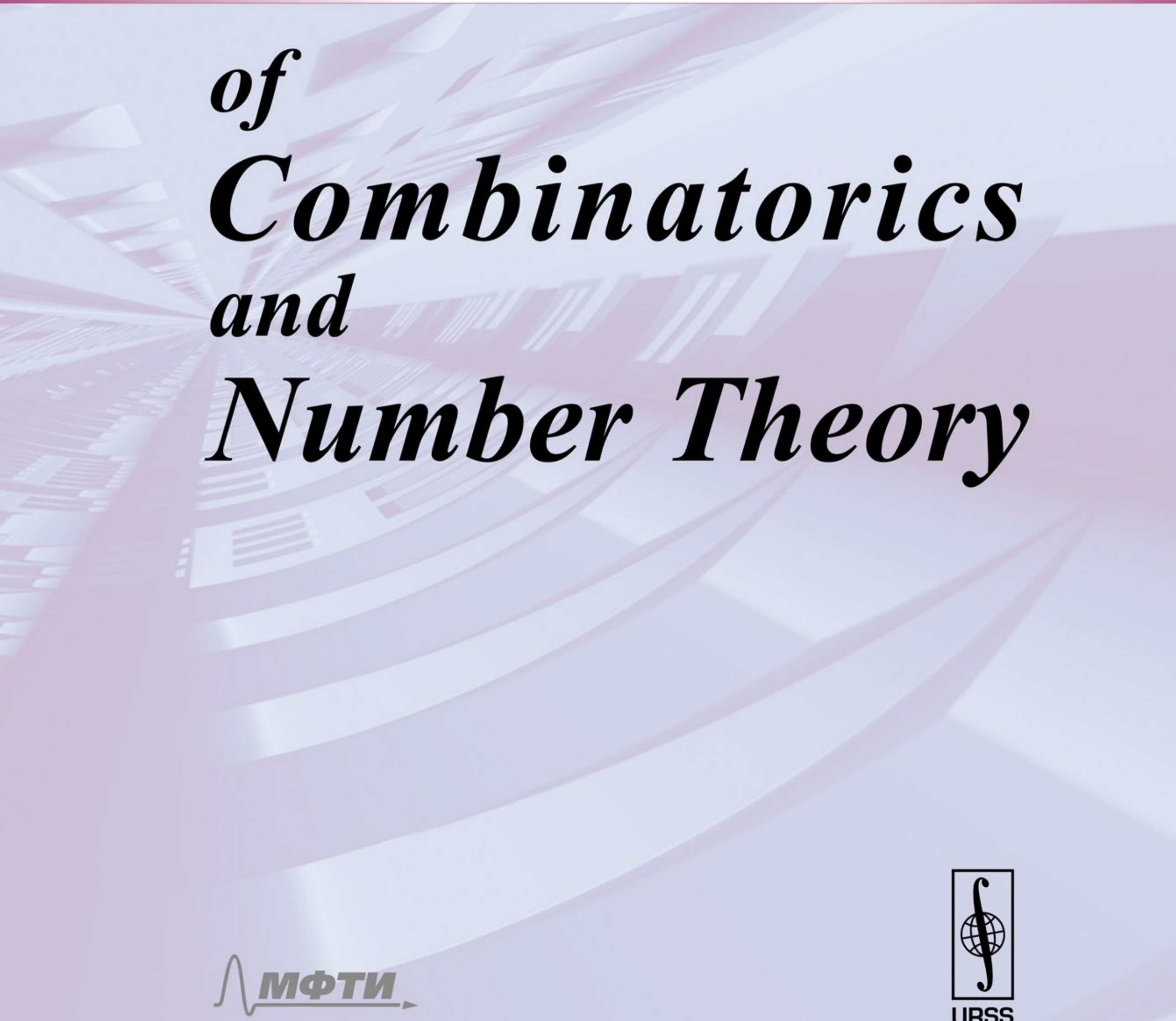
Reprint from

ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory



МФТИ

URSS

Volume 2 • Issue 2

2012

Moscow Journal of Combinatorics and Number Theory. 2012. Vol. 2. Iss. 2. 112 p.

The journal was founded in 2010.

Published by the Moscow Institute of Physics and Technology
with the support of Yandex and Microsoft.

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

Website of our journal
`http://mjcnt.phystech.edu`

E-mail
`mjcnt@phystech.edu`

Address of the Editorial Board
Moscow institute of physics
and technology (state university)
Faculty of Innovations
and High Technology,
Laboratorny Korpus, k. 209,
9, Institutskii pereulok,
Dolgoprudny,
Moscow Region,
Russia,
141700

Адрес редакции
Московский физико-технический
институт (государственный университет)
Факультет инноваций
и высоких технологий
Лабораторный корпус, к. 209,
Институтский переулок, д. 9,
г. Долгопрудный,
Московская область,
Российская Федерация,
141700

URSS Publishers
56, Nakhimovsky Prospekt,
Moscow,
Russia,
117335

Издательство «УРСС»
Нахимовский пр-т, 56
Москва,
Российская Федерация,
117335

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77–41900.

Формат 70 × 100/16. Печ. л. 7. Зак. № ВП-09.
Отпечатано в ООО «ЛЕНАНД».
117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.

ISSN 2220–5438

13031 ID 166877
9 785453 000364

Moscow Journal
of Combinatorics
and Number Theory
2012, vol. 2, iss. 2,
pp. 3–21, [pp. 91–109]

Arithmetical Results on Certain q-Series, IV

Peter Bundschuh (Köln), Keijo Väänänen (Oulu)

Abstract: As in Parts I–III, entire transcendental solutions f of functional equations $f(q^m z) = = R_0(z)f(z) + R_1(z)$ with polynomial coefficients R_0, R_1 are arithmetically studied. But whereas, in Part III, these coefficients satisfied the condition $R_0(0) \neq 0$, our present work concerns the case $R_0(0) = 0, R_1 \neq 0$ involving, in the arithmetic considerations, essential differences compared to the earlier one. Again lower bounds for the dimension of the K-vector space generated by 1 and the values of these f and their derivatives at certain powers of q are produced, K being $\mathbb{Q}$ or an imaginary quadratic number field. In favorable circumstances, linear independence can be obtained, even in a quantitative form.

Keywords: Nesterenko-type dimension estimates; linear independence; quantitative versions

AMS Subject classification: 11J72, 11J82

Received: 06.02.2012; **revised:** 28.07.2012

1. Introduction and results

Let K be the field of rational numbers or an imaginary quadratic field, and denote by O_K its ring of integers.

