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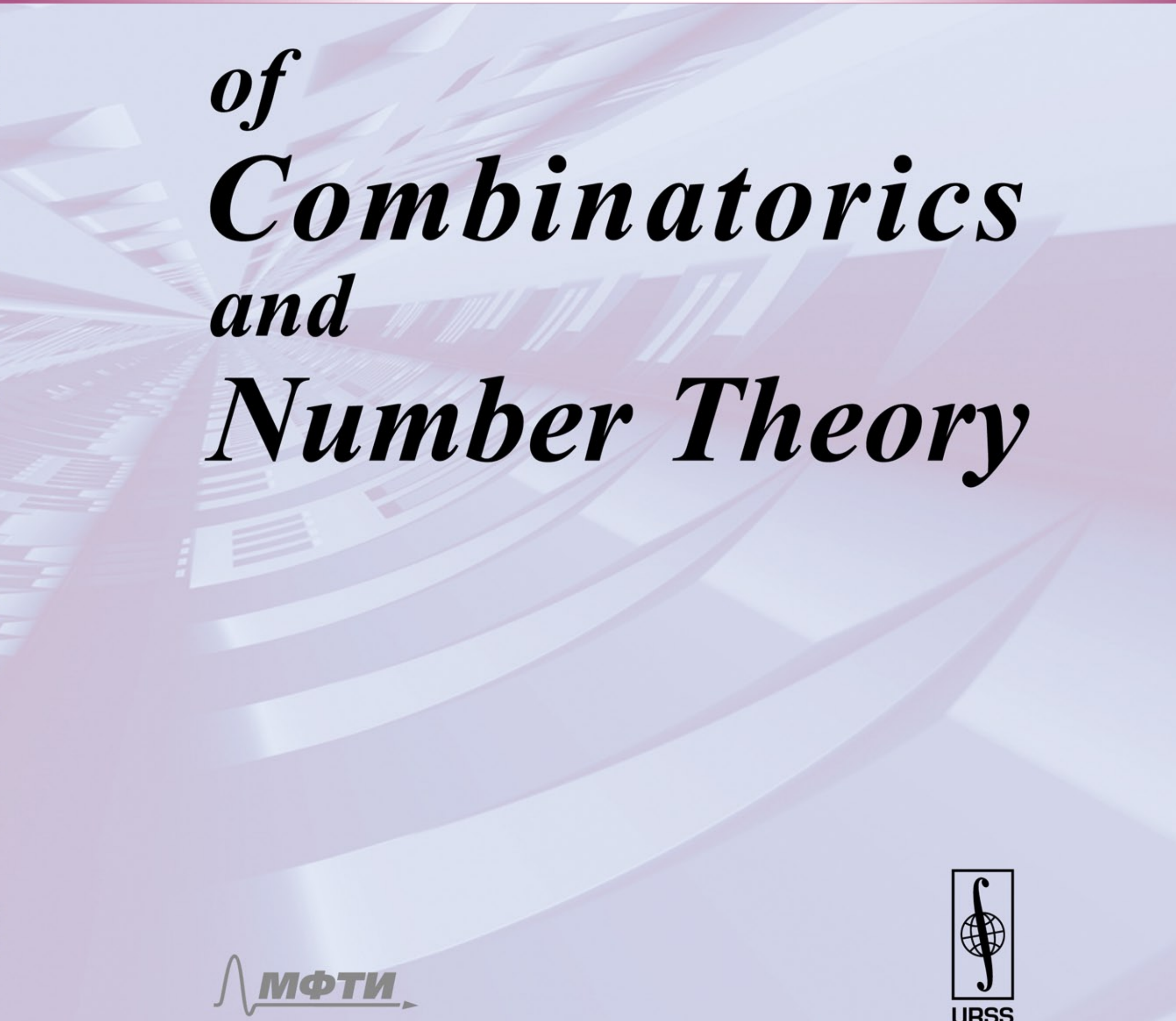

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The distribution of second degrees in the Bollobás–Riordan random graph model

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Abstract: We consider a random graph process which was suggested by Barabási and Albert. This process is a model of real-world networks. At each step we add one vertex and one edge. The probability that a new vertex will be connected to some vertex v is proportional to the degree of v. In [6] Bollobás and Riordan proved that in this model the degree sequence has a power law distribution. Here we consider so-called second degrees of vertices. Roughly speaking, a second degree of a vertex is the number of vertices at a distance two of this vertex. The distribution of second degrees is of interest because it is a good approximation of PageRank (we are not only interested in the degree of a vertex but also in popularity of its neighbors). We obtain that the distribution of second degrees also obeys a power law.

Keywords: random graphs; preferential attachment; power law distribution; second degrees

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1. Introduction

In this paper we consider some properties of random graphs. The standard random graph model $\mathfrak{G}(n, m)$ was introduced by Erdős and Rényi in [8]. In this model we randomly choose one graph from all graphs with n vertices and m edges. The similar model $\mathfrak{G}(n, p)$ was suggested by Gilbert in [10]. Here n vertices are joined independently with probability $0 < p < 1$. Many papers deal with the classical models. The main results can be found in [4], [9], [12].

Recently there has been interest in modeling complex real-world networks. Real structures differ from standard random graphs. One of the main characteristics of random graphs is their degree sequence. In many real-world structures the degree sequence has a power law distribution. Standard random graph models do not have this property. So Barabási and Albert suggested a new model in [2]. Then Bollobás and Riordan gave more precise definition of this model. Many models of real-world networks and main results can be found in [5].

Many papers deal with different variations of preferential attachment. We mention here the paper by Rudas, Tóth and Valkó (see [13]). The authors consider a quite generic model of a random tree and prove some interesting results concerning a neighborhood structure of a random vertex. Also one can find a neighborhood analysis in preferential attachment models in the preprint [3] on the weak graph limit.

This paper deals with the Bollobás–Riordan model. Now let us describe this model. Let n be a number of vertices in our graph and m be a fixed parameter. We begin with the case $m = 1$. We inductively construct a random graph G_1^n. Start with G_1^1 the graph with one vertex and one loop. Similarly we can start with G_1^0 the graph with no vertices. Assume that we already constructed the graph G_1^{t-1}. At the next step we add one vertex t and one edge between vertices t and i, where i is chosen randomly with

$$\mathrm{P}(i = s) = \begin{cases} d_{G_1^{t-1}}(s)/(2t-1) & \text{if } 1 \leqslant s \leqslant t-1, \\ \\ 1/(2t-1) & \text{if } s = t. \end{cases}$$

Here $d_{G_1^t}(s)$ is the degree of the vertex s in G_1^t. By $d(s)$ denote the degree of s in the graph G_1^n. In other words, the probability that a new vertex will be connected to the vertex i is proportional to the current degree of i. Therefore this process is said to be *preferential attachment*. To obtain G_m^n with $m > 1$ we construct G_1^{mn}. Then we identify the vertices $1, \ldots, m$ to form the first vertex; we identify the vertices $m+1, \ldots, 2m$ to form the second vertex; and so on. After this procedure, edges from G_1^n connect «big» vertices in G_m^n. Let $\mathfrak{G}_m^n$ be the probability space of constructed graphs.

Many papers deal with the Bollobás–Riordan model. The diameter of this random graph was considered in [7]. In [6] Bollobás and Riordan proved that the degree sequence has a power law distribution.

Theorem 1. *If $m \geqslant 1$ is fixed, then there exists a function $\varphi(n) = o(n)$ such that for any $m \leqslant d \leqslant n^{1/15}$ we have*

$$\lim_{n\to\infty} \mathrm{P}\left(\left|\#_m^n(d) - \frac{2nm(m+1)}{d(d+1)(d+2)}\right| > \frac{\varphi(n)}{d(d+1)(d+2)}\right) = 0.$$

Here $\#_m^n(d)$ is the number of vertices in G_m^n with degree equal to d.

