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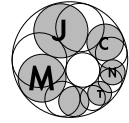
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Distance graphs with large chromatic number and arbitrary girth

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Abstract: In this article we consider a problem related to two famous combinatorial topics. One of them concerns the chromatic number of the space. The other deals with graphs having big girth (the length of the shortest cycle) and large chromatic number. Namely, we prove that for any $l \in \mathbb{N}$ there exists a sequence of distance graphs in \mathbb{R}^n with girth at least l and the chromatic number equal to $(c + \bar{o}(1))^n$ with $c > 1$.

Keywords: distance graphs; forbidden subgraphs; girth; chromatic number

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1. Introduction

1.1. History and related problems

In this article we study distance graphs (see [4]) of a certain type. Fix some $a > 0$. We say that $G = (V, E)$ is an a -distance graph in \mathbb{R}^n , if V is a subset of \mathbb{R}^n and

$$E \subseteq \{\{\mathbf{x}, \mathbf{y}\} : \mathbf{x}, \mathbf{y} \in V, |\mathbf{x} - \mathbf{y}| = a\}.$$

Remark 1. If we consider an a -distance graph in \mathbb{R}^n , then we can apply homothety and transform it into a 1-distance graph (which is also called unit distance graph). So we won't distinguish a -distance graphs for different a .

Such graphs arise naturally in the context of the problem of finding the chromatic number of the space. This famous question was posed by Nelson in 1950: what is the minimum number $\chi(\mathbb{R}^2)$ of colors needed to color all points of the plane so that no two points at distance one receive the same color? Although this question doesn't sound too difficult, it hasn't got an answer yet. The best we know is that $4 \leq \chi(\mathbb{R}^2) \leq 7$. One may ask the same question for higher-dimensional spaces. Here is the formal definition (see [2]):

$$\chi(\mathbb{R}^n) = \min\{m \in \mathbb{N} : \mathbb{R}^n = H_1 \cup \dots \cup H_m : \forall i, \forall \mathbf{x}, \mathbf{y} \in H_i \mid \mathbf{x} - \mathbf{y} \neq 1\}.$$

There are quite a few results about this quantity (see the surveys [16], [18] and also [12]), e.g. there are nontrivial lower bounds for the value of $\chi(\mathbb{R}^n)$, $n \leq 24$. We will be interested in the behavior of $\chi(\mathbb{R}^n)$ as $n \rightarrow \infty$. The following asymptotic lower and upper bounds are due to A. Raigorodskii [16] and D. Larman, C. Rogers [14] respectively:

$$(\zeta_{low} + \bar{o}(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + \bar{o}(1))^n, \text{ where } \zeta_{low} = 1,239\dots$$

The connection between the chromatic number of the space \mathbb{R}^n and distance graphs in \mathbb{R}^n is intimate. On the one hand, it follows from the definitions that for any such distance graph G , $\chi(G) \leq \chi(\mathbb{R}^n)$. On the other hand, N. G. de Bruijn and P. Erdős [5] proved that there exists a distance graph G' in \mathbb{R}^n with finite number of vertices such that $\chi(G') = \chi(\mathbb{R}^n)$.

The second question that lies at the basis of this article is the following. Can we construct graphs with arbitrarily large chromatic number and arbitrary girth (the length of the shortest cycle)? The positive answer to this question was given by P. Erdős [9]. He proved that such graphs exist, although the proof was probabilistic, so there was no explicit construction. Later, L. Lovász [15] managed to construct such graphs.

It is natural to ask how big can the chromatic number of a distance graph be if we additionally require that the graph has no cliques (complete subgraphs) or cycles of fixed size. The question, whether there is a distance graph in the plane with chromatic number 4 and without triangles (which are both cliques of size 3 and cycles of length 3) was asked by P. Erdős [10]. It was answered positively. Moreover, P. O'Donnell ([7], [8]) proved that for any $l \in \mathbb{N}$ there exists a distance graph in the plane with chromatic number 4 and girth greater than l .

We consider the following three families of distance graphs in \mathbb{R}^n : $\mathcal{C}(n, k)$ is the family of all distance graphs that do not contain complete subgraphs of size k ;

$\mathcal{G}_{odd}(n, k)$ is the family of all distance graphs that do not contain odd cycles of length $\leq k$; $\mathcal{G}(n, k)$ is the family of all distance graphs that do not contain cycles of length $\leq k$. We obviously have the following inclusion:

$$\mathcal{G}(n, l) \subset \mathcal{G}_{odd}(n, l) \subset \mathcal{C}(n, k) \subset \mathcal{C}(n, k'),$$

where $l \geq 3$ and $k' \geq k \geq 3$.