In our earlier works [2], [5], [6], linear independence properties of the values of entire transcendental solutions of functional equations of the type

$$f(q^m z) = R_0(z)f(z) + R_1(z), \quad |q| > 1, \tag{1}$$

with $m \in \mathbb{N} := \{1, 2, ...\}$, $q \in K$ and polynomials $R_0, R_1 \in K[z]$ satisfying $R_0(0) \neq 0$ were considered. In this note we study the same question in the case $R_0(0) = 0, R_1 \neq 0$. The analytic part of the proof in [6] (see also [7]) works also in this case, but some modifications are needed in the arithmetic part.

If we now write $R_0(z) = z^r R(z)$ with vanishing order $r := \operatorname{ord} R_0$ ($\in \mathbb{N}$) of R_0 at the origin ($\Leftrightarrow R(0) \neq 0$), and $\ell := \deg R_0$, then the solution of (1) is of the form

$$f(z) = \sum_{n=1}^{\infty} R_1(zq^{-nm}) \prod_{j=1}^{n-1} R_0(zq^{-jm}) = \\ = \sum_{n=0}^{\infty} \frac{R_1(zq^{-(n+1)m}) \prod_{j=1}^{n} R(zq^{-jm})}{q^{mr\binom{n+1}{2}}} z^{nr}. \tag{2}$$

Thus the function f is a linear combination with constant coefficients of the functions

$$f_h(z) = z^h \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{n} R(zq^{-jm})}{q^{mr\binom{n+1}{2}}} (z^r q^{-mh})^n, \quad h = 0, 1, ... \tag{3}$$

We do not know of any earlier results on arithmetical properties of the values of these functions except in the case $r = \ell$, which is connected to the Tschakaloff function. We now get the following result.

Theorem 1. *Assume that* $q \in K$ *with* $|q| > 1$ *is the quotient* u/v *of* $u, v \in O_K^{\times} := O_K \setminus \{0\}$, *and let* $\eta := (\log|v|)/(\log|u|)$. *Suppose that* f *is an entire transcendental solution of the functional equation* (1) *with* $R_0, R_1 \in K[z] \setminus \{0\}$, $R_0(z) = z^r R(z)$ *with* $r = \operatorname{ord} R_0 \in \mathbb{N}$ *and* $\ell = \deg R_0$. *Let* $\alpha \in K$ *satisfy the conditions* $R_0(\alpha q^{-j}) \neq 0$ *for every rational integer* $j \geqslant m$. *Then we have the dimension estimate*

$$\dim_K \left\{ K + \sum_{\sigma=0}^{k-1} \sum_{\mu=0}^{m-1} K f^{(\sigma)}(\alpha q^{-\mu}) \right\} \geqslant (1-\eta) C(k, \ell, m, r),$$

where

$$C(k,\ell,m,r) := \sup \frac{(h+km)^2 + (2g-1)k\ell m + 2h\ell g + \ell r g^2}{\ell((\ell-r)kmg^2 + km + 2h + rg^2 + 6\pi^{-2}(k-1)m + \delta h(h/r - 2(1-g)))},$$

$\delta := 0$, *if* $h/(2r) \leqslant 1 - g$, *and* $\delta := 1$ *otherwise, and the supremum is taken over all* $h \geqslant 0$ *and* $0 \leqslant g \leqslant 1$ *satisfying* $(1-g)\ell < h + km$.

Remark. 1) We tacitly may always suppose that u, v have only units from $O_K^\times$ as common divisors.

2) By taking $h = 0$ in the rational expression of the estimate in Theorem 1 we get

$$C(k, \ell, m, r) \geqslant \sup \frac{k^2m^2 + (2g-1)k\ell m + \ell r g^2}{\ell((\ell - r)kmg^2 + rg^2 + km + 6\pi^{-2}(k-1)m)} =: C_0(k, \ell, m, r),$$

where the supremum is now taken over all $0 \leqslant g \leqslant 1$ satisfying $(1-g)\ell < km$. The proof of Theorem 2 in [6] implies immediately a dimension estimate, where $C(k, \ell, m, r)$ above is replaced by $C_0(k, \ell, m, 0)$ which equals, using the notations of [6],

$$\widetilde{C}(k, \ell, m) = \frac{k^2m - k\ell + \sqrt{(k^2m - k\ell)^2 + 4k\ell(k + 6\pi^{-2}(k-1))}}{2\ell(k + 6\pi^{-2}(k-1))}.$$

In fact, this proof does not use Lemma 1 of [6], which is not valid in the present case. However we see easily that the rational expression in the definition of $C_0(k, \ell, m, r)$ is an increasing function of r in the interval $0 \leqslant r \leqslant \ell$, and therefore our Theorem 1 gives an improvement of this result.

3) By using the values $g = 0, h = 2(\ell - 1)$, we obtain

$$C(k, \ell, m, \ell - 1) \geqslant \frac{(km + 2(\ell-1))^2 - k\ell m}{\ell(d(k)km + 4(\ell - 1))} \quad \text{with} \quad d(k) := 1 + 6\pi^{-2}(1 - k^{-1}).$$

The right-hand side of this inequality is greater than 1 if

$$km > 2 + \frac{3 - d(k)}{2}\ell\left(\sqrt{1 + \frac{8(d(k)-1)}{(3-d(k))^2\ell}} - 1\right).$$

Taking $1 \leqslant d(k) < 1{,}6079...$ into account, we have the following consequence of Theorem 1: If $q \in O_K$ and if the function f satisfies (1) with $\deg R_0 \geqslant 2$ and $\operatorname{ord} R_0 \geqslant \deg R_0 - 1$, then at least one of the numbers $f^{(\sigma)}(\alpha q^{-\mu})$, $0 \leqslant \sigma < k$, $0 \leqslant \mu < m$, does not belong to K (assuming $km \geqslant 3$). In the particular case $k = 1$, we can assert that at least one of the numbers $f(\alpha q^{-\mu})$, $0 \leqslant \mu < m$, is not in K (assuming $m \geqslant 2$); notice here that the 'smallest' case $m = 2$ is a consequence of our next remark.

4) If $km = 2$, then $C(k, \ell, m, \ell - 1) > 1$ holds for any $\ell > 1$. Or, in letters: Suppose that q, f, α satisfy the conditions of Theorem 1, and furthermore that $q \in O_K$ and

$r = \ell - 1$. If $m = 2$ (or $m = 1$), then at least one of the numbers $f(\alpha), f(\alpha/q)$ (or of $f(\alpha), f'(\alpha)$, respectively) does not belong to K.