Recently Grechnikov substantially improved Theorem 1 (see [11]).

In this paper we consider second degrees of vertices in G_1^n. We estimate the expectation of the number of vertices with second degree equal to d. Also we prove a concentration result. This paper is organized as follows. In section 2 we give main definitions and results. In section 3 we prove all theorems.

2. Definitions and results

In this paper we study the random graph G_1^n. When we write $ij \in G_1^n$ we mean that G_1^n has the edge ij; when we write $t \in G_1^n$ we simply mean that t is a vertex of G_1^n. Given a vertex $t \in G_1^n$ we say that the *second degree* of the vertex t is

$$d_2(t) = \#\{ij : i \neq t, j \neq t, it \in G_1^n, ij \in G_1^n\}.$$

In other words, the second degree of t is the number of edges adjacent to the neighbors of t except for the edges adjacent to the vertex t.

Let $M_n^1(d)$ be the expectation of the number of vertices with degree d in G_1^n:

$$M_n^1(d) = \mathsf{E}\big(\#\{t \in G_1^n : d_{G_1^n}(t) = d\}\big).$$

By $X_n(d)$ denote the number of vertices with second degree d in G_1^n. By definition, put $M_n^2(d) = \mathsf{E}X_n(d)$.

The aim of this paper is to prove the following results.

THEOREM 2. *For any* $k > 1$ *we have*

$$M_n^2(k) = \frac{4n}{k^2}\left(1 + O\left(\frac{\log^2 k}{k}\right) + O\left(\frac{k^2}{n}\right)\right).$$

THEOREM 3. *For any* $\varepsilon > 0$ *there exists a function* $\varphi(n) = o(n)$ *such that*

$$\lim_{n\to\infty} \mathrm{P}\left(|X_n(k) - M_n^2(k)| \geqslant \frac{\varphi(n)}{k^2}\right) = 0$$

for any $1 \leqslant k \leqslant n^{1/6-\varepsilon}$.

This is a concentration result which means that the distribution of second degrees does also obey (asymptotically) a power law.

To prove Theorem 2, we need the following definition. Let $N_n(l, k)$ be the number of vertices in G_1^n with degree l, with second degree k, and without loops:

$$N_n(l, k) = \#\{t \in G_1^n : d(t) = l, d_2(t) = k, tt \notin G_1^n\}.$$

We shall prove the following theorem.

THEOREM 4. *In* G_1^n *we have*

$$\mathrm{E}N_n(l, k) = n\, c(l, k)\, (1 + \theta(n, l, k)),$$

where $|\theta(n, l, k)| < (2l + k - 1)^2/n$. *The constants* $c(l, k)$ *are defined as follows*:

$$\begin{aligned} c(l, 0) &= c(0, k) = 0, \\ c(1, k) &= \frac{2k^2 + 14k}{(k+1)(k+2)(k+3)(k+4)}, \\ c(l, k) &= c(l, k-1)\frac{l+k-1}{2l+k+2} + c(l-1, k)\frac{l-1}{2l+k+2}, \quad k > 0, \quad l > 1. \end{aligned}$$

We shall use the following lemmas to prove these theorems.

LEMMA 1. *Let* $d \geqslant 1$ *be natural*; *then*

$$M_n^1(d) = \frac{4n}{d(d+1)(d+2)}\left(1 + \widetilde{\theta}(n, d)\right),$$

where $|\widetilde{\theta}(n, d)| < d^2/n$.

Denote by $P_n(l, k)$ the number of vertices in G_1^n with a loop, with degree l, and with second degree k.

Lemma 2. *For any* n *we have*

$$\mathsf{E}P_n(l, k) \leqslant p(l, k),$$

where

$$p(2, 0) = 1, \quad p(l, k) = p(l, k-1)\frac{l+k-3}{2l+k-2} + p(l-1, k)\frac{l-1}{2l+k-2},$$

$$l \geqslant 3, \quad k \geqslant 0.$$

For the other values of l *and* k *we have* $p(l, k) = 0$.

The next section is organized as follows. First we prove Theorem 4 and Theorem 2; then we prove the lemmas. Finally we give a proof of Theorem 3.

3. Proofs

3.1. Proof of Theorem 4

From the definition of G_1^n it follows that $N_n(l, 0) = N_n(0, k) = 0$. Indeed, since we have no vertices of degree 0, we see that $N_n(0, k) = 0$. Since vertices with loops are not counted in $N_n(l, k)$, it follows that we have no vertices of second degree 0 and $N_n(l, 0) = 0$. Therefore we have $\mathsf{E}N_n(l, 0) = \mathsf{E}N_n(0, k) = 0$.

Let us prove that $\mathsf{E}N_n(1, k) = n\, c(1, k)\, (1+\theta(n, 1, k))$. The proof is by induction on k. For $k = 0$ there is nothing to prove. Now assume that for $j < k$ we have

$$\mathsf{E}N_n(1, j) = n\, c(1, j)\, (1 + \theta(n, 1, j)),$$

where

$$|\theta(n, 1, j)| < \frac{(j+1)^2}{n},$$

$$c(1, j) = \frac{2j^2 + 14j}{(j+1)(j+2)(j+3)(j+4)}.$$

Denote by $N_i(l)$ the number of vertices with degree l in G_1^i.

We need some additional notation. Let X be a function on n (the number of vertices), l (the first degree we are interested in), k (the second degree we are interested in); then denote by $\theta_1(X)$, $\theta_2(X)$, $\theta_3(X), \ldots$ some functions on n, l, k such that $|\theta_i(X)| < X$.

Obviously, $\mathsf{E}N_1(1,k) = 0$. For $i \geqslant 1$ we have

$$\mathsf{E}(N_{i+1}(1,k)|N_i(1,k), N_i(1,k-1), N_i(k)) =$$

$$= N_i(1,k)\left(1 - \frac{k+2}{2i+1}\right) + \frac{kN_i(1,k-1)}{2i+1} + \frac{k\,N_i(k)}{2i+1}. \qquad (1)$$

Let us explain this equality. Suppose we have G_1^i. We add one vertex and one edge. There are $N_i(1,k)$ vertices with degree 1 and with second degree k in G_1^i. The probability that we «spoil» one of these vertices is $(k+2)/(2i+1)$. Also we have $N_i(1,k-1)$ vertices with degree 1 and with second degree $k-1$. The probability that one of these vertices has degree 1 and second degree k in G_1^{i+1} is $k/(2i+1)$. Finally, with probability equal to $kN_i(k)/(2i+1)$ the vertex $i+1$ has necessary degrees in G_1^{i+1}.

Using (3.1), Lemma 1, and inductive assumption we get

$$\mathsf{E}N_{i+1}(1,k) = \mathsf{E}N_i(1,k)\frac{2i-k-1}{2i+1} + \frac{k\mathsf{E}N_i(1,k-1)}{2i+1} + \frac{kM_i^1(k)}{2i+1} =$$

$$= \mathsf{E}N_i(1,k)\frac{2i-k-1}{2i+1} + \left(\frac{ikc(1,k-1)}{2i+1} + \frac{4i}{(2i+1)(k+1)(k+2)}\right)\left(1+\theta_1\left(\frac{k^2}{i}\right)\right).$$

Let us introduce some notation:

$$a_i = \frac{2i-k-1}{2i+1},$$

$$b_i = \frac{2i}{2i+1}\left(1+\theta_1\left(\frac{k^2}{i}\right)\right),$$

$$m = \frac{c(1,k-1)\,k}{2} + \frac{2}{(k+1)(k+2)}.$$

Using this notation, we have

$$\mathsf{E}N_{i+1}(1,k) = \mathsf{E}N_i(1,k)\,a_i + m\,b_i.$$

Let us prove the following equality by induction on n:

$$\mathsf{E}N_n(1,k) = \frac{2mn}{k+4}\left(1+\theta(n,1,k)\right).$$

For $n=1$ we have $\mathsf{E}N_1(1,k)=0$. Since we have the condition $|\theta(1,1,k)| < (k+1)^2$, we can take $\theta(1,1,k) = -1$.