Now we define the following quantities:

$$\begin{aligned} \zeta_k &= \liminf_{n \rightarrow \infty} \max_{G \in \mathcal{C}(n, k)} (\chi(G))^{1/n}, \\ \xi_k^{odd} &= \liminf_{n \rightarrow \infty} \max_{G \in \mathcal{G}_{odd}(n, k)} (\chi(G))^{1/n}, \\ \xi_k &= \liminf_{n \rightarrow \infty} \max_{G \in \mathcal{G}(n, k)} (\chi(G))^{1/n}. \end{aligned}$$

For example, the bound $\zeta_k \geq 1,1$ means that there exists a sequence of distance graphs $G_n \subset \mathbb{R}^n$, such that $\chi(G_n) \geq (1,1 + \bar{o}(1))^n$ and none of G_i contains cliques of size k .

The values ζ_k and ξ_k^{odd} were considered in several papers (see [17]). The most accurate estimates on ζ_k are due to A. Kupavskii [13] (see also [6], where both ξ_k^{odd} and ζ_k were considered).

There are two approaches to estimate the quantity ζ_k . The first one is probabilistic, so we don't obtain an explicit graph. However, using this technique one can see that $\zeta_k \geq c_k$, where $c_k > 1$ and $\lim_{k \rightarrow \infty} c_k = \zeta_{low}$. In [6] this method gave nontrivial bounds only for $k \geq 5$. A refinement of this method suggested in [13] works for $k \geq 3$.

The second approach is in some sense code-theoretic. It provides us with explicit constructions of such graphs and it works for $k \geq 3$. Moreover, it gives much better bounds for small k . But as k grows, this method becomes worse than the probabilistic one, and the bounds tend to some constant that is significantly smaller than ζ_{low} .

A way to obtain bounds on ξ_k^{odd} , $k \geq 5$, is also code-theoretic. In [6] it was proved that for any fixed k we have $\xi_k^{odd} > 1$.

Bounds on both ζ_k and ξ_k^{odd} that are slightly weaker than the code-theoretic ones can be derived from simple geometric observations (see [13]).

1.2. Main Result

In the previous subsection we considered values $\zeta_k, \xi_k^{odd}, \xi_k$. For the first two values we mentioned some non-trivial bounds. However, both previous probabilistic and code-theoretic approaches failed to provide any estimate for $\xi_k, k \geq 4$. The main result of this article is the following

THEOREM 1. *For any fixed $k \geq 3$ we have $\xi_k \geq 1 + \delta$, where $\delta = \delta(k)$ is a positive constant that depends only on k .*

We say that a graph H is a *forest* if it doesn't contain cycles. Consider a finite family $\mathcal{H} = \{H_1, \dots, H_m\}$ of graphs. Let $\mathcal{G}(n, \mathcal{H})$ be the family of all distance graphs in \mathbb{R}^n that do not contain any of $H_i \in \mathcal{H}$ as a subgraph. We define the quantity $\xi(\mathcal{H})$ as above:

$$\xi(\mathcal{H}) = \liminf_{n \rightarrow \infty} \max_{G \in \mathcal{G}(n, \mathcal{H})} (\chi(G))^{1/n}.$$

Theorem 1 provides us with the following appealing corollary:

COROLLARY 1. *For any finite family \mathcal{H} of graphs such that no $H_i \in \mathcal{H}$ is a forest we have $\xi(\mathcal{H}) \geq 1 + \delta$, where $\delta = \delta(\mathcal{H})$ is a positive constant that depends only on \mathcal{H} .*

PROOF. The proof is immediate. Let l_i be the length of the shortest cycle in H_i , where $H_i \in \mathcal{H}$. Then $\xi(\mathcal{H}) \geq \max_i \xi_{l_i} \geq 1 + \delta$, where δ depends only on \mathcal{H} . \square

Unfortunately, Theorem 1 says nothing about the family of graphs, on which this bound can be attained. So it is natural to raise the following problem:

Problem 1. Prove Theorem 1 using an explicit construction.

The second disadvantage of the method used in this article is the following. The graphs that finally can be obtained are not necessarily *complete distance graphs*, i.e. in the definition of their set of edges we have the strict inclusion (not all possible edges are drawn). Here is another question:

Problem 2. Prove Theorem 1 using complete distance graphs.

The rest of the article is organized as follows. In Section 2.1 we will give necessary definitions and state auxiliary results. In Section 2.2 we will give the proof of Theorem 1.

