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Positive integers: counterexample to W. M. Schmidt's conjecture

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Abstract: We show that there exist real numbers α_1, α_2 linearly independent over $\mathbb{Z}$ together with 1 such that for every non-zero integer vector (m_1, m_2) with $m_1 \geqslant 0$ and $m_2 \geqslant 0$ one has

$$||m_1\alpha_1 + m_2\alpha_2|| \geqslant 2^{-300}(\max(m_1, m_2))^{-\sigma}$$

with $\sigma = 1{,}94696^+$.

Keywords: Diophantine approximation with positive integers; W. M. Schmidt's problem; Diophantine exponents

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1. Introduction

Let $||\xi||$ denote the distance from real ξ to the nearest integer. Let

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

In [6] W. M. Schmidt proved the following result.

Theorem A. (*W. M. Schmidt*) *Let real numbers α_1, α_2 be linearly independent over $\mathbb{Z}$ together with 1. Then there exists a sequence of integer two-dimensional vectors $(x_1(i), x_2(i))$ such that*

1. $x_1(i), x_2(i) > 0$;
2. $||\alpha_1 x_1(i) + \alpha_2 x_2(i)|| \cdot (\max\{x_1(i), x_2(i)\})^{\varphi} \to 0$ as $i \to +\infty$.

W. M. Schmidt posed a conjecture that the exponent φ here may be replaced by $2 - \varepsilon$ with arbitrary positive ε (see [7]). In this paper we show this conjecture to be false.

Let $\sigma = 1{,}94696^{+}$ be the largest real root of the equation

$$x^4 - 2x^2 - 4x + 1 = 0. \tag{1}$$

THEOREM 1. *There exist real numbers α_1, α_2 such that they are linearly independent over $\mathbb{Z}$ together with 1 and for every integer vector $(m_1, m_2) \in \mathbb{Z}^2$ with m_1, $m_2 \geqslant 0$ and $\max(m_1, m_2) \geqslant 2^{200}$ one has*

$$||m_1\alpha_1 + m_2\alpha_2|| \geqslant \frac{1}{2^{300}(\max(m_1, m_2))^{\sigma}}.$$

Of course constants 2^{200}, 2^{300} in Theorem 1 may be reduced.

Theorem 1 will be proved in Sections 3–9.

2. Diophantine exponents

Put

$$\psi(t) = \psi_{\alpha_1,\alpha_2}(t) = \min_{m_1,m_2 \in \mathbb{Z},\ 0<\max(|m_1|,|m_2|)\leqslant t} ||m_1\alpha_1 + m_2\alpha_2||,$$

$$\psi^*(t) = \psi^*_{\alpha_1,\alpha_2}(t) = \min_{x\in\mathbb{Z},\ 0<x\leqslant t}\ \max_{j=1,2} ||x\alpha_j||$$

and

$$\psi_+(t) = \psi_{+:\alpha_1,\alpha_2}(t) = \min_{m_1,m_2\in\mathbb{Z}_+,\ 0<\max(m_1,m_2)\leqslant t} ||m_1\alpha_1 + m_2\alpha_2||.$$

Recall the definitions of Diophantine exponents

$$\omega = \sup\{\gamma : \quad \liminf_{t\to\infty} t^{\gamma}\psi_{\alpha_1,\alpha_2}(t) < \infty\},$$

$$\widehat{\omega} = \sup\{\gamma : \quad \limsup_{t\to\infty} t^{\gamma}\psi_{\alpha_1,\alpha_2}(t) < \infty\}$$

and

$$\omega^* = \sup\{\gamma : \quad \liminf_{t\to\infty} t^{\gamma}\psi^*_{\alpha_1,\alpha_2}(t) < \infty\}.$$

We introduce Diophantine exponents

$$\omega_+ = \sup\{\gamma : \quad \liminf_{t\to\infty} t^\gamma \psi_{+;\alpha_1,\alpha_2}(t) < \infty\},$$

and

$$\widehat{\omega}_+ = \sup\{\gamma : \quad \limsup_{t\to\infty} t^\gamma \psi_{+;\alpha_1,\alpha_2}(t) < \infty\}.$$

In fact W. M. Schmidt proved that for α_1, α_2 under consideration one has the inequality

$$\omega_+ \geqslant \max\left(\frac{\widehat{\omega}}{\widehat{\omega}-1}; \widehat{\omega} - 1 + \frac{\widehat{\omega}}{\omega}\right) \tag{2}$$

from which we immediately deduce $\omega_+ \geqslant \varphi$. From Schmidt's argument one can easily see that for α_1, α_2 linearly independent together with 1 one has

$$\widehat{\omega}_+ \geqslant \frac{\omega}{\omega - 1}.$$

We would like to note here that Thurnheer (see Theorem 2 from [8]) showed that for α_1, α_2 linearly independent together with 1 in the case

$$\frac{1}{2} \leqslant \omega^* = \omega^*(\Theta) \leqslant 1$$

one has

$$\omega_+ \geqslant \frac{\omega^*+1}{4\omega^*} + \sqrt{\left(\frac{\omega^*+1}{4\omega^*}\right)^2 + 1} \tag{3}$$

(inequality (3) is a particular case of a general result obtained by Thurnheer).