Now put $t = k+1$. This is needed for the sequel. Assume that

$$\mathsf{E}N_i(1,k) = \frac{2mi}{t+3}\left(1+\theta(i,1,t-1)\right).$$

Then

$$\mathsf{E}N_{i+1}(1,k) = \mathsf{E}N_i(1,k)\,a_i + m\,b_i =$$

$$= \frac{2mi(2i-t)}{(2i+1)(t+3)}\left(1+\theta_2\left(\frac{t^2}{i}\right)\right) + \frac{2mi}{2i+1}\left(1+\theta_1\left(\frac{(t-1)^2}{i}\right)\right) =$$

$$= \frac{2m}{t+3}\left(i+1-\frac{1}{2i+1}+\theta_3\left(\frac{(2i-t)t^2}{2i+1}\right)+\theta_4\left(\frac{(t-1)^2(t+3)}{2i+1}\right)\right).$$

If $t \geqslant 1$ and $2i-t \geqslant 0$, then

$$\frac{1}{2i+1}+\frac{t^2|2i-t|}{2i+1}+\frac{(t-1)^2(t+3)}{2i+1} < t^2.$$

Therefore,

$$\mathsf{E}N_{i+1}(1,k) = \frac{2m(i+1)}{t+3}\left(1+\theta_5\left(\frac{t^2}{i+1}\right)\right).$$

In this case, we can put $\theta(i+1,1,k) = \theta_5\big(t^2/(i+1)\big)$.

If $t \geqslant 1$ and $2i-t \leqslant -2$, then we do not have enough edges in G_1^i and $\mathsf{E}N_{i+1}(l,k) = 0$. In this case, we can put $\theta(i+1,l,k) = -1$.

We consider the case $2i-t=-1$ later.

We get

$$\mathsf{E}N_n(1,k) = \frac{2mn}{k+4}\left(1+\theta(n,1,k)\right).$$

Note that

$$\frac{2m}{k+4} = \frac{4}{(k+1)(k+2)(k+4)} + \frac{2\,c(1,k-1)\,k}{2(k+4)} =$$

$$= \frac{4}{(k+1)(k+2)(k+4)} + \frac{2(k-1)^2+14(k-1)}{(k+1)(k+2)(k+3)(k+4)} =$$

$$= \frac{2k^2+14k}{(k+1)(k+2)(k+3)(k+4)} = c(1,k).$$

This completes the proof for $\mathrm{E}N_n(1,k)$.

Consider the case $l, k > 1$. Assume that we already proved that $\mathrm{E}N_n(i,j) = n\,c(i,j)\,(1+\theta(n,i,j))$ for all i and j, such that $i < l$, $j \leqslant k$ or $i \leqslant l$, $j < k$. Put $t = 2l+k-1$. Obviously, $\mathrm{E}N_1(l,k) = 0$. For $i \geqslant 1$ we have

$$\mathrm{E}N_{i+1}(l,k)=\mathrm{E}N_i(l,k)\left(1-\frac{2l+k}{2i+1}\right)+\frac{(l-1)\mathrm{E}N_i(l-1,k)}{2i+1}+\frac{(l+k-1)\mathrm{E}N_i(l,k-1)}{2i+1}=$$

$$=\mathrm{E}N_i(l,k)\frac{2i-t}{2i+1}+\left(\frac{(l-1)c(l-1,k)i}{2i+1}+\frac{(l+k-1)c(l,k-1)i}{2i+1}\right)\left(1+\theta_1\left(\frac{(t-1)^2}{i}\right)\right).$$

Introduce some notation:

$$a_i = \frac{2i-t}{2i+1},$$

$$b_i = \frac{2i}{2i+1}\left(1+\theta_1\left(\frac{(t-1)^2}{i}\right)\right),$$

$$m = \frac{(l-1)\,c(l-1,k)}{2} + \frac{(l+k-1)\,c(l,k-1)}{2}.$$

We have

$$\mathrm{E}N_{i+1}(l,k) = \mathrm{E}N_i(l,k)\,a_i + m\,b_i.$$

It remains to prove the following statement by induction on n:

$$\mathrm{E}N_n(l,k) = \frac{2mn}{t+3}\left(1+\theta_5\left(\frac{t^2}{n}\right)\right) = \frac{2mn}{t+3}\left(1+\theta(n,l,k)\right).$$

The proof is the same as in the case of $l = 1$. In this case we have

$$\frac{2m}{t+3} = \frac{(l-1)\,c(l-1,k)}{2l+k+2} + \frac{(l+k-1)\,c(l,k-1)}{2l+k+2} = c(l,k).$$

Now we need to consider only the case $2i - t = -1$. We need to show that $\mathsf{E}N_{i+1}(l,k) = (i+1)c(l,k)(1+\theta(i+1,l,k))$. We have $2(i+1) = 2l+k$. In our graph G_1^{i+1} we have $i+1$ edges. Therefore the sum of all degrees is equal to $2l+k$. Suppose we have at least one vertex with degree l and second degree k. We do not count vertices with a loop in $N_{i+1}(l,k)$. Consequently l edges go out from this vertex. And there are $k/2$ edges between the neighbors of our vertex. And we have no other edges. Hence our vertex is joined to all other vertices in G_1^{i+1}. So $l = i$. Thus $k = 2$. It follows that we consider the vertex 2. And there is one edge from the vertex 2 to the vertex 1; also edges from the vertices $3, \ldots, i+1$ go to the vertex 2. So, there is only one graph with $N_{i+1}(l,k) \neq 0$. This graph has only one vertex with degree l and second degree k. Therefore the probability of this graph is equal to $\mathsf{E}N_{l+1}(l,2)$. We have

$$\mathsf{E}N_{l+1}(l,2) = \frac{2(l-1)!}{(2l+1)!!}.$$

Recall that $l = i$ and $k = 2$. Now we must only prove that

$$\mathsf{E}N_{l+1}(l,2) = (l+1)c(l,2)(1+\theta(l+1,l,2)).$$

Let us prove the inequality

$$c(l,2) \geqslant \frac{24(l-1)!}{5(2l+4)!!}.$$

It follows from the definition of $c(l,k)$ that

$$c(1,2) = \frac{1}{10},$$

$$c(l,2) \geqslant c(l-1,2)\frac{l-1}{2l+4}, \quad l \geqslant 2.$$

Obviously, $\theta(l+1,l,2) \geqslant -1$. Let us obtain the following upper bound:

$$\theta(l+1,l,2)+1 = \frac{\mathsf{E}N_{l+1}(l,2)}{(l+1)c(l,k)} \leqslant \frac{2(l-1)!5(2l+4)!!}{(2l+1)!!(l+1)24(l-1)!} =$$
$$= \frac{5(2l+4)!!}{12(2l+1)!!(l+1)} \leqslant \frac{(2l+1)^2}{(l+1)}.$$

This completes the proof.