To explain this assertion a bit, we insert $km = 2$ and $h = 2(1-g)(\ell - 1)$ into the quotient appearing in the definition of $C(k, \ell, m, \ell - 1)$ under the 'sup' and obtain, after some lengthy computation, this quotient as a rational function of g, ℓ and $k \in \{1, 2\}$. From its explicit representation we conclude that it is strictly greater than 1 if and only if

$$2g^2 > \left(3g^2 - 2g + \frac{6}{\pi^2}\left(1 - \frac{1}{k}\right)\right)\ell.$$

In case $k = 1$, this is true for, $g = 2/3$, say, and, in case $k = 2$, every $g \in [g_1, g_2] \subset]0, 1[$ is enough, where g_1, g_2 are the two roots of the polynomial $3g^2 - 2g + 3\pi^{-2}$.

5) Furthermore, the choice $g = 1/2, h = \ell - 2$ implies $C(1, \ell, 4, \ell - 2) > 1$. Thus, if q and f satisfy the conditions from the preceding remark, but with $\deg R_0 \geqslant 3$ and $\operatorname{ord} R_0 \geqslant \deg R_0 - 2$, then at least one of the four numbers $f(\alpha), f(\alpha q^{-1}), f(\alpha q^{-2}), f(\alpha q^{-3})$ does not belong to K.

6) Whereas, in the last two remarks, we obtained only exclusion statements, we notice here that the quantity $C_0(...)$ defined in Remark 2) satisfies

$$C_0(1, \ell, m, \ell) \geqslant \frac{m^2 + \ell m + \ell^2}{\ell^2 + \ell m} = \frac{m}{\ell} + \frac{\ell}{\ell + m}.$$

Therefore, we obtain the full linear independence in the Tschakaloff function case $\ell = r = 1$ if $k = 1$ and $\eta = 0$ (or even $\eta \geqslant 0$ but 'small'). Note that this does not follow from Theorem 2 in [6].

In the case considered in our last remark, i.e., $\ell = r = 1, k = 1$ and $\eta \geqslant 0$ but small, we can even formulate a measure for the linear independence of $f(\alpha), f(\alpha q^{-1}), ..., f(\alpha q^{-(m-1)})$ and 1 as follows.

THEOREM 2. *Let the assumptions of Theorem* 1 *be valid, and assume furthermore* $R_0(z) = a_0 z$ *with* $a_0 \in K^\times$, *and*

$$0 \leqslant \eta < \frac{2}{(m^2 + 2 + m\sqrt{m^2 + 4})}.$$

Then, for every $\varepsilon \in \mathbb{R}_+$, *there exists a constant* $C \in \mathbb{R}_+$ *depending at most on* $q, m, a_0, \alpha, \varepsilon$ *and on* R_1 *such that, for every*

$$\underline{\Lambda} := (\lambda_0, ..., \lambda_m) \in O_K^{m+1} \quad \textit{with} \quad |\underline{\Lambda}| := \max(|\lambda_0|, ..., |\lambda_m|) \geqslant C,$$

the following inequality holds

$$\Big| \sum_{\mu=0}^{m-1} \lambda_\mu f(\alpha q^{-\mu}) + \lambda_m \Big| \geqslant |\underline{\Lambda}|^{-\psi(m,\eta)-\varepsilon} \quad \textit{with}$$

$$\psi(m,\eta) := \frac{m(m+\sqrt{m^2+4}\,)}{2-(m^2+2+m\sqrt{m^2+4}\,)\eta}\,.$$

Remark. 1) Since $\psi(1,\eta) = (1+\sqrt{5})/(2-(3+\sqrt{5})\eta)$, Theorem 2 contains Stihl's Korollar 2 in [10] which, in turn, is precisely the first author's Satz 2 in [1].

2) It should be pointed out that, already in [3], a linear independence measure as in Theorem 2 but with $\psi(m,\eta)$ in the exponent of $|\underline{\Lambda}|$ replaced by

$$\theta(m,\eta) := \frac{2m-1+\sqrt{4m^2+1}}{2-(2m+1+\sqrt{4m^2+1})\eta}$$

was proved in the particular case $a_0 = q^m$, $R_1(z) = 1$, where f reduces to the Tschakaloff function. Note here that one has $\theta(m,\eta) \leqslant \psi(m,\eta)$ with equality if and only if $m = 1$; note also that, for large m, the quantity $\theta(m,0)$ is linear in m, whereas our $\psi(m,0)$ is quadratic in m.

Almost at the same time as [3], Stihl [11] produced, for more general functions, linear independence measures with exponents reducing in the Tschakaloff case to $\theta(m,\eta)$; moreover, Stihl included also derivatives of the functions in question. Since the mid-1980s, Katsurada [8], Wallisser and the second author [13], and Rochev [9] extended Stihl's results in various directions and by different methods. But all their extensions have the common property that, in the Tschakaloff case and letting derivatives aside, the above $\theta(m,\eta)$ appears as 'linear independence exponent'.

In our proofs of Theorems 1 and 2, we use heavily the proof of [6] and we just indicate the needed modifications keeping the notations of [6]. We also note as in [6] that it is enough to consider the case $\alpha = 1$ and deg $R_1 \leqslant \ell$.

2. Three lemmas

In the present case $r \in \mathbb{N}$, the denominators of the Taylor series coefficients of f behave differently from the case $r = 0$ in [6]. If

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

satisfies (1), then we obtain the following result on the coefficients c_n.

LEMMA 1. *Let $s \in O_K^{\times}$ be such that $sR_0, sR_1 \in O_K[z]$. Then, for every $t \in \mathbb{N}$, the products*

$$s^t q^{mr\binom{t+1}{2}} c_n \quad (0 \leqslant n < rt)$$

are polynomials in $O_K[q]$ of degree at most

$$mr\binom{t+1}{2} \quad (\textit{in } q).$$

PROOF. Supposing

$$R_0(z) = z^r R(z) = z^r \sum_{j=0}^{\ell-r} a_j z^j$$

with

$$a_0 a_{\ell-r} \neq 0, \quad R_1(z) = \sum_{j=0}^{\mu} b_j z^j$$

we obtain from (1)

$$Q^n c_n = \sum_{j=0}^{\ell-r} a_j c_{n-r-j} + b_n \quad (n = 0, 1, \dots)\,, \tag{4}$$

where $Q := q^m$, $c_j := 0$ if $j < 0$, and $b_n := 0$ if $n > \mu$. Clearly, we have

$$sQ^n c_n = sb_n \in O_K$$

for $0 \leqslant n < r$. Next we show the intermediate assertion that, for any $t \in \mathbb{N}$, the products