A lower bound for ω_+ in terms of ω was obtained by the author in [4]. It was based on the original Schmidt's argument from [6]. However the choice of parameters in [4] was not optimal. Here we explain the optimal choice. From Schmidt's proof and Jarník's result

$$\omega \geqslant \widehat{\omega}(\widehat{\omega} - 1)$$

(see [1] and a recent paper [3]) one can easily see that

$$\omega_+ \geqslant \max\Big\{g\colon \max_{y,z\geqslant 1:\, y^{\widehat{\omega}-1}\leqslant z \leqslant y^{\omega/\widehat{\omega}}} \quad \max_{y^{-\omega}\leqslant x \leqslant z^{-\widehat{\omega}}} \min\left(x^{1-g}z^{-g}; xy^{-1}z^{g+1}\right) \leqslant 1\Big\}. \tag{4}$$

This inequality immediately follows from Schmidt's argument, see Lemma 1 and Lemma 2 from [4]. The right hand side of (4) can be easily calculated. We divide the set

$$\mathfrak{A} = \left\{(\omega, \widehat{\omega}) \in \mathbb{R}^2 : \quad \widehat{\omega} \geqslant 2, \quad \omega \geqslant \widehat{\omega}(\widehat{\omega} - 1)\right\}$$

of all admissible values of $(\omega, \widehat{\omega})$ into two parts:

$$\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2,$$

$$\mathfrak{A}_1 = \left\{(\omega, \widehat{\omega}) \in \mathbb{R}^2 : \quad 2 \leqslant \widehat{\omega} \leqslant \varphi^2, \quad \omega \geqslant \frac{\widehat{\omega}(\widehat{\omega} - 1)}{3\widehat{\omega} - \widehat{\omega}^2 - 1}\right\},$$

$$\mathfrak{A}_2 = \mathfrak{A} \setminus \mathfrak{A}_1.$$

If $(\omega, \widehat{\omega}) \in \mathfrak{A}_1$ then

$$\omega_+ \geqslant G(\omega) = \frac{1}{2}\left(\frac{\omega + 1}{\omega} + \sqrt{\left(\frac{\omega + 1}{\omega}\right)^2 + 4}\right)$$

(the function $G(\omega)$ on the right hand side decreases from $G(2) = 2$ to $G(+\infty) = \varphi$). If $(\omega, \widehat{\omega}) \in \mathfrak{A}_2$ then

$$\omega_+ \geqslant \widehat{\omega} - 1 + \frac{\widehat{\omega}}{\omega}.$$

So we get the following result.

THEOREM 2. *Let real numbers α_1, α_2 be linearly independent over $\mathbb{Z}$ together with* 1. *Then*

$$\omega_+ \geqslant \max\left(\frac{1}{2}\left(\frac{\omega + 1}{\omega} + \sqrt{\left(\frac{\omega + 1}{\omega}\right)^2 + 4}\right); \widehat{\omega} - 1 + \frac{\widehat{\omega}}{\omega}\right).$$

This theorem gives the best bound in terms of ω, $\widehat{\omega}$ which one can deduce from Schmidt's argument from [6].

3. The construction

We shall deal with the Euclidean norm for simplicity reason. So we use $|\cdot|$ for the Euclidean norm of two- or three-dimensional vectors. By angle$(\mathbf{u}, \mathbf{v})$ we denote the angle between vectors $\mathbf{u}$, $\mathbf{v}$.

Define

$$\tau = \frac{1+\sigma^2}{2\sigma} = 1{,}23029^+. \tag{5}$$

Note that

$$\sigma\tau - 1 > \tau. \tag{6}$$

Put

$$\omega = \tau + 1. \tag{7}$$

FUNDAMENTAL LEMMA. *There exist real numbers* $\alpha_1,\ \alpha_2 \in \mathbb{R}$ *linearly independent together with* 1 *over* $\mathbb{Z}$ *and such that there exists a sequence of integer vectors*

$$\mathbf{m}_0 = (1, 1, -1), \quad \mathbf{m}_\nu = (m_{0,\nu}, m_{1,\nu}, m_{2,\nu}) \in \mathbb{Z}^3, \quad \nu = 1, 2, 3\dots .$$

satisfying the following conditions (*i*)–(*v*).

(*i*) *For any* $\nu \geqslant 1$ *the triple* $\mathbf{m}_{\nu-1},\ \mathbf{m}_\nu,\ \mathbf{m}_{\nu+1}$ *consists of linearly independent vectors*, *and each two-dimensional sublattice*

$$\mathcal{L}_\nu = \langle \mathbf{m}_\nu, \mathbf{m}_{\nu+1} \rangle_{\mathbb{Z}}$$

is complete, *that is*

$$\mathbb{Z}^3 \cap \operatorname{span} \mathcal{L}_\nu = \mathcal{L}_\nu, \quad \nu = 0, 1, 2, 3, \dots .$$

(*ii*) *Define*

$$\overline{\mathbf{m}}_\nu = (m_{1,\nu}, m_{2,\nu}) \in \mathbb{Z}^2 \quad M_\nu = |\overline{\mathbf{m}}_\nu|$$

and

$$\zeta_\nu = m_{0,\nu} + m_{1,\nu}\alpha_1 + m_{2,\nu}\alpha_2.$$

For every $\nu \geqslant 0$ *one has*

$$\frac{1}{2^{10} M_{\nu+1}^{\omega}} \leqslant \zeta_\nu \leqslant \frac{2}{M_{\nu+1}^{\omega}}. \tag{8}$$