3.2. Proof of Theorem 2

From Theorem 4 we have the constants $c(l,k)$. Imagine that we have a table with $c(l,k)$, where l is the number of a row and k is the number of a column. The sum of all numbers in the table is equal to 1. The sum of numbers in l-th row is equal to

$$\frac{4}{l(l+1)(l+2)}.$$

It can easily be checked using the definition of $c(l,k)$. But we need to calculate $M_n^2(k)$, so we are interested in the sum of all numbers in k-th column. More precisely,

$$M_n^2(k) = \sum_{l=1}^{\infty} \mathsf{E}N_n(l,k) + \sum_{l=1}^{\infty} \mathsf{E}P_n(l,k).$$

First we estimate $\sum_{l=1}^{\infty} c(l,k)$. Recall that

$$c(l,0) = 0,$$

$$c(1,k) = \frac{2k^2+14k}{(k+1)(k+2)(k+3)(k+4)},$$

$$c(l,k) = c(l,k-1)\frac{l+k-1}{2l+k+2} + c(l-1,k)\frac{l-1}{2l+k+2}, \quad k>0, l>1.$$

Note that there exists a function $C(k) \geqslant 0$ such that for all $l \geqslant k \geqslant 0$ and $l \geqslant 1$ the inequality

$$c(l,k) \leqslant C(k)2^{-l}\frac{(l-1)!}{(l-k)!} \tag{2}$$

holds. Indeed, the case of $k = 0$ is obvious with $C(k) = 0$. In the case of $k \geqslant 1$ we define $C(k)$ so that $C(k) \geqslant C(k-1)$ and (2) holds for $l = k$. We have

$$(2l + k + 2)\frac{c(l, k)}{C(k)} \leqslant \frac{C(k-1)}{C(k)}(l + k - 1)2^{-l}\frac{(l-1)!}{(l-k+1)!} + 2^{-l+1}\frac{(l-1)!}{(l-k-1)!} \leqslant$$

$$\leqslant 2^{-l}\frac{(l-1)!}{(l-k)!}\left(\frac{l+k-1}{l-k+1} + 2(l-k)\right) \leqslant (2l + k + 2)2^{-l}\frac{(l-1)!}{(l-k)!}.$$

This proves (2).

In particular, the series

$$\sum_{l=1}^{\infty} l^N c(l, k)$$

converges for all N and k.

Let us make some transformations:

$$(2l + k + 2)c(l, k) = (l + k - 1)c(l, k - 1) + (l - 1)c(l - 1, k),$$

$$\sum_{l=2}^{\infty}(2l + k + 2)c(l, k) = \sum_{l=2}^{\infty}(l + k - 1)c(l, k - 1) + \sum_{l=1}^{\infty} lc(l, k),$$

$$\sum_{l=2}^{\infty}(l + k + 2)c(l, k) = \sum_{l=2}^{\infty}(l + k - 1)c(l, k - 1) + c(1, k).$$

Put

$$x_k = \sum_{l=2}^{\infty} c(l, k).$$

Then $x_0 = 0$ and for $k \geqslant 1$ we have

$$(k + 2)x_k = (k - 1)x_{k-1} + c(1, k) + \sum_{l=2}^{\infty} l(c(l, k - 1) - c(l, k)),$$

$$(k + 2)(k + 1)kx_k = (k - 1)(k + 1)kx_{k-1} + (k + 1)kc(1, k) +$$

$$+ \sum_{l=2}^{\infty} l(k(k + 1)c(l, k - 1) - k(k + 1)c(l, k)),$$

$$
\begin{aligned}
&(k+2)(k+1)kx_k = \\
&= \sum_{s=1}^{k}(s(s+1)(s+2)x_s - (s-1)s(s+1)x_{s-1}) = \sum_{s=1}^{k} s(s+1)c(1,s) + \\
&\quad + \sum_{l=2}^{\infty} l\left(\sum_{s=1}^{k}(s(s+1)c(l,s-1) - s(s+1)c(l,s))\right) = \sum_{s=1}^{k} s(s+1)c(1,s) + \\
&\quad + \sum_{l=2}^{\infty} l\left(\sum_{s=1}^{k}((s+1)(s+2) - s(s+1))c(l,s) - (k+1)(k+2)c(l,k)\right) = \\
&= \sum_{s=1}^{k} s(s+1)c(1,s) + \sum_{l=2}^{\infty} l\left(\sum_{s=1}^{k} 2(s+1)c(l,s) - (k+1)(k+2)c(l,k)\right),
\end{aligned}
$$

$$
\begin{aligned}
x_k = &\frac{1}{k(k+1)(k+2)}\sum_{s=1}^{k} s(s+1)c(1,s) + \\
&+ \frac{2}{k(k+1)(k+2)}\sum_{l=2}^{\infty} l\left(\sum_{s=1}^{k}(s+1)c(l,s)\right) - \frac{1}{k}\sum_{l=2}^{\infty} lc(l,k). \qquad (3)
\end{aligned}
$$

Put

$$y_k = \sum_{l=2}^{\infty} lc(l,k).$$

Then

$$x_k = \frac{1}{k(k+1)(k+2)}\sum_{s=1}^{k} s(s+1)c(1,s) + \frac{2}{k(k+1)(k+2)}\sum_{s=1}^{k}(s+1)y_s - \frac{1}{k}y_k.$$

Make some transformations:

$$(2l+k+2)lc(l,k) = (l+k-1)lc(l,k-1) + l(l-1)c(l-1,k),$$

$$\sum_{l=2}^{\infty}(2l+k+2)lc(l,k) = \sum_{l=2}^{\infty}(l+k-1)lc(l,k-1) + \sum_{l=1}^{\infty} l(l+1)c(l,k),$$

$$\sum_{l=2}^{\infty}(l+k+1)lc(l,k) = \sum_{l=2}^{\infty}(l+k-1)lc(l,k-1) + 2c(1,k),$$

$$ky_k + \sum_{l=2}^{\infty}(l+1)lc(l,k) = (k-2)y_{k-1} + \sum_{l=2}^{\infty} l(l+1)c(l,k-1) + 2c(1,k),$$

$$k(k-1)y_k = \sum_{s=1}^{k}(s(s-1)y_s - (s-1)(s-2)y_{s-1}) =$$

$$= \sum_{s=1}^{k}\left((s-1)\sum_{l=2}^{\infty} l(l+1)c(l,s-1) - (s-1)\sum_{l=2}^{\infty}(l+1)lc(l,s) + 2(s-1)c(1,s)\right) =$$

$$= 2\sum_{s=1}^{k}(s-1)c(1,s) + \sum_{l=2}^{\infty} l(l+1)\sum_{s=1}^{k} c(l,s) - k\sum_{l=2}^{\infty} l(l+1)c(l,k).$$

For $k \geqslant 2$ we have

$$y_k = \frac{2}{k(k-1)}\sum_{s=1}^{k}(s-1)c(1,s) + $$
$$+ \frac{1}{k(k-1)}\sum_{l=2}^{\infty} l(l+1)\sum_{s=1}^{k} c(l,s) - \frac{1}{k-1}\sum_{l=2}^{\infty} l(l+1)c(l,k).$$

Let

$$z_k = \sum_{l=2}^{\infty} l(l+1)c(l,k).$$

Then for $k \geqslant 2$

$$y_k = \frac{2}{k(k-1)}\sum_{s=1}^{k}(s-1)c(1,s) + \frac{1}{k(k-1)}\sum_{s=1}^{k} z_s - \frac{1}{k-1} z_k.$$

Make similar transformations

$$(2l+k+2)l(l+1)c(l,k) = (l+k-1)l(l+1)c(l,k-1) + (l+1)l(l-1)c(l-1,k),$$

$$\sum_{l=2}^{\infty}(2l+k+2)l(l+1)c(l,k) = \sum_{l=2}^{\infty}(l+k-1)l(l+1)c(l,k-1) + \sum_{l=1}^{\infty} l(l+1)(l+2)c(l,k),$$

$$\sum_{l=2}^{\infty}(l+k)l(l+1)c(l,k) = \sum_{l=2}^{\infty}(l+k-1)l(l+1)c(l,k-1) + 6c(1,k),$$

$$\sum_{s=1}^{k}\sum_{l=2}^{\infty}(l+s)l(l+1)c(l,s)=\sum_{s=0}^{k-1}\sum_{l=2}^{\infty}(l+s)l(l+1)c(l,s)+\sum_{s=1}^{k}6c(1,s),$$

$$\sum_{l=2}^{\infty}(l+k)l(l+1)c(l,k)=\sum_{s=1}^{k}6c(1,s).$$

Since

$$c(1,s)=O\left(\frac{1}{s^2}\right),$$

we have

$$0\leqslant z_k\leqslant\frac{1}{k}\sum_{l=2}^{\infty}(l+k)l(l+1)c(l,k)=O\left(\frac{1}{k}\sum_{s=1}^{k}\frac{1}{s^2}\right)=O\left(\frac{1}{k}\right),$$

$$\sum_{s=1}^{k}(s-1)c(1,s)=O\left(\sum_{s=1}^{k}\frac{1}{s}\right)=O(\log k),$$

$$y_k=O\left(\frac{\log k}{k^2}\right),$$

$$\sum_{s=1}^{k}(s+1)y_s=O\left(\sum_{s=1}^{k}\frac{\log s}{s}\right)=O(\log^2 k),$$

$$x_k=\frac{1}{k(k+1)(k+2)}\sum_{s=1}^{k}s(s+1)c(1,s)+O\left(\frac{\log^2 k}{k^3}\right).$$