2. Proof of Theorem 1

2.1. Preliminaries

As a basis of our construction we will take a family $\mathcal{G} = \{G_{4i} : i \in \mathbb{N}\}$ of distance graphs, where $G_{4n} = (V_{4n}, E_{4n})$, and

$$V_{4n} = \{\mathbf{x} = (x_1, \dots, x_{4n}) : x_i \in \{0, 1\}, x_1 + \dots + x_{4n} = 2n\},$$

$$E_{4n} = \{\{\mathbf{x}, \mathbf{y}\} : (\mathbf{x}, \mathbf{y}) = n\}.$$

Here (\cdot) denotes the Euclidean scalar product. In the next subsection we will prove that for any $k \in \mathbb{N}$ there exists a family of graphs H_{4i} such that for each i the graph H_{4i} is a subgraph of G_{4i} and H_{4i} has girth greater than k . Moreover, $\chi(H_{4i}) = (c + \delta(4i))^{4i}$, where $c > 1$ and $\delta(i) \rightarrow 0$ as $i \rightarrow \infty$. This is all we need to prove since in any dimension of the form $4i + j$, $j = 1, 2, 3$ we can consider a plane of codimension j and embed an isometric copy of H_{4i} there. As a result we obtain a sequence of graphs with desired properties in all dimensions.

It is easy to see that

$$|V_{4n}| = \binom{4n}{2n} = (2 + \bar{o}(1))^{4n} \quad \text{and} \quad |E_{4n}| = \binom{4n}{2n} \binom{2n}{n}^2 = (4 + \bar{o}(1))^{4n}.$$

We will use the following result from the paper [11]:

THEOREM 2. *For any $\epsilon > 0$ there exists $\delta > 0$ such that for any subset S of V_{4n} , $|S| \geq (2 - \delta)^{4n}$, the number of edges in S (the cardinality of $E_{4n}|_S$) is greater than $(4 - \epsilon)^{4n}$.*

Remark 2. We do not give any numerical bounds for ξ_k since they are very difficult to derive. The reason is that we use Theorem 2, in which there is no explicit dependency between ϵ and δ .

We will also need Lovász Local Lemma (see [1]):

THEOREM 3. *Let A_1, \dots, A_m be events in an arbitrary probability space and $J(1), \dots, J(m)$ be subsets of $\{1, \dots, m\}$. Suppose there are real numbers γ_i such that $0 < \gamma_i < 1$, $i = 1, \dots, m$. Suppose the following conditions hold:*

1. A_i is independent of algebra generated by $\{A_j, j \notin J(i) \cup \{i\}\}$.

$$2. \ P(A_i) \leq \gamma_i \prod_{j \in J(i)} (1 - \gamma_j).$$

Then

$$P\left(\bigwedge_{i=1}^m \overline{A_i}\right) \geq \prod_{i=1}^m (1 - \gamma_i) > 0.$$

We will use the following version of local lemma (see [3]):

LEMMA 1. *Let A_1, \dots, A_m and $J(1), \dots, J(m)$ be as in Theorem 3. Suppose there are real numbers δ_i such that $0 < \delta_i P(A_i) < 0,69$, $i = 1, \dots, m$. Suppose the following condition holds:*

$$\ln \delta_i \geq \sum_{j \in J(i)} 2\delta_j P(A_j). \quad (1)$$

Then

$$P\left(\bigwedge_{i=1}^m \overline{A_i}\right) \geq \prod_{i=1}^m (1 - \delta_i P(A_i)) > 0.$$

PROOF. This form of local lemma is easy to derive from Theorem 3. We just need to verify that the inequality 2 from Theorem 3 follows from the inequality (1). Indeed, we have the following inequality:

$$\ln \delta_i \geq \sum_{j \in J(i)} 2\delta_j P(A_j) \geq \sum_{j \in J(i)} -\ln(1 - \delta_j P(A_j)),$$

since $\ln(1 - t) \geq -t - t^2 \geq -2t$ for $0 < t < 0,69$ (see [3]). We take an exponent of both sides:

$$\delta_i \geq \prod_{j \in J(i)} (1 - \delta_j P(A_j))^{-1}.$$

Finally, we substitute $\delta_i = \gamma_i/P(A_i)$. □

Recall that *the independence number* $\alpha(G)$ of a graph $G = (V, E)$ is the size of a maximum set $S \subset V$ such that for any $v, w \in S$ we have $\{v, w\} \notin E$.