(*iii*) $2^{100} \leqslant M_1 \leqslant 2^{200}$ *and for every* $\nu \geqslant 1$ *one has*

$$2^{20} M_\nu \leqslant M_{\nu+1} \tag{9}$$

and

$$H_\nu \leqslant M_{\nu+1} \leqslant 2^5 H_\nu, \quad H_\nu = \frac{M_\nu^{\sigma\tau-1}}{2^{20}}. \tag{10}$$

(*iv*) *For every* $\nu \geqslant 0$ *one has* $m_{1,\nu} \cdot m_{2,\nu} < 0$; *moreover for the vectors*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

one has

$$\text{angle}(\overline{\mathbf{m}}_\nu, \pm \mathbf{e}_j) \geqslant \frac{1}{4}, \quad j = 1, 2. \tag{11}$$

(*v*) *For every* $\nu \geqslant 0$ *for vectors*

$$\overline{\mathbf{m}}_\nu = (m_{1,\nu}, m_{2,\nu}), \quad \overline{\mathbf{m}}_{\nu+1} = (m_{1,\nu+1}, m_{2,\nu+1})$$

one has

$$\text{angle}(\overline{\mathbf{m}}_\nu, \pm \overline{\mathbf{m}}_{\nu+1}) \geqslant \frac{1}{4}.$$

We give a proof of Fundamental Lemma in Sections 6–8. It uses a standard argument related to an inductive construction of special singular (in the sense of A. Khintchine) vectors (quite similar but much easier construction was used in [2]). Inequality (6) is of major importance. Many different properties of singular vectors are discussed in our recent survey [5].

For every ν we define two-dimensional lattice $\Lambda_\nu = \langle \overline{\mathbf{m}}_\nu, \overline{\mathbf{m}}_{\nu+1} \rangle_{\mathbb{Z}} \subset \mathbb{Z}^2$. Let D_ν be the fundamental volume of the lattice Λ_ν. Obviously

$$D_\nu \leqslant M_\nu M_{\nu+1}. \tag{12}$$

From the condition (*v*) one has

$$D_\nu \geqslant \frac{M_\nu M_{\nu+1}}{2^5}. \tag{13}$$

In the sequel we use the following notation. For an integer vector

$$\mathbf{m} = (m_0, m_1, m_2) \in \mathbb{Z}^3$$

we define

$$\zeta = \zeta(\mathbf{m}) = m_0 + m_1\alpha_1 + m_2\alpha_2, \quad \overline{\mathbf{m}} = \overline{\mathbf{m}}(\mathbf{m}) = (m_1, m_2) \in \mathbb{Z}^2$$

and

$$M = M(\mathbf{m}) = |\overline{\mathbf{m}}|.$$

In Sections 3, 4, 5 below we suppose that α_1, α_2 are the numbers from Fundamental Lemma.

4. Linearly independent vectors

We prove a lemma concerning a lower bound for the value of $|\zeta(\mathbf{m})|$ in the case when the vector $\mathbf{m} \in \mathbb{Z}^3$ is linearly independent of vectors $\mathbf{m}_\nu, \mathbf{m}_{\nu+1}$.

Consider the segment

$$\mathcal{I}_\nu = \left[(4M_\nu M_{\nu+1})^{1/\sigma}, \frac{M_{\nu+1}^\tau}{2^4} \right] \tag{14}$$

(inequalities (6) and (9) show that the left endpoint of the segment is less than the right endpoint indeed).

LEMMA 1. *Suppose that a vector* $\mathbf{m} \in \mathbb{Z}^3$ *is linearly independent of vectors* $\mathbf{m}_\nu, \mathbf{m}_{\nu+1}$ *and*

$$M \in \mathcal{I}_\nu. \tag{15}$$

Then

$$|\zeta(\mathbf{m})| \geqslant M^{-\sigma}.$$

PROOF. Consider the determinant

$$\Delta = \begin{vmatrix} m_0 & m_1 & m_2 \\ m_{0,\nu} & m_{1,\nu} & m_{2,\nu} \\ m_{0,\nu+1} & m_{1,\nu+1} & m_{2,\nu+1} \end{vmatrix} = \begin{vmatrix} \zeta(\mathbf{m}) & m_1 & m_2 \\ \zeta_\nu & m_{1,\nu} & m_{2,\nu} \\ \zeta_{\nu+1} & m_{1,\nu+1} & m_{2,\nu+1} \end{vmatrix}.$$

We see from (8, 7) that

$$1 \leqslant |\Delta| \leqslant 2|\zeta(\mathbf{m})| M_\nu M_{\nu+1} + 2^3 M M_{\nu+1}^{-\tau}.$$

From the inequality $M \leqslant M_{\nu+1}^{\tau}/2^4$ which follows from (15) we see that

$$2^3 M M_{\nu+1}^{-\tau} \leqslant \frac{1}{2}.$$

That is why

$$|\zeta(\mathbf{m})| M_\nu M_{\nu+1} \geqslant \frac{1}{4}.$$

Now we take into account the lower bound for M from (15) and the lemma follows. □

5. Vectors dependent with $\mathbf{m}_\nu$, $\mathbf{m}_{\nu+1}$

Condition (i) means that each integer vector $\mathbf{m} \in \mathbb{Z}^3$ which is linearly dependent together with $\mathbf{m}_\nu$, $\mathbf{m}_{\nu+1}$ can be written in a form

$$\mathbf{m} = \lambda \mathbf{m}_\nu + \mu \mathbf{m}_{\nu+1}$$

with integer λ and μ. So if $\mathbf{m} \in \mathbb{Z}^3$ is linearly dependent together with $\mathbf{m}_\nu$, $\mathbf{m}_{\nu+1}$ then for «cut» vectors we have the equality

$$\overline{\mathbf{m}} = \lambda \overline{\mathbf{m}}_\nu + \mu \overline{\mathbf{m}}_{\nu+1} \tag{16}$$

with integer λ and μ.