Finally,

$$c(1,s)=\frac{2}{s(s+1)}+O\left(\frac{1}{s^3}\right),\quad\text{so}\quad\sum_{s=1}^{k}s(s+1)c(1,s)=2k+O(\log k)$$

and

$$x_k=\frac{2}{(k+1)(k+2)}+O\left(\frac{\log^2 k}{k^3}\right)=\frac{2}{k^2}+O\left(\frac{\log^2 k}{k^3}\right),$$

$$\sum_{l=1}^{\infty}c(l,k)=c(1,k)+x_k=\frac{4}{k^2}+O\left(\frac{\log^2 k}{k^3}\right).$$

Now we can estimate $M_n^2(k)$:

$$M_n^2(k) = \sum_{l=1}^{\infty} c(l,k)\, n(1+\theta(n,l,k)) + \sum_{l=1}^{\infty} \mathsf{E} P_n(l,k).$$

The first sum:

$$\sum_{l=1}^{\infty} c(l,k)\, n = \frac{4n}{k^2} + O\left(\frac{n\log^2 k}{k^3}\right).$$

The second sum:

$$\sum_{l=1}^{\infty} c(l,k)\, n\, |\theta(n,l,k)| \leqslant \sum_{l=1}^{\infty} c(l,k)\, (2l+k)^2 =$$

$$= \sum_{l=1}^{\infty} 4l^2 c(l,k) + \sum_{l=1}^{\infty} 4lkc(l,k) + \sum_{l=1}^{\infty} k^2 c(l,k) =$$

$$= 4c(1,k) + \sum_{l=2}^{\infty} 4l(l+1)c(l,k) - \sum_{l=2}^{\infty} 4lc(l,k) +$$

$$+ 4kc(1,k) + \sum_{l=2}^{\infty} 4lkc(l,k) + \sum_{l=1}^{\infty} k^2 c(l,k) =$$

$$= (4+4k)c(1,k) + 4z_k + (4k-4)y_k + \sum_{l=1}^{\infty} k^2 c(l,k) =$$

$$= O\left(\frac{1}{k} + \frac{1}{k} + \frac{\log k}{k} + \frac{\log^2 k}{k} + 1\right) = O\left(1\right).$$

The third sum:

$$\sum_{l=1}^{\infty} \mathsf{E} P_n(l,k) \leqslant \sum_{l=1}^{\infty} p(l,k).$$

Recall that

$$p(2,0) = 1, \quad p(l,k) = p(l,k-1)\frac{l+k-3}{2l+k-2} + p(l-1,k)\frac{l-1}{2l+k-2},$$

$$k \geqslant 0, \quad l \geqslant 3.$$

For the other values of l and k we have $p(l, k) = 0$. We can estimate $p(l, k)$:

$$p(l, k) \leqslant \frac{6}{l(l+1)}.$$

Indeed, it is easy to check that the function $6/(l(l+1))$ follows the recurrent relation. So when $l = 2$ and $k = 0$ we use the fact that

$$p(l, k) = 1 \leqslant \frac{6}{l(l+1)},$$

and then we proceed by induction. Hence the series

$$\sum_{l=2}^{\infty} p(l, k)$$

converges. In other words,

$$\sum_{l=2}^{\infty} p(l, k) = O(1).$$

Therefore

$$M_n^2(k) = \frac{4n}{k^2} + O\left(\frac{n \log^2 k}{k^3}\right) + O(1) + O(1) = \frac{4n}{k^2}\left(1 + O\left(\frac{\log^2 k}{k}\right) + O\left(\frac{k^2}{n}\right)\right).$$

This completes the proof.

Now we must only prove Lemma 1 and Lemma 2.

3.3. Proof of Lemma 1

In [6] Bollobás and Riordan computed the expectation of the number of vertices with degree d. But they only looked at $d \leqslant n^{1/15}$ and proved that

$$\mathsf{E}M_n^1(d) \sim \frac{4n}{d(d+1)(d+2)}.$$

We are interested in $\mathsf{E}M_n^1(d)$ for any d. In addition, we want to estimate $|\widetilde{\theta}(n, d)|$. Therefore we compute $M_n^1(d)$ in this paper.

The proof is by induction on d. First we need to consider 2 cases: $d = 1$ and $d = 2$.

Consider the case $d = 1$. Obviously, $M_0^1(1) = 0$. Assume that

$$M_i^1(1) = \frac{2i}{3}\big(1 + \widetilde{\theta}(i, 1)\big).$$

Then

$$M_{i+1}^1(1) = M_i^1(1)\left(1 - \frac{1}{2i+1}\right) + \frac{2i}{2i+1} =$$

$$= \frac{2i}{3}\left(1 + \widetilde{\theta}(i, 1)\right)\frac{2i}{2i+1} + \frac{2i}{2i+1} = \frac{2}{3}\left(i + 1 - \frac{1}{2i+1} + \frac{2i^2}{2i+1}\widetilde{\theta}(i, 1)\right) =$$

$$= \frac{2(i+1)}{3}\left(1 - \frac{1}{(2i+1)(i+1)} + \frac{2i^2}{(2i+1)(i+1)}\widetilde{\theta}(i, 1)\right).$$

Put

$$\widetilde{\theta}(i+1, 1) = \frac{2i^2}{(2i+1)(i+1)}\widetilde{\theta}(i, 1) - \frac{1}{(2i+1)(i+1)}.$$

Note that

$$|\widetilde{\theta}(i+1, 1)| \leqslant \frac{2i}{(2i+1)(i+1)} + \frac{1}{(2i+1)(i+1)} \leqslant \frac{1}{i+1}.$$

This completes the proof for $d = 1$.

The case $d = 2$ is somewhat different. Obviously, $M_0^1(2) = 0$. Suppose

$$M_i^1(2) = \frac{i}{6}\big(1 + \widetilde{\theta}(i, 2)\big).$$

Then

$$M_{i+1}^1(2) = M_i^1(2)\left(1 - \frac{2}{2i+1}\right) + M_i^1(1)\frac{1}{2i+1} + \frac{1}{2i+1} =$$

$$= \frac{i}{6}\left(1 + \widetilde{\theta}(i, 2)\right)\frac{2i-1}{2i+1} + \frac{2i}{3(2i+1)}\left(1 + \widetilde{\theta}(i, 1)\right) + \frac{1}{2i+1} =$$

$$= \frac{1}{6}\left(i + 1 + \frac{5}{2i+1} + \frac{(2i-1)i}{2i+1}\widetilde{\theta}(i, 2) + \frac{4i}{2i+1}\widetilde{\theta}(i, 1)\right) =$$

$$= \frac{i+1}{6}\left(1 + \frac{5}{(2i+1)(i+1)} + \frac{(2i-1)i}{(2i+1)(i+1)}\widetilde{\theta}(i, 2) + \frac{4i}{(2i+1)(i+1)}\widetilde{\theta}(i, 1)\right).$$

Put

$$\widetilde{\theta}(i+1,2) = \frac{5}{(2i+1)(i+1)} + \frac{(2i-1)i}{(2i+1)(i+1)}\widetilde{\theta}(i,2) + \frac{4i}{(2i+1)(i+1)}\widetilde{\theta}(i,1).$$

Note that $\widetilde{\theta}(i,1) < 0$. Hence

$$|\widetilde{\theta}(i+1,2)| \leqslant \left|\frac{(2i-1)i}{(2i+1)(i+1)}\widetilde{\theta}(i,2)\right| + \\ + \max\left\{\left|\frac{5}{(2i+1)(i+1)}\right|, \left|\frac{4i}{(2i+1)(i+1)}\widetilde{\theta}(i,1)\right|\right\}.$$

We got necessary bounds for $\widetilde{\theta}(i,2)$ and $\widetilde{\theta}(i,1)$. Thus, it is easy to check that

$$|\widetilde{\theta}(i+1,2)| \leqslant \frac{4}{i+1}.$$

This completes the proof for $d = 2$.