$$s^t Q^{r\binom{t+1}{2}} c_n \quad (0 \leqslant n < rt)$$

are polynomials in $O_K[Q]$ of degree at most

$$r\binom{t+1}{2} \quad \text{(in } Q\text{)}$$

which implies our lemma. This intermediate assertion is valid for $t = 1$ (see the formula after (4)); assuming it is already valid for some $t \in \mathbb{N}$, we verify it for $t+1$. To this purpose, we consider the equation

$$Q^{rt+i}c_{rt+i} = \sum_{j=0}^{\ell-r} a_j c_{r(t-1)+i-j} + b_{rt+i}$$

for $0 \leqslant i < r$. On multiplying it by $s{\cdot}s^t Q^{r\binom{t+1}{2}}$, we see from the right-hand side and the induction hypothesis (notice $r(t-1)+i-j < rt$) that

$$s^{t+1}Q^{r\binom{t+1}{2}+rt+i}c_{rt+i}$$

is in $O_K[Q]$ of degree at most

$$r\binom{t+1}{2} + r(t+1) = r\binom{t+2}{2} \quad \text{(in } Q\text{)}$$

completing our induction. □

Our next lemma, identical with the Nesterenko-type Lemma 2 in [6], directly prepares the dimension estimate in Theorem 1.

LEMMA 2. *Let* $\mathbb{E}$ *be* $\mathbb{R}$ *or* $\mathbb{C}$ *according as* K *is* $\mathbb{Q}$ *or an imaginary quadratic field. Further, let*

$$d \in \mathbb{N} \setminus \{1\} \quad \textit{and} \quad \underline{\omega} = (\omega_1, \ldots, \omega_d) \in \mathbb{E}^d \setminus \{\underline{0}\}.$$

Finally, assume that there exist $N_0 \in \mathbb{N}, \tau \in \mathbb{R}_+$, *an unbounded increasing function* $F : \mathbb{N} \to \mathbb{R}_+$, *and a sequence* $(\Lambda_N)_{N \geqslant N_0}$ *of linear forms over* O_K *in* d *variables with*

(*i*) $\limsup_{N\to\infty} F(N+1)/F(N) \leqslant 1$,

(*ii*) $\log \|\underline{\Lambda}_N\|_2 \leqslant F(N)$ *for every* $N \geqslant N_0$,

(*iii*) $\log |\Lambda_N(\underline{\omega})| = -(\tau + o(1))F(N)$ *for every large* $N \geqslant N_0$, *where* $\|\underline{\Lambda}_N\|_2$ *denotes the euclidean* (*or* l_2) *norm of the coefficient vector of the linear form* Λ_N. *Then the following dimension estimate holds*

$$\dim_K K\omega_1 + \dots + K\omega_d \geqslant 1 + \tau.$$

Our proof of Theorem 2 will be prepared by Lemma 3^* below which is a quantitative version of Lemma 2. This version generalizes Lemma 3 from [6], where hypotheses and conclusion were given in terms of $\|\,.\,\|_2$ (as in Lemma 2) since its proof in [4] and [12] used essentially the ℓ_2-norm on $\mathbb{E}^d$. But sometimes one wants to express the conclusion of such a result in terms of a different norm on $\mathbb{E}^d$, say, the supremum norm$^{1)}$ $\|\,.\,\|_\infty$ as it is quite usual in linear independence measures.

To state Lemma 3^*, we first recall the following standard result on norms in $\mathbb{E}^d$ ('norm equivalence'). If $\|\,.\,\|$ denotes an arbitrary norm on $\mathbb{E}^d$, $\|\,.\,\|_2$ the ℓ_2-norm, then there are $\alpha, \beta \in \mathbb{R}_+$, depending at most on d and $\|\,.\,\|$, such that

$$\alpha\|\underline{x}\| \leqslant \|\underline{x}\|_2 \leqslant \beta\|\underline{x}\| \tag{5}$$

holds for any $\underline{x} \in \mathbb{E}^d$.

LEMMA 3^*. *Let* $\mathbb{E}$, d *and* $\underline{\omega}$ *be as in Lemma* 2, *and let* $\|\,.\,\|$ *be an arbitrary norm on* $\mathbb{E}^d$. *Assume that there exist* $N_0, N_1 \in \mathbb{N}$ *with* $N_0 < N_1$, *unbounded and monotonically increasing functions* $b^*, g, g^*, h^*\colon \mathbb{N} \to \mathbb{R}_+$, *and, for every* $N \in \{N_0, \dots, N_1\}$, *a linear form* Λ_N *over* O_K *in* d *variables such that the following conditions hold for all* $N \in \{N_0, \dots, N_1\}$:

(*i*) $h^*(N) \leqslant -\log|\Lambda_N(\underline{\omega})| \leqslant g^*(N)$,

(*ii*) $g^*(N) + \log\|\underline{\Lambda}_N\| \leqslant g(N)$,

(*iii*) $\max(\log\|\underline{\Lambda}_N\|, g(N+1) - h^*(N) + \log 2d) \leqslant b^*(N)$.

If Λ *is a non-trivial linear form over* O_K *in* d *variables, then the following additional condition*

(*iv*) $g(N_1+1) - (d-1)b^*(N_1) > \log\|\underline{\Lambda}\| + g(N_0) + d\log\beta + \max(0, -\log\|\underline{\omega}\|_2)$

implies the inequality

$$|\Lambda(\underline{\omega})| > \frac{\alpha}{2\beta}\|\underline{\Lambda}\|\exp(-g(N_1+1)).$$

---

[1] Note that we write $|\underline{\Lambda}|$ instead of $\|\underline{\Lambda}\|_\infty$ except for the present discussions on norms.

Proof. By virtue of (5), we deduce from (ii) and (iii)

$$g^*(N) + \log ||\underline{\Lambda}_N||_2 \leqslant g^*(N) + \log ||\underline{\Lambda}_N|| + \log \beta \leqslant g(N) + \log \beta$$

and

$$\log ||\underline{\Lambda}_N||_2 \leqslant \log ||\underline{\Lambda}_N|| + \log \beta \leqslant b^*(N) + \log \beta,$$

respectively. Hence we try to apply Lemma 3 from [6] with $G^* := g^*, H^* := h^*, G(N) := g(N) + \log \beta, B^*(N) := b^*(N) + \log \beta$. We next have to fulfill the second condition in (iii) of [6, Lemma 3]:

$$\begin{gathered} G(N+1) - H^*(N) + \log 2d = \\ = g(N+1) - h^*(N) + \log \beta + \log 2d \leqslant b^*(N) + \log \beta = B^*(N)\,, \end{gathered}$$

where we applied the second condition in (iii) of Lemma 3*. Up to here, all conditions (i), (ii), (iii) in Lemma 3 from [6] are satisfied.