LEMMA 2. *Suppose that a nonzero vector* $\mathbf{m} = (m_0, m_1, m_2) \in \mathbb{Z}^3$ *satisfies the condition* $m_1, m_2 \geqslant 0$ *Suppose that vectors* $\mathbf{m}$, $\mathbf{m}_\nu$, $\mathbf{m}_{\nu+1}$ *are linearly dependent for some* ν. *Then*

$$|\zeta(\mathbf{m})| \geqslant 2^{-300} M^{-\sigma}.$$

PROOF. We can split two-dimensional lattice Λ_ν into a countable union of one-dimensional lattices $\Lambda_{\nu,\mu}$ in the following way:

$$\Lambda_\nu = \bigsqcup_{\mu \in \mathbb{Z}} \Lambda_{\nu,\mu}, \quad \Lambda_{\nu,\mu} = \{\mathbf{z} = (z_1, z_2) \in \Lambda_\nu : \quad \mathbf{z} = \lambda \overline{\mathbf{m}}_\nu + \mu \overline{\mathbf{m}}_{\nu+1}, \ \lambda \in \mathbb{Z}\}.$$

By the condition (iv) there are no non-zero points $(z_1, z_2) \in \Lambda_{\nu,0}$ satisfying $z_1 \cdot z_2 \geqslant 0$.

Suppose that $\mu \neq 0$. As the fundamental volume of Λ_ν is equal to D_ν we see that the Euclidean distance between any two neighbouring lines aff $\Lambda_{\nu,\mu}$ and aff $\Lambda_{\nu,\mu+1}$ is equal to

$$\frac{D_\nu}{\sqrt{m_{1,\nu}^2 + m_{2,\nu}^2}}.$$

That is why the conditions

$$(m_1, m_2) \in \Lambda_{\nu,\mu}, \quad m_1, m_2 \geqslant 0 \tag{17}$$

imply

$$\max(m_1, m_2) \geqslant \frac{|\mu| D_\nu}{2M_\nu} \geqslant \frac{|\mu| M_{\nu+1}}{2^8} \tag{18}$$

(in the last inequality we use (13)).

From the other hand conditions (17) together with (11) from (*iv*) lead to the inequality

$$|\lambda| \geqslant \frac{|\mu| M_{\nu+1}}{2^8 M_\nu}$$

for the coefficient λ from (16). So (we apply lower bound from (*ii*) for ζ_ν and upper bound from (*ii*) for $\zeta_{\nu+1}$) we see that

$$|\zeta(\mathbf{m})| = |\lambda\zeta_\nu + \mu\zeta_{\nu+1}| \geqslant |\mu| \left(\frac{M_{\nu+1}}{2^8 M_\nu} \zeta_\nu - \zeta_{\nu+1} \right) \geqslant |\mu| \left(\frac{1}{2^{18} M_\nu M_{\nu+1}^\tau} - \frac{2}{M_{\nu+2}^{\tau+1}} \right).$$

Now we apply lower bound (9) from (*iii*). It gives $M_{\nu+2}^{\tau+1} \geqslant 2^{20} M_\nu M_{\nu+1}^\tau$. So

$$|\zeta(\mathbf{m})| \geqslant \frac{|\mu|}{2^{19} M_\nu M_{\nu+1}^\tau}.$$

Now we should use lower bound for $M_{\nu+1}$ in terms of M_ν. For $\nu \geqslant 1$ the lower bound (10) from (*iii*) gives

$$|\zeta(\mathbf{m})| \geqslant 2^{-19-\frac{20}{\sigma\tau-1}} |\mu| M_{\nu+1}^{-\tau-\frac{1}{\sigma\tau-1}} = 2^{-19-\frac{20}{\sigma\tau-1}} |\mu| M_{\nu+1}^{-\sigma} > 2^{-200} |\mu| M_{\nu+1}^{-\sigma} \tag{19}$$

(here we use the definition of σ as a root of (1) and (5) to see that $\tau + 1/(\sigma\tau - 1) = \sigma$). But for $\nu = 0$ this is true also, as $M_0 = 1$ and $\sigma \geqslant \tau$: for a vector $\mathbf{m} \in \Lambda_{0,\mu}$ one has

$|\zeta(\mathbf{m})| \geqslant 2^{-19}|\mu|M_1^{-\tau} \geqslant 2^{-200}|\mu|M_1^{-\sigma}$. Now we combine (18,19) to get

$$|\zeta(\mathbf{m})| \geqslant \frac{|\mu|^{1+\sigma}}{2^{300}M^{\sigma}} > \frac{1}{2^{300}M^{\sigma}}.$$