Suppose $d \geqslant 3$ and we can prove the theorem for all smaller degrees. This case is proved by induction on i. For $i = 0$ we have $M_0^1(d) = 0$. Assume that

$$M_i^1(d) = \frac{4i}{d(d+1)(d+2)}\big(1 + \widetilde{\theta}(i,d)\big).$$

Then

$$M_{i+1}^1(d) = M_i^1(d)\left(1 - \frac{d}{2i+1}\right) + M_i^1(d-1)\frac{d-1}{2i+1} = \\ = \frac{4i(2i+1-d)}{d(d+1)(d+2)(2i+1)}\big(1+\widetilde{\theta}(i,d)\big) + \frac{4i}{d(d+1)(2i+1)}\big(1+\widetilde{\theta}(i,d-1)\big) = \\ = \frac{4(i+1)}{d(d+1)(d+2)}\left(1 - \frac{1}{(2i+1)(i+1)} + \right. \\ \left. + \frac{i(2i+1-d)}{(2i+1)(i+1)}\widetilde{\theta}(i,d) + \frac{i(d+2)}{(2i+1)(i+1)}\widetilde{\theta}(i,d-1)\right).$$

If $2i+1-d \geqslant 0$, we can put

$$\widetilde{\theta}(i+1,d) = -\frac{1}{(2i+1)(i+1)} + \frac{i(2i+1-d)}{(2i+1)(i+1)}\widetilde{\theta}(i,d) + \frac{i(d+2)}{(2i+1)(i+1)}\widetilde{\theta}(i,d-1).$$

We obtain the following estimate:

$$|\widetilde{\theta}(i+1,d)| \leqslant \frac{1}{(2i+1)(i+1)} + \\ + \frac{i(2i+1-d)}{(2i+1)(i+1)}|\widetilde{\theta}(i,d)| + \frac{i(d+2)}{(2i+1)(i+1)}|\widetilde{\theta}(i,d-1)| \leqslant \frac{d^2}{i+1}.$$

If $2i+2-d \leqslant 0$, we have no vertices with degree d in G_1^{i+1}. Indeed, in G_1^{i+1} the sum of all degrees is $2i+2$. If $d > 2i+2$, we obviously do not have enough edges. If $d = 2i+2$, then it is easy to check that we can not have any vertices with degree d $(d > 2)$. So we can put $\widetilde{\theta}(i+1,d) = -1$. This concludes the proof.

3.4. Proof of Lemma 2

Obviously, $\mathsf{E}P_n(0,k) = \mathsf{E}P_n(1,k) = 0$. For all $k > 0$ we have $\mathsf{E}P_n(2,k) = 0$. For $k = 0$ we have

$$\mathsf{E}P_n(2,0) = \sum_{i=1}^{n} \frac{1}{2i-1} \prod_{j=i+1}^{n} \frac{2j-3}{2j-1} = \sum_{i=1}^{n} \frac{1}{2n-1} = \frac{n}{2n-1} \leqslant 1.$$

The rest of the proof is by induction. Consider $l \geqslant 3$, $k \geqslant 0$. Assume that we already proved that $\mathsf{E}P_n(i,j) \leqslant p(i,j)$ for all i and j, such that $i < l$, $j \leqslant k$ or $i \leqslant l$, $j < k$.

Trivially, $P_1(l,k) = 0$. It is easily shown that $\mathsf{E}P_{i+1}(l,k) = 0$ if $2i+4 < 2l+k$.

If $2i+4 = 2l+k$ and $P_{i+1}(l,k) \neq 0$, then $l = i+2$ and $k = 0$. And we have only one graph with $P_{i+1}(l,k) \neq 0$. Arguing as in the end of Section 3.1, we see that the probability of this graph is

$$\frac{(l-1)!}{(2l-1)!!}.$$

From the recurrent relation we have

$$p(l,0) = \frac{1}{2^{l-2}}.$$

In our case we get

$$\mathsf{E}P_{i+1}(l,k) = \frac{(l-1)!}{(2l-1)!!} < \frac{1}{2^{l-2}} = p(l,0).$$

If $2i+3 \geqslant 2l+k$, then

$$\mathsf{E}P_{i+1}(l,k)=\mathsf{E}P_i(l,k)\left(1-\frac{2l+k-2}{2i+1}\right)+\mathsf{E}P_i(l,k-1)\frac{l+k-3}{2i+1}+\mathsf{E}P_i(l-1,k)\frac{l-1}{2i+1}.$$

Using the recurrent relation for $p(l,k)$ and induction on i it is easy to prove that $\mathsf{E}P_n(l,k) \leqslant p(l,k)$. This concludes the proof of Lemma 2.

3.5. Proof of Theorem 3

This proof is similar to the proof given in [6]. But our case is more complicated. We need the Azuma—Hoeffding inequality (see [1]):

LEMMA 3. *Let* $(X_i)_{i=0}^n$ *be a martingale such that* $|X_i - X_{i-1}| \leqslant c$ *for any* $1 \leqslant i \leqslant n$. *Then*

$$\mathsf{P}\left(|X_n - X_0| \geqslant x\right) \leqslant 2e^{-\frac{x^2}{2c^2n}}$$

for any $x > 0$.

Suppose we are given an $\varepsilon > 0$. Fix $n \geqslant 3$ and k: $1 \leqslant k \leqslant n^{1/6-\varepsilon}$. Consider the random variables $X^i(k) = \mathsf{E}(X_n(k)|G_1^i)$, $i = 0, \dots, n$. Let us explain the notation $\mathsf{E}(X_n(k)|G_1^i)$. We construct the graph $G_1^n \in \mathfrak{G}_1^n$ by induction. For any $t \leqslant n$ there exists a unique $G_1^t \in \mathfrak{G}_1^t$ such that G_1^n is obtained from G_1^t. So $\mathsf{E}(X_n(k)|G_1^t)$ is the expectation of the number of vertices with second degree k in G_1^n if at the step t we have the graph G_1^t.

Note that $X^0(k) = \mathsf{E}X_n(k)$ and $X^n(k) = X_n(k)$. From the definition of G_1^n it follows that $X^i(k)$ is a martingale.

We will prove below that for any $i = 1, \dots, n$

$$|X^i(k) - X^{i-1}(k)| \leqslant 10\,k\,\log n.$$

Theorem 3 follows from this statement immediately. Put $c = 10\,k\log n$. Then from Azuma—Hoeffding inequality it follows that

$$\mathsf{P}\big(|X_n(k) - \mathsf{E}X_n(k)| \geqslant k\sqrt{n}\,\log^2 n\big) \leqslant 2\exp\left\{-\frac{n\,k^2\,\log^4 n}{200\,n\,k^2\log^2 n}\right\} = o(1).$$

If $k \leqslant n^{1/6-\varepsilon}$, then the value of n/k^2 is considerably greater than $k\ \log^2 n\,\sqrt{n}$. This means that we have

$$\frac{\bigl(k\sqrt{n}\ \log^2 n\bigr)}{(n/k^2)} = o(1).$$

This is exactly what we need.