If Λ is a non-trivial linear form over O_K in d variables, then condition (iv) of Lemma 3* implies that, in

$$\begin{aligned} & G(N_1+1) - (d-1)B^*(N_1) - G(N_0) - \log ||\underline{\Lambda}||_2 \geqslant \\ & \quad \geqslant g(N_1+1) - (d-1)b^*(N_1) - g(N_0) - d \log \beta - \log ||\underline{\Lambda}|| \end{aligned}$$

the right-hand side is greater than $\max(0, -\log ||\underline{\omega}||_2)$ such that condition (iv) of [6, Lemma 3] is also satisfied. Thus, this lemma gives us the desired inequality

$$|\Lambda(\underline{\omega})| > \frac{1}{2} ||\underline{\Lambda}||_2 \exp(-G(N_1+1)) \geqslant \frac{\alpha}{2\beta} ||\underline{\Lambda}|| \exp(-g(N_1+1))$$

using the lower bound in (5). □

Remark. Clearly, one could also eliminate $||\underline{\omega}||_2$ from (iv) in favour of $||\underline{\omega}||$ by formulating a condition in terms of $||\underline{\omega}||$ such that, using (5), our present hypothesis (iv) is implied. This is only a question of taste noting that, for applications like ours, the three last summands on the right-hand side of (iv) are ‘constants’.

3. Proof of Theorem 1

To prove Theorems 1 and 2, we use the construction of [6]. Thus, linear forms used here are obtained from the integral

$$I(N) := \frac{1}{2\pi i} \oint\limits_{\Gamma(N)} \frac{f(q^G z)dz}{z^L \prod_{j=0}^{M+\beta_N+1}(z-q^j)^{k_j}},$$

where $L = [hN], h \geqslant 0, M = mN$ and $G = [-gM], 0 \leqslant g \leqslant 1$, are integers depending on a large parameter $N \in \mathbb{N}$, β_N is a suitable integer satisfying $0 \leqslant \beta_N < \ell$, $k_j = k$ for $0 \leqslant j \leqslant M, k_j = 1$ for $j > M$, and $\Gamma(N)$ is a positively oriented circle:

$$|z| = R > |q|^{\widetilde{M}},$$

where $\widetilde{M} := M + \beta_N + 1$. By the residue theorem we get

$$I(N) = \sum_{\sigma+\sigma_0+\ldots+\sigma_{\widetilde{M}}=L-1} (-1)^{k(M+1)+\widetilde{M}-M} c_\sigma q^{G\sigma - \sum\limits_{j=0}^{\widetilde{M}} j(k_j+\sigma_j)} \prod_{j=0}^{M} \binom{k-1+\sigma_j}{\sigma_j} +$$

$$+ \sum_{j=0}^{M} \sum_{\substack{\sigma+\sigma_0+\ldots+\sigma_{\widetilde{M}} \\ =k-1}} q^{G\sigma_j} \frac{f^{(\sigma_j)}(q^{G+j})}{\sigma_j! q^{j(L+\sigma)}} (-1)^\sigma \binom{L-1+\sigma}{\sigma} \prod_{\substack{i=0 \\ i \neq j}}^{\widetilde{M}} \binom{k_i-1+\sigma_i}{\sigma_i} \frac{(-1)^{\sigma_i}}{(q^j-q^i)^{k_i+\sigma_i}} +$$

$$+ \sum_{j=M+1}^{\widetilde{M}} f(q^{G+j}) q^{-jL} \prod_{\substack{i=0 \\ i \neq j}}^{\widetilde{M}} (q^j - q^i)^{-k_i}, \tag{6}$$

where both multiple sums are over all non-negative σ's satisfying the given conditions. Exactly as in [6] we see that $I(N)$ is a linear form in 1 and $f^{(\sigma)}(q^{-\mu}), 0 \leqslant \sigma < k$, $0 \leqslant \mu < m$, with coefficients from K,

$$I(N) \in K + \sum_{\sigma=0}^{k-1} \sum_{\mu=0}^{m-1} K f^{(\sigma)}(q^{-\mu}).$$

Also the denominators of these coefficients can be studied as in [6] except the consideration of $\Omega_1(N)$ coming from the terms $f^{(\sigma)}(q^{G+j})$ with $G + j \leqslant 0$ in the second sum in (6) and the powers of q in the first sum in (6).

If $G+j=-\lambda m-\mu\leqslant 0$, then, by (8) of [6],

$$q^{-\lambda m\sigma}f^{(\sigma)}(q^{-\lambda m-\mu})=$$
$$=\sum_{\tau=0}^{\sigma}\binom{\sigma}{\tau}f^{(\tau)}(q^{-\mu})\left(\frac{1}{R_{0,\lambda}(q^{-\lambda m}z)}\right)^{(\sigma-\tau)}\Bigg|_{z=q^{-\mu}}-\left(\frac{R_{1,\lambda}(q^{-\lambda m}z)}{R_{0,\lambda}(q^{-\lambda m}z)}\right)^{(\sigma)}\Bigg|_{z=q^{-\mu}},$$

where

$$R_{0,\lambda}(z)=\prod_{\kappa=0}^{\lambda-1}R_0(q^{\kappa m}z),$$

$$R_{1,\lambda}(z)=\sum_{\kappa=0}^{\lambda-1}R_1(q^{\kappa m}z)\prod_{\nu=\kappa+1}^{\lambda-1}R_0(q^{\nu m}z).$$