Lemma 2 is proved. □

6. Proof of Theorem 1

We take α_1, α_2 from Fundamental Lemma. Consider an integer vector

$$\mathbf{m} = (m_0, m_1, m_2) \quad \text{with} \quad m_1, m_2 \geqslant 0.$$

We may suppose that $|\zeta(\mathbf{m})| = ||m_1\alpha_1 + m_2\alpha_2||$. If for some ν vectors

$$\mathbf{m}, \quad \mathbf{m}_\nu, \quad \mathbf{m}_{\nu+1} \tag{20}$$

are linearly dependent then application of Lemma 2 proves Theorem 1. So we may suppose that all triples (20) consist of linearly independent vectors for every $\nu \geqslant 0$. Now to prove Theorem 1 we may use Lemma 1. It is enough to show that

$$\bigcup_{\nu \geqslant 0} \mathcal{I}_\nu \supset [2^{200}, +\infty)$$

(segments $\mathcal{I}_\nu$ are defined in (14)). But this follows from the condition $M_1 \leqslant 2^{200}$ and the inequality

$$(4M_\nu M_{\nu+1})^{1/\sigma} \leqslant M_\nu^{\tau}/2^4.$$

The last inequality is a corollary of the right inequality from (10). □

7. Fundamental Lemma: beginning of proof

For the inner product of two vectors $\mathbf{w} = (w_0, w_1, w_2)$ and $\xi = (\xi_0, \xi_1, \xi_2)$ we write

$$\mathbf{w} \cdot \xi = w_0\xi_0 + w_1\xi_1 + w_2\xi_2.$$

Let $\mathbf{m} \in \mathbb{Z}^3$ be an integer vector. The formulation of Fundamental Lemma deals with the values of $M = M(\mathbf{m}) = |\overline{\mathbf{m}}|$. To describe the ideas of the proof it is much more convenient to consider the Euclidean norm $\mathrm{M} = |\mathbf{m}|$ of the vector $\mathbf{m}$ itself than the Euclidean norm $M = |\overline{\mathbf{m}}|$ of the «cut» vector $\overline{\mathbf{m}} \in \mathbb{Z}^2$. Of course values of M and M are of the same order for all integer vectors $\mathbf{m}$ under consideration. We may assume that $M \leqslant \mathrm{M} \leqslant 8M$ (we may suppose that in the sequel all the numbers $\xi_{1,j}/\xi_{0,j}$, $\xi_{2,j}/\xi_{0,j}$ and α_1, α_2 below are from the interval $(0, 2)$).

We take two unit vectors

$$\mathbf{y}^1 = \left(0, -\cos\left(\frac{1}{3}\right), \sin\left(\frac{1}{3}\right)\right), \quad \mathbf{y}^2 = \left(0, -\sin\left(\frac{1}{3}\right), \cos\left(\frac{1}{3}\right)\right).$$

So $\overline{\mathbf{y}}^1, \overline{\mathbf{y}}^2 \in \mathbb{R}^2$ satisfy the conditions

$$y_1^1 < 0, \quad y_2^1 > 0, \quad y_1^2 < 0, \quad y_2^2 > 0; \quad \mathrm{angle}(\overline{\mathbf{y}}^1, -\mathbf{e}_1) = \frac{1}{3},$$
$$\mathrm{angle}(\overline{\mathbf{y}}^2, \mathbf{e}_2) = \frac{1}{3}, \quad \mathrm{angle}(\overline{\mathbf{y}}^1, \overline{\mathbf{y}}^2) = \frac{\pi}{2} - \frac{2}{3}. \tag{21}$$

Let

$$\mathfrak{S} = \{\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 : \quad |\mathbf{x}| = 1\}$$

be the unit sphere. We construct a sequence of nested closed sets

$$\mathcal{B}_\nu \subset \mathfrak{S} \cap \{\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 : \quad 1/2 \leqslant x_0 \leqslant 1\} \tag{22}$$

by induction. Their unique common point

$$\mathbf{x}^* = (x_0^*, x_1^*, x_2^*) \in \bigcap_\nu \mathcal{B}_\nu$$

will define real numbers $\alpha_1 = x_1^*/x_0^*, \alpha_2 = x_2^*/x_0^*$ which satisfy the conclusion of Fundamental Lemma.

The base of inductive process is trivial. The only thing we should point out is that we must suppose that

$$\mathrm{angle}\,(\mathbf{m}_1, \mathbf{y}^1) \leqslant \frac{1}{2^{20}}, \quad \mathrm{angle}\,(\mathbf{m}_2, \mathbf{y}^2) \leqslant \frac{1}{2^{20}}. \tag{23}$$

This is possible as we do not have any important condition for integer vectors $\mathbf{m}$ with small value of M.

To proceed the inductive step we suppose that the following objects are already constructed:

1) primitive integer vectors $\mathbf{m}_j = (m_{0,j}, m_{1,j}, m_{2,j})$, $0 \leqslant j \leqslant \nu$ with $\mathrm{M}_j = |\mathbf{m}_j|$; we suppose that these vectors satisfy the condition (*iii*); we suppose that every triple $\mathbf{m}_{j-1}$, $\mathbf{m}_j$, $\mathbf{m}_{j+1}$ consists of linearly independent vectors; moreover, we suppose that

$$\mathrm{angle}(\mathbf{m}_j, \mathbf{y}^1) \leqslant \frac{1}{2^8} \tag{24}$$

for odd j, and

$$\mathrm{angle}(\mathbf{m}_j, \mathbf{y}^2) \leqslant \frac{1}{2^8} \tag{25}$$

for even j;