2.2. Proof of Theorem 1

Fix natural numbers $k \geq 3$ and n . Let $\gamma \in (0, 1)$ be a constant that will be defined later, and set $p = \gamma^{4n}$. Consider a random subgraph G of the graph G_{4n} in which all edges are chosen independently and uniformly with the probability of each

edge to occur equal to p . Namely, we have the probability space $(\Omega_{4n}, \mathcal{B}_{4n}, P_{4n})$, where

$$\Omega_{4n} = \{G = (V_{4n}, E), \quad E \subseteq E_{4n}\}, \quad \mathcal{B}_{4n} = 2^{\Omega_{4n}},$$

$$P_{4n}(G) = p^{|E|}(1-p)^{|E_{4n}|-|E|} \quad \text{for } G = (V_{4n}, E).$$

Denote $N = |V_{4n}|$. We define two families of events on Ω_{4n} . Firstly, for some l we enumerate all l -element subsets of V_{4n} and introduce the events

$$X_i = \{\textit{i}th \textit{l}-element \textit{subset is independent}\}, \quad i = 1, \dots, C_N^l.$$

Secondly, for each $s = 3, \dots, k$ we enumerate all (labeled) cycles of length s in G_{4n} and introduce the events

$$Y_j^s = \{\textit{j}th \textit{s}-tuple is an \textit{s}-cycle\}, \quad j = 1, \dots, c_s(G_{4n}),$$

where $c_s(G_{4n})$ is the number of labeled s -cycles in G_{4n} .

Take $l = (2 - \delta)^{4n}$, where $\delta > 0$ is again some constant that will be defined later. The statement of Theorem 1 will follow from the inequality

$$P\left(\bigwedge_{i=1}^{C_N^l} \overline{X_i} \wedge \bigwedge_{s=3}^k \left(\bigwedge_{j=1}^{c_s(G_{4n})} \overline{Y_j^s}\right)\right) > 0. \tag{2}$$

Indeed, we obtain from (2) that there exists a subgraph G' in G_{4n} such that it does not contain cycles of length $\leq k$ and at the same time $\alpha(G') \leq l$. The above means that $G' \in \mathcal{G}(4n, k)$ and

$$\chi(G') \geq \frac{N}{l} = \left(\frac{2}{2-\delta} + \bar{o}(1)\right)^{4n} = \left(1 + \delta' + \bar{o}(1)\right)^{4n},$$

where δ' is a positive constant which will be seen to depend only on k .

To prove (2) we shall use Lemma 1. But before we apply it we have to estimate the probabilities of the events X_i, Y_i^s .

We start with X_i . Put $a_i = |E(G_{4n}|_{W_i})|$, where W_i is an i -th l -element subset of V_{4n} . In other words, a_i is the number of edges in W_i in the graph G_{4n} . Then

$$P(X_i) = (1-p)^{a_i} \leq e^{-pa_i} = e^{-\gamma^{4n} a_i}.$$

Fix $\epsilon = \epsilon(k) > 0$ and choose $\delta = \delta(\epsilon, k)$ as in Theorem 2. Then obviously $a_i \geq (4 - \epsilon)^{4n}$.

We go on to Y_i^s . It is easy to see that for each s -tuple Q_i^s we have

$$P(Y_i^s) = p^s = \gamma^{4ns}.$$

We also need to analyze the dependencies between the events.

For each X_i let us estimate the number of Y_j^s on which it may depend. Note that if X_i and Y_j^s are dependent then the corresponding sets W_i and Q_j^s must have a common edge. The number of ways to make an s -cycle out of a fixed edge is not bigger than $2^{(s-2)4n}$. Thus the number of Y_j^s on which X_i depends does not exceed $a_i 2^{(s-2)4n}$.

For all i, s , each Y_i^s and X_i depend on not more than C_N^l events X_j .

It remains to estimate for each $Y_i^{s_1}$ the number of the events $Y_j^{s_2}$ on which it depends. If they are dependent, $Q_i^{s_1}$ and $Q_j^{s_2}$ must have a common edge. Then it is easy to see that the number of such events does not exceed

$$s_1 2^{4n(s_2-2)} = 2^{4n(s_2-2)(1+\bar{\sigma}(1))}.$$

For each event $E \in \{X_i, Y_i^s\}$ we split the set $J(E)$ (see Lemma 1) into parts. First one ($J^x(E)$) contains all events of the type X_j . The other parts ($J_s^y(E)$) consist of the events of the type Y_j^s . We want to apply Lemma 1, so we rewrite the conditions (1) for our events:

$$\left\{ \begin{array}{l} (X_i) \quad \ln \delta_i^x \geq 2 \sum_{j \in J^x(X_i)} \delta_j^x e^{-\gamma^{4n} a_j} + 2 \sum_{s=3}^k \sum_{j \in J_s^y(X_i)} \delta_j^y(s) \gamma^{4ns}, \\ (Y_i^{s_1}) \quad \ln \delta_i^{y(s_1)} \geq 2 \sum_{j \in J^x(Y_i^{s_1})} \delta_j^x e^{-\gamma^{4n} a_j} + 2 \sum_{s=3}^k \sum_{j \in J_s^y(Y_i^{s_1})} \delta_j^y(s) \gamma^{4ns}. \end{array} \right. \quad (3)$$