We recall that $s\in O_K^{\times}$ satisfies $sR_0, sR_1\in O_K[z]$. Then, see [6, p. 96],

$$s^{\lambda}\left(\prod_{\kappa=1}^{\lambda}q^{\ell\kappa m}\right)R_{0,\lambda}(q^{-\lambda m}z)=:\widetilde{R}_{0,\lambda}(z)$$

and

$$s^{\lambda}\left(\prod_{\kappa=1}^{\lambda}q^{\ell\kappa m}\right)R_{1,\lambda}(q^{-\lambda m}z)=:\widetilde{R}_{1,\lambda}(z)$$

are polynomials in q and z with coefficients in O_K. Thus

$$\left(\frac{1}{R_{0,\lambda}(q^{-\lambda m}z)}\right)^{(\sigma-\tau)}\Bigg|_{z=q^{-\mu}}=s^{\lambda}\left(\prod_{\kappa=1}^{\lambda}q^{\ell\kappa m}\right)\left(\frac{1}{\widetilde{R}_{0,\lambda}(z)}\right)^{(\sigma-\tau)}\Bigg|_{z=q^{-\mu}},$$

$$\left(\frac{R_{1,\lambda}(q^{-\lambda m}z)}{R_{0,\lambda}(q^{-\lambda m}z)}\right)^{(\sigma)}\Bigg|_{z=q^{-\mu}}=\left(\frac{\widetilde{R}_{1,\lambda}(z)}{\widetilde{R}_{0,\lambda}(z)}\right)^{(\sigma)}\Bigg|_{z=q^{-\mu}},$$

and therefore

$$\left(q^{\mu\ell\lambda}\widetilde{R}_{0,\lambda}(q^{-\mu})\right)^{\sigma+1}=\left(q^{\mu(\ell-r)\lambda}s^{\lambda}\prod_{\kappa=1}^{\lambda}q^{(\ell-r)\kappa m}R(q^{-\kappa m-\mu})\right)^{\sigma+1}$$

is a denominator for $f^{(\sigma)}(q^{G+j})$ with $G+j=-\lambda m-\mu\leqslant 0$. Since $\lambda\leqslant gN+1/m$, the product $\Omega_1(N)f^{(\sigma)}(q^{G+j})$ is, for all possible $G+j$, a linear form in 1 and the

$f^{(\sigma)}(q^{-\mu})$ with coefficients from $O_K[q]$ if we choose

$$\Omega_1(N) := \begin{cases} s^{\mathrm{O}(N)} q^{\mathrm{O}(N)} & \text{if } g = 0, \\ s^{\mathrm{O}(N)} q^{\mathrm{O}(N)} \prod\limits_{\mu=0}^{m-1} \prod\limits_{\kappa=1}^{gN+\mathrm{O}(1)} R(q^{-\kappa m-\mu})^k q^{(\ell-r)\kappa mk} & \text{if } 0 < g \leqslant 1. \end{cases}$$

The study of $\Omega_2(N)$ and $\Omega_3(N)$ follows [6] unless considering $\Omega_2(N)$, where we use Lemma 1 to estimate the needed power of q in the first sum of (6). For this, note that if $\sigma \leqslant L-1$ is of the form $\sigma = rt + i, 0 \leqslant i < r$, then

$$mr\binom{t+2}{2} = \frac{m\sigma^2}{2r} + \mathrm{O}(\sigma).$$

Therefore

$$-G\sigma + \sum_{j=0}^{\widetilde{M}} j(k_j+\sigma_j) + mr\binom{t+2}{2} \leqslant ML + \frac{1}{2}kM(M+1) +$$
$$+\sigma\left(\frac{m\sigma}{2r} - M - G\right) + \mathrm{O}(N) \leqslant ML + \frac{1}{2}kM(M+1) + \delta L\left(\frac{mL}{2r} - M - G\right) + \mathrm{O}(N),$$

where $\delta := 0$, if $h/(2r) \leqslant 1-g$, and $\delta := 1$ otherwise. By Lemma 1 and [6], we may thus choose

$$\Omega_2(N) := q^{ML+kM(M+1)/2+\delta L(\frac{mL}{2r}-M-G)+\mathrm{O}(N)},$$
$$\Omega_3(N) := (k-1)!s^{\mathrm{O}(N)} \cdot \prod_{i=1}^{M+\ell} \Phi_i(q)^{k-1} \prod_{\rho=1}^{M+\ell} (q^\rho - 1)^k,$$

where Φ_n is the nth cyclotomic polynomial, and then the choice

$$\widetilde{\Omega}(N) := \Omega_1(N)\Omega_2(N)\Omega_3(N)$$

makes $\widetilde{\Omega}(N)I(N)$ a linear form in 1 and $f^{(\sigma)}(q^{-\mu}), 0 \leqslant \sigma < k, 0 \leqslant \mu < m$, having coefficients from $O_K[q]$. To estimate the degree (in q) of these coefficients, we first note that

$$\deg \widetilde{\Omega}(N) \leqslant \frac{1}{2}k(\ell - r)m^2g^2N^2 + LM + kM^2 +$$
$$+ \delta L\left(\frac{mL}{2r} - M - G\right) + \frac{3}{\pi^2}(k-1)M^2 + \mathrm{O}(N\log N).$$

Then we may proceed as in [6], but now, when considering the above terms with $G + j = -\lambda m - \mu$, we have to take into account that

$$\deg_q \widetilde{R}_{1,\lambda}(z) - \deg_q \widetilde{R}_{0,\lambda}(z) \leqslant \sum_{\kappa=1}^{\lambda} \ell\kappa m - \deg_q \widetilde{R}_{0,\lambda}(z) \leqslant$$
$$\leqslant \sum_{\kappa=1}^{\lambda} r\kappa m \leqslant \frac{1}{2}rmg^2N^2 + \mathrm{O}(N),$$

which then gives the following analogue of Lemma 5 of [6].

LEMMA 4. *If* $(1-g)\ell < h + km$ *holds, then* $\widetilde{\Omega}(N)I(N)$ *is a linear form in* 1 *and the* $f^{(\sigma)}(q^{-\mu})$, $0 \leqslant \sigma < k, 0 \leqslant \mu < m$, *with coefficients in* $O_K[q]$. *For all sufficiently large* N, *the degrees* (*with respect to* q) *of these coefficient polynomials are bounded above by the expression*

$$\left(\frac{1}{2}k(\ell - r)m^2g^2 + hm + \frac{1}{2}km^2 + \frac{1}{2}rmg^2 + \right.$$
$$\left. + \delta hm\left(\frac{h}{2r} - 1 + g\right) + \frac{3}{\pi^2}(k-1)m^2\right)N^2 + \mathrm{O}(N\log N)$$

to be denoted by $D(N)$.

By defining

$$\Omega(N) := v^{D(N)}\widetilde{\Omega}(N)$$

we obtain the linear form

$$\Lambda_N = \Omega(N)I(N) \in O_K + \sum_{\sigma=0}^{k-1}\sum_{\mu=0}^{m-1} O_K f^{(\sigma)}(q^{-\mu}).$$

Supposing $(1-g)\ell < h+km$ we deal with this analogously to Section 6 of [6]. This time, the maximum norm of the coefficient vector $\underline{\Lambda}_N$ of this linear form satisfies

$$\log|\underline{\Lambda}_N| \leqslant \frac{m}{2}\Gamma_0 N^2 \log|u| + \mathrm{O}(N\log N), \tag{7}$$

with

$$\Gamma_0 := k(\ell-r)mg^2 + 2h + km + rg^2 + \delta h(\frac{h}{r} - 2(1-g)) + \frac{6}{\pi^2}(k-1)m \tag{8}$$

and the same estimate holds for $||\underline{\Lambda}_N||_2$ with a slightly larger O-constant depending only on k and m.