2) two-dimensional complete sublattices $\mathcal{L}_j = \langle \mathbf{m}_j, \mathbf{m}_{j+1} \rangle_{\mathbb{Z}}$, $j = 0, \ldots, \nu - 1$ with fundamental volumes d_j satisfying inequalities

$$\frac{\mathrm{M}_j \mathrm{M}_{j+1}}{2^5} \leqslant d_j \leqslant \mathrm{M}_j \mathrm{M}_{j+1} \tag{26}$$

(here the right inequality is trivial, the left one means that the angle between vectors $\mathbf{m}_{j+1}$, $\mathbf{m}_j$ is bounded from below);

3) vectors $\xi_j = (\xi_{0,j}, \xi_{1,j}, \xi_{2,j}) \in \mathfrak{S}$ such that $\xi_j \perp \mathbf{m}_j$, $\xi_j \perp \mathbf{m}_{j+1}$, $0 \leqslant j \leqslant \nu - 1$, and one-dimensional linear subspaces $\Xi_j = \mathrm{span}\, \xi_j$, $0 \leqslant j \leqslant \nu - 1$;

4) two-dimensional linear subspaces

$$\ell_j^0 = \left\{ \mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{m}_j = 0 \right\}, \quad 0 \leqslant j \leqslant \nu - 1,$$

and two-dimensional affine subspaces

$$\ell_j^1 = \left\{ \mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{m}_j = \frac{1}{2\mathrm{M}_{j+1}^{\omega}} \right\}, \quad 0 \leqslant j \leqslant \nu - 1;$$

5) spherical domains

$$\mathcal{C}_j = \left\{ \mathbf{x} \in \mathfrak{S} : \quad \mathrm{dist}\,(\mathbf{x}, \Xi_j) \leqslant \frac{1}{\mathrm{M}_{j+1}^{\omega} \mathrm{M}_j} \right\}, \quad 0 \leqslant j \leqslant \nu - 1$$

(here dist $(\cdot,\cdot)$ denotes the Euclidean distance between sets or between a set and a point) and closed sets

$$\mathcal{G}_j = \left\{ \mathbf{x} \in \mathfrak{S} \cap \mathcal{C}_j : \quad \mathbf{x} \cdot \mathbf{m}_j \geqslant \frac{1}{2\mathrm{M}^{\omega}_{j+1}} \right\} \subset \mathfrak{S}, \quad 0 \leqslant j \leqslant \nu - 1$$

(so a part of the boundary of $\mathcal{G}_j$ belongs to ℓ^1_j);

6) we suppose that the vector $\mathbf{m}_\nu$ is defined, so we can consider linear subspace

$$\ell^0_\nu = \left\{ (x_0, x_1, x_2) \in \mathbb{R}^3 : m_{0,\nu}x_0 + m_{1,\nu}x_1 + m_{2,\nu}x_2 = 0 \right\};$$

we suppose that linear subspaces ℓ^0_j for every j from the range $1 \leqslant j \leqslant \nu$ satisfy the condition

$$\ell^0_j \cap \mathcal{G}_{j-1} \neq \varnothing, \quad 1 \leqslant j \leqslant \nu;$$

moreover we suppose that for any j within the range $1 \leqslant j \leqslant \nu$ there is a point $\eta_j = (\eta_{0,j}, \eta_{1,j}.\eta_{2,j}) \in \ell^0_j \cap \mathcal{G}_{j-1}$ such that the sets

$$\mathcal{B}_j = \left\{ \mathbf{x} \in \mathfrak{S} : \quad |\mathbf{x} - \eta_j| \leqslant \frac{1}{2^6 \mathrm{M}^{\omega}_j \mathrm{M}_{j-1}} \right\}$$

satisfy the condition

$$\mathcal{B}_j \subset \mathcal{G}_{j-1} \subset \mathcal{C}_{j-1} \subset \mathcal{B}_{j-1}. \tag{27}$$

Here we should note that $\mathcal{B}_0 \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_{\nu-1} \supset \mathcal{B}_\nu$.

Now we show that the vectors $\mathbf{m}_j$, $1 \leqslant j \leqslant \nu$ and every pair α_1, α_2 of the form $\alpha_1 = x_1/x_0$, $\alpha_2 = x_2/x_0$, $\mathbf{x} = (x_0, x_1, x_2) \in B_\nu$ satisfy all the conditions (*i*)–(*v*) of Fundamental Lemma which are defined up to the $(\nu - 1)$-th step.

The condition (*i*) is satisfied as in 1) we suppose linear independence of any triple of consecutive vectors, and the completeness of the corresponding lattice is stated in 2).

The condition (*ii*) follows from the fact that $\mathbf{x} \in \mathcal{B}_j \subset \mathcal{G}_{j-1}$ where $\mathcal{G}_{j-1}$, $j \leqslant \nu$ are defined in 5). Indeed for such $\mathbf{x} = (x_0, x_1, x_2)$ one has

$$\frac{1}{2\mathrm{M}^{\omega}_{j+1}} \leqslant m_{0,j}x_0 + m_{1,j}x_1 + m_{2,j}x_2 \leqslant \frac{1}{\mathrm{M}^{\omega}_{j+1}}, \quad 0 \leqslant j \leqslant \nu - 1.$$

So as $1/2 \leqslant x_0 \leqslant 1$ (see (22)) we have

$$\frac{1}{2\mathrm{M}_{j+1}^{\omega}} \leqslant m_{0,j} + m_{1,j}\alpha_1 + m_{2,j}\alpha_2 \leqslant \frac{2}{\mathrm{M}_{j+1}^{\omega}}, \quad 0 \leqslant j \leqslant \nu - 1.$$

As $\mathrm{M} \leqslant M \leqslant 8\mathrm{M}$ and $\omega \leqslant 3$ we have

$$\frac{1}{2^{10} M_{j+1}^{\omega}} \leqslant m_{0,j} + m_{1,j}\alpha_1 + m_{2,j}\alpha_2 \leqslant \frac{2}{M_{j+1}^{\omega}}, \quad 0 \leqslant j \leqslant \nu - 1.$$

Inequalities (8) are established.