It remains to estimate the quantity $|X^i(k) - X^{i-1}(k)|$. The proof is by a direct calculation.

Fix $1 \leqslant i \leqslant n$ and some graph G_1^{i-1}. Note that

$$\bigl|\mathsf{E}\bigl(X_n(k)|G_1^i\bigr) - \mathsf{E}\bigl(X_n(k)|G_1^{i-1}\bigr)\bigr| \leqslant$$
$$\leqslant \max_{\widetilde{G}_1^i \supset G_1^{i-1}} \bigl\{\mathsf{E}\bigl(X_n(k)|\widetilde{G}_1^i\bigr)\bigr\} - \min_{\widetilde{G}_1^i \supset G_1^{i-1}} \bigl\{\mathsf{E}\bigl(X_n(k)|\widetilde{G}_1^i\bigr)\bigr\}.$$

Put

$$\widehat{G}_1^i = \arg\max \mathsf{E}(X_n(k)|\widetilde{G}_1^i), \quad \overline{G}_1^i = \arg\min \mathsf{E}(X_n(k)|\widetilde{G}_1^i).$$

We need to estimate the difference

$$\mathsf{E}(X_n(k)|\widehat{G}_1^i) - \mathsf{E}(X_n(k)|\overline{G}_1^i).$$

Using the notation $N_n(l,k)$ and $P_n(l,k)$ from Section 2, we get

$$\mathsf{E}(X_n(k)|\widehat{G}_1^i) = \sum_{l=1}^{\infty} \mathsf{E}(N_n(l,k)|\widehat{G}_1^i) + \sum_{l=1}^{\infty} \mathsf{E}(P_n(l,k)|\widehat{G}_1^i),$$

$$\mathsf{E}(X_n(k)|\overline{G}_1^i) = \sum_{l=1}^{\infty} \mathsf{E}(N_n(l,k)|\overline{G}_1^i) + \sum_{l=1}^{\infty} \mathsf{E}(P_n(l,k)|\overline{G}_1^i).$$

For $i \leqslant t \leqslant n$ put

$$\delta_t(l,k) = \mathsf{E}(N_t(l,k)|\widehat{G}_1^i) - \mathsf{E}(N_t(l,k)|\overline{G}_1^i), \quad \delta_t'(l,k) = \delta_t(l,k)I(\delta_t(l,k) > 0),$$

$$\epsilon_t(l,k) = \mathsf{E}(P_t(l,k)|\widehat{G}_1^i) - \mathsf{E}(P_t(l,k)|\overline{G}_1^i), \quad \epsilon_t'(l,k) = \epsilon_t(l,k)I(\epsilon_t(l,k) > 0),$$

$$\delta_t(k) = \mathsf{E}(N_t(k)|\widehat{G}_1^i) - \mathsf{E}(N_t(k)|\widehat{G}_1^i)), \quad \delta_t'(k) = \delta_t(k)I(\delta_t(k) > 0).$$

Note that

$$\mathsf{E}(X_n(k)|\widehat{G}_1^i) - \mathsf{E}(X_n(k)|\overline{G}_1^i) = \sum_{l=1}^{\infty} \delta_n(l,k) + \sum_{l=1}^{\infty} \epsilon_n(l,k) \leqslant$$

$$\leqslant \sum_{l=1}^{\infty} \delta'_n(l,k) + \sum_{l=1}^{\infty} \epsilon'_n(l,k) \leqslant \sum_{l=1}^{\infty}\sum_{j=0}^{k} \left(\delta'_n(l,j) + \epsilon'_n(l,j)\right).$$

Let us estimate this double sum.

First suppose that $n = i$. Fix G_1^{i-1}. Graphs $\widehat{G}_1^i$ and $\overline{G}_1^i$ are obtained from the graph G_1^{i-1}. We add the vertex i and one edge $i\widehat{q}$ or $i\bar{q}$, respectively. New edge changes only the degree of $\widehat{q}$ or $\bar{q}$ and the second degree of neighbors of $\widehat{q}$ or $\bar{q}$, respectively. Consider $\widehat{G}_1^i$. Fix l and $j \leqslant k$. We are interested in measuring the growth of the number of vertices with degree l and second degree j at the step i. First i can become a vertex of second degree j with $j \leqslant k$. Secondly the vertex $\widehat{q}$ can become a vertex of second degree j with $j \leqslant k$. Thirdly the second degree of neighbors of $\widehat{q}$ increases. If $\widehat{q}$ has at least $k+1$ neighbors in G_1^{i-1}, then after the step i these vertices have second degree bigger than k and we do not count them. If $\widehat{q}$ has at most k neighbors in G_1^{i-1}, then at most k vertices change their second degrees at the step i. Arguing as above, we consider $\overline{G}_1^i$. We are interested in measuring the decrease of the values $N_{i-1}(l,j)$ and $P_{i-1}(l,j)$. First $\bar{q}$ has new degree after the step i. Secondly some neighbors of $\bar{q}$ can have second degree $j \leqslant k$ in G_1^{i-1} (so the number of the neighbors of $\bar{q}$ in G_1^{i-1} is not bigger than $k+1$). Let us sum all the just-mentioned numbers. We have

$$\sum_{l=1}^{\infty}\sum_{j=0}^{k} \left(\delta'_i(l,j) + \epsilon'_i(l,j)\right) \leqslant 1 + 1 + k + 1 + (k+1) = 2k+4.$$

The case $n = i$ is complete. Now consider t: $i \leqslant t \leqslant n-1$. Note that

$$\mathsf{E}\left(N_{t+1}(1)|G_1^i\right) = \mathsf{E}\left(N_t(1)|G_1^i\right)\left(1 - \frac{1}{2t+1}\right) + \frac{2t}{2t+1},$$

$$\mathsf{E}\left(N_{t+1}(2)|G_1^i\right) = \mathsf{E}\left(N_t(2)|G_1^i\right)\left(1 - \frac{2}{2t+1}\right) + \mathsf{E}\left(N_t(1)|G_1^i\right)\frac{1}{2t+1} + \frac{1}{2t+1},$$

$$\mathsf{E}\left(N_{t+1}(j)|G_1^i\right) = \mathsf{E}\left(N_t(j)|G_1^i\right)\left(1 - \frac{j}{2t+1}\right) + \mathsf{E}\left(N_t(j-1)|G_1^i\right)\frac{j-1}{2t+1}, \quad j \geqslant 3,$$

$$\mathsf{E}\big(N_{t+1}(1,j)|G_1^i\big) =$$

$$= \mathsf{E}\big(N_t(1,j)|G_1^i\big)\left(1-\frac{j+2}{2t+1}\right) + \frac{j\,\mathsf{E}\big(N_t(1,j-1)|G_1^i\big)}{2t+1} + \frac{j\,\mathsf{E}\big(N_t(j)|G_1^i\big)}{2t+1},$$

$$\mathsf{E}\left(N_{t+1}(l,j)|G_1^i\right) = \mathsf{E}\left(N_t(l,j)|G_1^i\right)\left(1-\frac{2l+j}{2t+1}\right) + \frac{(l-1)\,\mathsf{E}\left(N_t(l-1,j)|G_1^i\right)}{2t+1} +$$

$$+\frac{(l+j-1)\mathsf{E}\left(N_t(l,j-1)|G_1^i\right)}{2t+1}, \quad l \geqslant 2,$$

$$\mathsf{E}\big(P_{t+1}(2,0)|G_1^i\big) = \mathsf{E}\big(P_t(2,0)|G_1^i\big)\left(1-\frac{2}{2t+1}\right) + \frac{1}{2t+1},$$

$$\mathsf{E}\big(P_{t+1}(l,j)|G_1^i\big) = \mathsf{E}\big(P_t(l,j)|G_1^i\big)\left(1-\frac{2l+j-2}{2t+1}\right) +$$

$$+\,\mathsf{E}\big(P_t(l,j-1)|G_1^i\big)\frac{l+j-3}{2t+1} + \mathsf{E}\big(P_t(l-1,j)|G_1^i\big)\frac{l-1}{2t+1}, \quad l \geqslant 3.$$