Lemma 4 from [6] remains valid also now, and therefore we find

$$\log|\Lambda_N| = D(N)\log|v| + \frac{m}{2}\Big(k(\ell-r)mg^2 + 2h + 2km + \delta h(\frac{h}{r} - 2(1-g)) + \\ + \frac{6}{\pi^2}(k-1)m - \frac{1}{\ell}(h+km)^2 - 2g(h+km)\Big)N^2\log|q| + \mathrm{O}(N\log N),$$

the two terms with minus signs originating from the asymptotic evaluation of $|I(N)|$. Using (8) and the definition of $D(N)$ in our present Lemma 4, the last expression can be written as

$$\log|\Lambda_N| = -\frac{m}{2}\Big((1-\eta)\Gamma_1 - \Gamma_0\Big)N^2\log|u| + \mathrm{O}(N\log N) \tag{9}$$

with

$$\Gamma_1 := \frac{1}{\ell}(h+km)^2 + (2g-1)km + 2gh + rg^2. \tag{10}$$

If now g, h satisfy the conditions stated in Theorem 1, then Lemma 2, (7) and (9) lead to

$$\dim_K\left\{K + \sum_{\sigma=0}^{k-1}\sum_{\mu=0}^{m-1} Kf^{(\sigma)}(q^{-\mu})\right\} \geqslant (1-\eta)\frac{\Gamma_1}{\Gamma_0}\,, \tag{11}$$

whence the assertion of Theorem 1, if we take the definitions of Γ_0, Γ_1 in (8) and (10) into account.

4. Proof of Theorem 2

Since we obviously want to express the lower bound of the linear form

$$\Lambda(\underline{\omega}) := \lambda_0\omega_0 + \ldots + \lambda_m\omega_m$$

at the point

$$\underline{\omega} := (f(\alpha), \ldots, f(\alpha q^{-(m-1)}), 1)$$

in terms of $|\underline{\Lambda}|$, the maximum norm of the coefficient vector

$$\underline{\Lambda} := (\lambda_0, \ldots, \lambda_m) \in O_K^{m+1},$$

we will apply Lemma 3^* with this norm.

First, by (9), to satisfy (i) of Lemma 3^*, we may choose

$$\begin{aligned} g^*(N) &:= \frac{m}{2}((1-\eta)\Gamma_1 - \Gamma_0)N^2 \log|u| + \theta_1 N \log N, \\ h^*(N) &:= \frac{m}{2}((1-\eta)\Gamma_1 - \Gamma_0)N^2 \log|u| - \theta_1 N \log N, \end{aligned}$$

with Γ_0, Γ_1 from (8) and (10) taken with $k = \ell = r = 1$ satisfying the condition $(1-\eta)\Gamma_1 > \Gamma_0$ which has to be sharpened a bit later (see (14)). Here θ_1 (and then $\theta_2, \theta_3, \ldots$) is a suitable positive constant independent of N. Next, by using (7), we have

$$g^*(N) + \log|\underline{\Lambda}_N| \leqslant \frac{m}{2}(1-\eta)\Gamma_1 N^2 \log|u| + \theta_2 N \log N =: g(N)$$

giving (ii) of Lemma 3^*. Further,

$$\begin{aligned} &g(N+1) - h^*(N) + \log 2(m+1) \leqslant \\ &\leqslant \frac{m}{2}(1-\eta)\Gamma_1(N+1)^2 \log|u| - \frac{m}{2}((1-\eta)\Gamma_1 - \Gamma_0)N^2 \log|u| + \\ &\quad + \theta_3 N \log N \leqslant \frac{m}{2}\Gamma_0 N^2 \log|u| + \theta_4 N \log N =: b^*(N). \end{aligned}$$

Choosing θ_4 possibly a little greater, we have also $\log|\underline{\Lambda}_N| \leqslant b^*(N)$, by (7), whence (iii) of Lemma 3^* is valid. Clearly, we may choose N_0 such that, for $N \geqslant N_0$, all four functions g^*, h^*, g, b^* increase monotonically.

Suppose now $\underline{\Lambda}$ as above with $|\underline{\Lambda}|$ large enough. Then, towards condition (iv) of Lemma 3^*, we may choose N_1 as smallest integer $> N_0$ such that

$$g(N_1+1) > mb^*(N_1) + \log|\underline{\Lambda}| + \theta_5 \tag{12}$$

holds with

$$\theta_5 := g(N_0) + (m+1)\log\beta + \max(0, -\log\|\underline{\omega}\|_2),$$

where we can take $\beta = \sqrt{m+1}$ (and later $\alpha = 1$) in the present situation. That such a choice of N_1 is possible can be seen from

$$g(N+1) - mb^*(N) > \frac{m}{2}((1-\eta)\Gamma_1 - m\Gamma_0)N^2\log|u| - \theta_6 N\log N \tag{13}$$

if we suppose from now on[2)]

$$(1-\eta)\Gamma_1 > m\Gamma_0. \tag{14}$$

Thus, according to our definition of N_1 in (12) and the monotonicity of b^*, we obtain that $g(N_1) \leqslant mb^*(N_1) + \log|\underline{\Lambda}| + \theta_5$ holds, whence

$$\Gamma N_1^2\log|u| \leqslant \log|\underline{\Lambda}| + \theta_7 N_1\log N_1 \quad \text{with} \quad \Gamma := \frac{m}{2}((1-\eta)\Gamma_1 - m\Gamma_0) \quad (>0). \tag{15}$$

Now we conclude from Lemma 3^*

$$\begin{aligned}
&\log|\Lambda(\underline{\omega})| > -\log 2\sqrt{m+1} + \\
&\quad + \log|\underline{\Lambda}| - \frac{m}{2}(1-\eta)\Gamma_1(N_1+1)^2\log|u| - \theta_2(N_1+1)\log(N_1+1) \geqslant \\
&\geqslant \log|\underline{\Lambda}| - \frac{m}{2}(1-\eta)\Gamma_1 N_1^2\log|u| - \theta_8 N_1\log N_1 \geqslant \\
&\geqslant \left(1 - \frac{m}{2}(1-\eta)\frac{\Gamma_1}{\Gamma}\right)\log|\underline{\Lambda}| - \theta_9 N_1\log N_1,
\end{aligned}$$

the last inequality by (15). From (15) one can easily deduce that, given any $\varepsilon \in \mathbb{R}_+$, there exists a constant $C \in \mathbb{R}_+$ such that $\theta_9 N_1\log N_1 \leqslant \varepsilon\log|\underline{\Lambda}|$ if $|\underline{\Lambda}| \geqslant C$.