The condition (*iii*) is a part of condition 1).

The conditions (*iv*) and (*v*) follows from the conditions on angles (7) and (24), (25).

So everything is good up to the $(\nu - 1)$-th step. Our task is to define an integer vector $\mathbf{m}_{\nu+1}$ and all related objects of the ν-th step.

8. Lemmata

Consider $\mathbf{n} = (n_0, n_1, n_2) \in \mathbb{Z}^3$ such that the triple $\mathbf{n}$, $\mathbf{m}_{\nu-1}$, $\mathbf{m}_\nu$ forms a basis of $\mathbb{Z}^3$. Such a vector does exist as the lattice $\mathcal{L}_{\nu-1}$ is complete. We may suppose that

$$\max(|n_1|, |n_2|) \leqslant \mathrm{M}_\nu, \quad \mathbf{n} \cdot \xi_{\nu-1} < 0. \tag{28}$$

We consider the two-dimensional lattices

$$\mathcal{L}_{\nu-1,\mu} = \{\mathbf{z} = \lambda_1 \mathbf{m}_{\nu-1} + \lambda_2 \mathbf{m}_\nu + \mu\mathbf{n}, \quad \lambda_1, \lambda_2 \in \mathbb{Z}\}$$

and the two-dimensional affine subspaces

$$\mathfrak{L}_{\nu-1,\mu} = \mathrm{aff}\, \mathcal{L}_{\nu-1,\mu}.$$

Write

$$\mathfrak{L}_{\nu-1} = \mathfrak{L}_{\nu-1,0}.$$

Note that

$$\mathbb{Z}^3 = \bigsqcup_{\mu \in \mathbb{Z}} \mathcal{L}_{\nu-1,\mu}.$$

In fact $\mathbf{n} \in \mathcal{L}_{\nu-1,1}$. The Euclidean distance between the neighbouring affine subspaces $\mathfrak{L}_{\nu-1,\mu}$ and $\mathfrak{L}_{\nu-1,\mu+1}$ is equal to $d_{\nu-1}^{-1}$. Put

$$\mu_* = [8d_{\nu-1}H_\nu \mathrm{M}_\nu^{-\omega}\mathrm{M}_{\nu-1}^{-1}]. \tag{29}$$

In fact μ_* is of the size

$$\mu_* \asymp M_\nu^t, \quad t = \sigma\tau - \omega > 0$$

(here the latter inequality follows from (6)).

Now we define the two-dimensional linear subspace $\ell_\nu^* \subset \mathbb{R}^3$ and a point $\mathbf{w}_\nu \in \operatorname{aff} \mathcal{L}_{\nu-1,\mu_*}$ as follows. Consider one-dimensional affine subspace

$$\pi_\nu = \ell_{\nu-1}^0 \cap \mathfrak{L}_{\nu-1} \subset \mathbb{R}^3.$$

Define

$$\ell_\nu^* = \operatorname{span}(\pi_\nu \cup \eta_\nu).$$

As $\xi_{\nu-1} \perp \mathfrak{L}_{\nu-1}$ and $\xi_{\nu-1} \in \ell_{\nu-1}^0$ we have $\xi_{\nu-1} \perp \pi_\nu$ and $\xi_{\nu-1} \perp \mathbf{m}_{\nu-1}$. Note that ℓ_ν^* can be obtained from $\ell_{\nu-1}^0$ by a rotation around π_ν. Let φ be the angle of this rotation. Suppose that η_ν' is the image of the point $\xi_{\nu-1}$ under this rotation. It will be important for us that

$$\eta_\nu' \in \mathcal{G}_{\nu-1}. \tag{30}$$

Let $\mathbf{w}_\nu \in \operatorname{aff} \mathcal{L}_{\nu-1,\mu_*}$ be the unique point such that $\mathbf{w}_\nu \perp \ell_\nu^*$. We see that $\mathbf{w}_\nu \perp \eta_\nu'$. We see that the points

$$\mathbf{m}_{\nu-1}, \quad \mathbf{w}_\nu, \quad \xi_{\nu-1}, \quad \eta_\nu' \tag{31}$$

belong to a common two-dimensional linear subspace $\mathfrak{P}$ which is orthogonal to the one-dimensional subspace π_ν. Moreover,

$$\operatorname{angle}(\mathbf{m}_{\nu-1}, \mathbf{w}_\nu) = \operatorname{angle}(\xi_{\nu-1}, \eta_\nu') = \varphi \in \left(0, \frac{2}{\mathrm{M}_\nu^\omega \mathrm{M}_{\nu-1}}\right). \tag{32}$$

Now we define the disk

$$\mathcal{D}_\nu = \left\{\mathbf{w} \in \operatorname{aff} \mathcal{L}_{\nu-1,\mu_*} : \quad |\mathbf{w} - \mathbf{w}_\nu| \leqslant \frac{H_\nu}{2^{10}}\right\}.$$

We need three more lemmas.