We obtained the same equalities in proofs of Theorem 4, Lemma 1, and Lemma 2. Replace G_1^i by $\widehat{G}_1^i$ or $\overline{G}_1^i$ in these equalities. Substracting the equalities with $\overline{G}_1^i$ from the equalities with $\widehat{G}_1^i$ and using the inequality $(a+b)I(a+b>0) \leqslant aI(a>0) + {}+bI(b>0)$, we get

$$\delta'_{t+1}(j) \leqslant \delta'_t(j)\left(1-\frac{j}{2t+1}\right) + \delta'_t(j-1)\frac{j-1}{2t+1},$$

$$\delta'_{t+1}(1,j) \leqslant \delta'_t(1,j)\left(1-\frac{j+2}{2t+1}\right) + \frac{j\delta'_t(1,j-1)}{2t+1} + \frac{j\delta'_t(j)}{2t+1},$$

$$\delta'_{t+1}(l,j) \leqslant \delta'_t(l,j)\left(1-\frac{2l+j}{2t+1}\right) + \frac{(l-1)\delta'_t(l-1,j)}{2t+1} + \frac{(l+j-1)\delta'_t(l,j-1)}{2t+1}, \quad l \geqslant 2,$$

$$\epsilon'_{t+1}(2,0) \leqslant \epsilon'_t(2,0)\left(1-\frac{2}{2t+1}\right),$$

$$\epsilon'_{t+1}(l,j) \leqslant \epsilon'_t(l,j)\left(1-\frac{2l+j-2}{2t+1}\right) + \frac{(l-1)\epsilon'_t(l-1,j)}{2t+1} +$$

$$+\frac{(l+j-3)\epsilon'_t(l,j-1)}{2t+1}, \quad l \geqslant 3.$$

Now we can estimate the sum

$$\sum_{l=1}^{\infty}\sum_{j=0}^{k}\delta'_{t+1}(l,j)+\sum_{l=2}^{\infty}\sum_{j=0}^{k}\epsilon'_{t+1}(l,j)\leqslant$$

$$\leqslant\sum_{j=1}^{k}\left(\delta'_t(1,j)\left(1-\frac{j+2}{2t+1}\right)+\frac{j\delta'_t(1,j-1)}{2t+1}+\frac{j\delta'_t(j)}{2t+1}\right)+\epsilon'_t(2,0)\left(1-\frac{2}{2t+1}\right)+$$

$$+\sum_{l=2}^{\infty}\sum_{j=1}^{k}\left(\delta'_t(l,j)\left(1-\frac{2l+j}{2t+1}\right)+\frac{(l-1)\delta'_t(l-1,j)}{2t+1}+\frac{(l+j-1)\delta'_t(l,j-1)}{2t+1}\right)+$$

$$+\sum_{l=3}^{\infty}\sum_{j=0}^{k}\left(\epsilon'_t(l,j)\left(1-\frac{2l+j-2}{2t+1}\right)+\frac{(l-1)\epsilon'_t(l-1,j)}{2t+1}+\frac{(l+j-3)\epsilon'_t(l,j-1)}{2t+1}\right)=$$

$$=\sum_{j=1}^{k}\delta'_t(1,j)-\sum_{j=1}^{k}\frac{(j+2)\delta'_t(1,j)}{2t+1}+\sum_{j=1}^{k-1}\frac{(j+1)\delta'_t(1,j)}{2t+1}+\sum_{j=1}^{k}\frac{j\delta'_t(j)}{2t+1}+$$

$$+\sum_{l=2}^{\infty}\sum_{j=1}^{k}\delta'_t(l,j)-\sum_{l=2}^{\infty}\sum_{j=1}^{k}\frac{(2l+j)\delta'_t(l,j)}{2t+1}+\sum_{l=1}^{\infty}\sum_{j=1}^{k}\frac{l\delta'_t(l,j)}{2t+1}+$$

$$+\sum_{l=2}^{\infty}\sum_{j=1}^{k-1}\frac{(l+j)\delta'_t(l,j)}{2t+1}+\epsilon'_t(2,0)-\frac{2\epsilon'_t(2,0)}{2t+1}+$$

$$+\sum_{l=3}^{\infty}\sum_{j=0}^{k}\epsilon'_t(l,j)-\sum_{l=3}^{\infty}\sum_{j=0}^{k}\frac{(2l+j-2)\epsilon'_t(l,j)}{2t+1}+$$

$$+\sum_{l=3}^{\infty}\sum_{j=0}^{k}\frac{l\epsilon'_t(l,j)}{2t+1}+\frac{2\epsilon'_t(2,0)}{2t+1}+\sum_{l=3}^{\infty}\sum_{j=0}^{k-1}\frac{(l+j-2)\epsilon'_t(l,j)}{2t+1}=$$

$$=\sum_{l=1}^{\infty}\sum_{j=0}^{k}\Big(\delta'_t(l,j)+\epsilon'_t(l,j)\Big)+\sum_{j=1}^{k}\frac{j\delta'_t(j)}{2t+1}-\sum_{l=1}^{\infty}\delta'_t(l,k)\frac{l+k}{2t+1}-\sum_{l=3}^{\infty}\epsilon'_t(l,k)\frac{l+k-2}{2t+1}\leqslant$$

$$\leqslant\sum_{l=1}^{\infty}\sum_{j=0}^{k}\Big(\delta'_t(l,j)+\epsilon'_t(l,j)\Big)+\sum_{j=1}^{k}\frac{j\delta'_t(j)}{2t+1}.$$

It remains to estimate the sum

$$\sum_{j=1}^{k} \frac{j\delta_t'(j)}{2t+1}.$$

Note that for any $t \geqslant i$ we have

$$\sum_{j=0}^{k} \delta_t'(j) \leqslant 3.$$

It is obvious for $t = i$ (when we add a new vertex i, we change only the degree of $\widehat{q}$ or $\bar{q}$). If $t+1 > i$, then

$$\sum_{j=1}^{k} \delta_{t+1}'(j) \leqslant \sum_{j=1}^{k} \left(\delta_t'(j)\left(1 - \frac{j}{2t+1}\right) + \delta_t'(j-1)\frac{j-1}{2t+1}\right) = \\ = \sum_{j=1}^{k} \delta_t'(j) - \delta_t'(k)\frac{k}{2t+1}.$$

In other words, $\sum_{j=1}^{k} \delta_t'(j)$ is not increasing when t is growing.

So we get

$$\sum_{l=1}^{\infty}\sum_{j=0}^{k} \left(\delta_{t+1}'(l,j) + \epsilon_{t+1}'(l,j)\right) \leqslant \sum_{l=1}^{\infty}\sum_{j=0}^{k} \left(\delta_t'(l,j) + \epsilon_t'(l,j)\right) + \frac{3k}{2t+1}.$$

Thus we have

$$|\mathsf{E}(X_n(k)|G_1^{i-1}) - \mathsf{E}(X_n(k)|G_1^{i})| \leqslant \sum_{l=1}^{\infty}\sum_{j=0}^{k} \left(\delta_i'(l,j) + \epsilon_i'(l,j)\right) + \sum_{t=i}^{n-1} \frac{3k}{2t+1} \leqslant \\ \leqslant 2k + 4 + \sum_{t=1}^{n-1} \frac{3k}{2t+1} \leqslant 2k + 5 + \frac{3}{2}k\log n \leqslant 10\,k\log n.$$

This concludes the proof of Theorem 3.

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