[2)] Note that this is equivalent to $(1-\eta)(\Gamma_1/\Gamma_0) > m$, hence that the vector space appearing in (11) has maximal dimension $m+1$.

Therefore, our above chain of inequalities leads to

$$\log|\Lambda(\underline{\omega})| > \left(1-(1-\eta)\frac{m\Gamma_1}{2\Gamma}-\varepsilon\right)\log|\underline{\Lambda}| =$$
$$= -\left(\frac{m}{(1-\eta)(\Gamma_1/\Gamma_0)-m}+\varepsilon\right)\log|\underline{\Lambda}|. \tag{16}$$

This estimate resembles quite a lot to the conclusion of our Theorem 2. The only remaining point is to maximize the quotient (see (8) and (10))

$$\frac{\Gamma_1}{\Gamma_0} = \frac{(h+m)^2+(2g-1)m+2gh+g^2}{m+2h+g^2+\delta h(h+2g-2)} =: Q(g,h) \tag{17}$$

for all (g, h) in the vertical half-strip $\mathcal{H} := [0,1]\times\mathbb{R}_{\geqslant 0}$, where, in case $m = 1$, we have to let aside the point $(0,0)$ due to the required condition $(1-g)\ell < h+km$ reducing here to $1 < g+h+m$.

Below we will give some few hints how one can show that $\sup\limits_{(g,h)\in\mathcal{H}} Q(g,h)$ is, indeed, a maximum, and its value equals

$$\frac{1}{2}\left(m+\sqrt{m^2+4}\,\right). \tag{18}$$

Note that, this value taken for Γ_1/Γ_0, hypothesis (14) is equivalent to

$$\eta < \frac{2}{m^2+2+m\sqrt{m^2+4}},$$

and this is an assumption of Theorem 2. Furthermore, taking (18) for Γ_1/Γ_0, shows

$$\frac{m}{(1-\eta)(\Gamma_1/\Gamma_0)-m} = \frac{m(m+\sqrt{m^2+4}\,)}{2-(m^2+2+m\sqrt{m^2+4}\,)\eta}$$

for the quotient appearing on the right-hand side of (16), whence our Theorem 2, of course, modulo the assertion concerning the supremum of our continuous function Q on $\mathcal{H}$ to be explained subsequently.

To prove this assertion, we consider the rational function

$$P(x) := \frac{(x+m)^2-m}{x^2+m}$$

of x. It is strictly increasing in $x \geqslant 0$ until the point

$$x = x_m := 1 + \frac{\sqrt{m^2+4} - m}{2}\,,$$

thereafter it strictly decreases, and we have

$$P(x_m) = \frac{m + \sqrt{m^2+4}}{2}\,,$$

the value in (18). Since $P(g+h) = Q(g,h)$ if $(g,h) \in \mathcal{H}$ satisfies $h \geqslant 2(1-g)$, we know that the maximum of Q in that upper part of $\mathcal{H}$ is $P(x_m)$, and it remains only to show that $Q(g,h) < P(x_m)$ for those $(g,h) \in \mathcal{H}$ with $h < 2(1-g)$.

Indeed, if $0 < h < 2(1-g)$, then we have to apply (17) with $\delta = 0$, whence the denominator of the right-hand side of (17) reduces to $m + 2h + g^2$ which is strictly greater than $(g+h)^2 + m$ leading to $Q(g,h) < P(g+h)$. Finally, again from (17), we obtain $Q(g,0) = P(g)$ $(< P(x_m))$ for any $g \in [0,1]$.

Acknowledgements. The authors would like to gratefully acknowledge the careful reading and the helpful comments of the anonymous referees.

Bibliography

1. ***P. Bundschuh,*** *Verschärfung eines arithmetischen Satzes von Tschakaloff*, Portugal. Math. **33** (1974), 1–17.
2. ***P. Bundschuh,*** *Arithmetical results on certain q-series*, *I*, Int. J. Number Theory **4** (2008), 25–43.
3. ***P. Bundschuh, I. Shiokawa,*** *A measure for the linear independence of certain numbers*, Results Math. **7** (1984), 130–144.
4. ***P. Bundschuh, T. Töpfer,*** *Über lineare Unabhängigkeit*, Monatsh. Math. **117** (1994), 17–32.
5. ***P. Bundschuh, K. Väänänen,*** *Arithmetical results on certain q-series*, *II*, Int. J. Number Theory **5** (2009), 1231–1245.
6. ***P. Bundschuh, K. Väänänen,*** *Arithmetical results on certain q-series*, *III*, Moscow J. Combinatorics and Number Theory **1** (2011), 83–106.
7. ***P. Bundschuh, K. Väänänen,*** *q-series: dimension estimates, linear independence*, Publ. Math. Debrecen **79** (2011), 367–378.
8. ***M. Katsurada,*** *Linear independence measures for certain numbers*, Results Math. **14** (1988), 318–329.

9. ***I. P. Rochev,*** *New linear independence measures for values of q-hypergeometric series*, arXiv: math/1006.5413v1 (2010).
10. ***T. Stihl,*** *Irrationalitätsmaße für Werte der Lösungen einer Funktionalgleichung von Poincaré*, Arch. Math. (Basel) **41** (1983), 531–537.
11. ***T. Stihl,*** *Arithmetische Eigenschaften spezieller Heinescher Reihen*, Math. Ann. **268** (1984), 21–41.
12. ***T. Töpfer,*** *Über lineare Unabhängigkeit in algebraischen Zahlkörpern*, Results Math. **25** (1994), 139–152.
13. ***K. Väänänen, R. Wallisser,*** *Zu einem Satz von Skolem über lineare Unabhängigkeit von Werten gewisser Thetareihen*, Manuscripta Math. **65** (1989), 199–212.

PETER BUNDSCHUH

Mathematisches Institut
Universität zu Köln
Weyertal 86–90
50 931 Köln, Germany,
pb@math.uni-koeln.de

KEIJO VÄÄNÄNEN

Department of Mathematical Sciences
University of Oulu
P. O. Box 3000
90 014 Oulu, Finland,
kvaanane@sun3.oulu.fi