Fix a constant $f = f(\epsilon, \delta, k) > 0$, which will be defined later, and put

$$\delta_i^y(s) = e, \quad \delta_i^x = e^{\gamma^{4n(1+f)} a_i}.$$

It is easy to see that for sufficiently large n we have

$$0 < \delta_i^x P(X_i) < 0,69, \quad 0 < \delta_i^y(s) P(Y_i^s) < 0,69$$

(see Lemma 1). Then for any j

$$\delta_j^x e^{-\gamma^{4n} a_j} = e^{\gamma^{4n(1+f)} a_j - \gamma^{4n} a_j} = e^{-(\gamma - \bar{o}(1))^{4n} a_j} \leq e^{-(\gamma - \bar{o}(1))^{4n} (4-\epsilon)^{4n}} = e^{-((4-\epsilon)\gamma - \bar{o}(1))^{4n}}.$$

We also have

$$C_N^l \leq \left(\frac{eN}{l} \right)^l \leq \left(\frac{2}{2-\delta} + \bar{o}(1) \right)^{4n(2-\delta)^{4n}} = e^{(2-\delta + \bar{o}(1))^{4n}}.$$

Thus for any i

$$\begin{aligned} \sum_{j \in J^x(X_i)} \delta_j^x e^{-\gamma^{4n} a_j} &\leq \sum_{j \in J^x(X_i)} e^{-((4-\epsilon)\gamma - \bar{o}(1))^{4n}} \leq \\ &\leq C_N^l e^{-((4-\epsilon)\gamma - \bar{o}(1))^{4n}} \leq e^{(2-\delta + \bar{o}(1))^{4n} - ((4-\epsilon)\gamma - \bar{o}(1))^{4n}} = \bar{o}(1), \end{aligned}$$

if

$$\gamma > \frac{2-\delta}{4-\epsilon}. \quad (4)$$

Similarly, if (4) holds, then

$$\sum_{j \in J^x(Y_i^s)} \delta_j^x e^{-\gamma^{4n} a_j} = \bar{o}(1).$$

Thereby, if we suppose that (4) holds, then the inequalities (3) will follow from the system

$$\begin{cases} (X_i) & \gamma^{4n(1+f)} a_i \geq 2 \sum_{s=3}^k e a_i 2^{4n(s-2)(1+\bar{o}(1))} \gamma^{4ns}, \\ (Y_i^{s_1}) & 1 \geq 2 \sum_{s=3}^k e 2^{4n(s-2)(1+\bar{o}(1))} \gamma^{4ns}. \end{cases} \quad (5)$$

One can see that since $\gamma < 1$, both inequalities of (5) are consequences of the following. For any function $g(n) = \bar{o}(1)$, any $s = 3, \dots, k$ and all sufficiently big n should hold

$$2^{4n(s-2)(1+g(n))} \gamma^{4n(s-1-f)} = \bar{o}(1). \quad (6)$$

In turn, to prove this it is enough to check the inequality

$$s - 2 + (s - 1 - f) \log_2 \gamma < 0 \tag{7}$$

for $s = 3, \dots, k$. Any γ ,

$$0 < \gamma < 2^{-\frac{k-2}{k-1-f}}, \tag{8}$$

satisfies (7), and also the system (5). Therefore, the system (3) is satisfied if both (4) and (8) hold:

$$\frac{2 - \delta}{4 - \epsilon} < \gamma < 2^{-\frac{k-2}{k-1-f}}. \tag{9}$$

Lastly, we can choose the parameters. We choose $\epsilon = \epsilon(k)$ so that

$$\frac{2}{4 - \epsilon} < 2^{-\frac{k-2}{k-1}}.$$

Then we choose $f = f(\epsilon, \delta, k)$ small enough so that

$$\frac{2 - \delta}{4 - \epsilon} < 2^{-\frac{k-2}{k-1-f}}.$$

Finally, we choose

$$\gamma = \gamma(f, \epsilon, \delta, k) \in \left(\frac{2 - \delta}{4 - \epsilon}, 2^{-\frac{k-2}{k-1-f}} \right).$$

We have verified all the conditions of Lemma 1, hence the inequality (2) holds and Theorem 1 is proved.

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