Lemma 3. *If* $\mathbf{w} \in \mathcal{D}_\nu$ *then*

$$H_\nu \leqslant |\overline{\mathbf{w}}| \leqslant 2^5 H_\nu. \tag{33}$$

Proof. Let

$$\rho = \mathrm{dist}(\eta'_\nu, \Xi_{\nu-1})$$

be the Euclidean distance from the point $\eta'_\nu \in \mathfrak{S}$ to the affine subspace $\Xi_{\nu-1}$. By (30) we have

$$\frac{1}{2\mathrm{M}_\nu^\omega \mathrm{M}_{\nu-1}} \leqslant \rho \leqslant \frac{1}{\mathrm{M}_\nu^\omega \mathrm{M}_{\nu-1}}. \tag{34}$$

Note that the Euclidean distance from $\mathbf{w}_\nu \in \mathfrak{L}_{\nu-1,\mu_*}$ to $\mathfrak{L}_{\nu-1,0} = \mathrm{span}\,\mathcal{L}_{\nu-1}$ is equal to $\mu_*/d_{\nu-1}$. We take into account that the points (31) belong to a common two-dimensional subspace $\mathfrak{P}$ and the equality (32) for the angles to deduce that

$$\frac{|\mathbf{w}_\nu|}{\mu_*/d_{\nu-1}} = \frac{1}{\rho}. \tag{35}$$

From (34,35) and (29) we see that

$$7H_\nu \leqslant |\mathbf{w}_\nu| \leqslant 2^4 H_\nu. \tag{36}$$

That is why for every $\mathbf{w}$ close to $\mathbf{w}_\nu$ we have

$$6H_\nu \leqslant |\mathbf{w}| \leqslant 2^5 H_\nu. \tag{37}$$

Hence for the «cut» vector $\overline{\mathbf{w}}$ we have (33). Lemma 3 is proved. □

For $\mathbf{w} = (w_0, w_1, w_2)$ we consider the two-dimensional linear subspace

$$\ell[\mathbf{w}] = \{\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 : \ \mathbf{x} \cdot \mathbf{w} = 0\}.$$

Consider a smaller ball $\mathcal{B}'_\nu \subset \mathcal{B}_\nu$ with the same center η_ν and radius

$$\frac{1}{2^7 \mathrm{M}_\nu^\omega \mathrm{M}_{\nu-1}}.$$

Lemma 4. *Suppose that* $\mathbf{w} \in \mathcal{D}_\nu$. *Then*

$$\ell[\mathbf{w}] \cap \ell_\nu^0 \cap \mathcal{B}'_\nu \neq \varnothing. \tag{38}$$

PROOF. As $\mathbf{w}_\nu \perp \eta_\nu$ and $(\mathbf{w} - \mathbf{w}_\nu) \perp \xi_{\nu-1}$ we have

$$\mathbf{w}_\nu \cdot \eta_\nu = (\mathbf{w} - \mathbf{w}_\nu) \cdot \xi_{\nu-1} = 0.$$

We make the following calculation:

$$\mathbf{w} \cdot \eta_\nu = (\mathbf{w} - \mathbf{w}_\nu) \cdot \eta_\nu = (\eta_\nu - \xi_{\nu-1}) \cdot (\mathbf{w} - \mathbf{w}_\nu).$$

As

$$|\mathbf{w} - \mathbf{w}_\nu| \leqslant \frac{H_\nu}{2^{10}} \quad \text{and} \quad |\eta_\nu - \xi_{\nu-1}| \leqslant \frac{3}{2M_\nu^\omega M_{\nu-1}}$$

by means of lower bound from (37) we have

$$\cos(\text{angle}(\mathbf{w}, \eta_\nu)) = \frac{|\mathbf{w} \cdot \eta_\nu|}{|\mathbf{w}|} \leqslant \frac{|\mathbf{w} - \mathbf{w}_\nu| \times |\eta_\nu - \xi_{\nu-1}|}{|\mathbf{w}|} \leqslant \frac{3}{2^{11} M_\nu^\omega M_{\nu-1}}.$$

That is why the angle between vectors $\mathbf{w}$ and η_ν is close to $\pi/2$ and we have $\ell[\mathbf{w}] \cap \mathcal{B}'_\nu \neq \varnothing$.

To get (38) we note that from (7) and (24,25) it follows that

$$\text{angle}(\mathbf{m}_{\nu-1}, \mathbf{m}_\nu) \geqslant \frac{1}{4}.$$

At the same time as $\mathbf{w} \in \mathcal{D}_\nu$ we have the inequality

$$\text{angle}(\mathbf{w}, \mathbf{w}_\nu) \leqslant \frac{1}{2^{10}}$$

(we take into account that for any $\mathbf{w} \in \mathcal{D}_\nu$ one has

$$|\mathbf{w} - \mathbf{w}_\nu| \leqslant \frac{H_\nu}{2^{10}}$$

and use lower bound from (37)). Moreower by (32) we see that $\text{angle}(\mathbf{m}_{\nu-1}, \mathbf{w}_\nu)$ is very small.

By the triangle inequality

$$\text{angle}(\mathbf{w}, \mathbf{m}_\nu) \geqslant \text{angle}(\mathbf{m}_{\nu-1}, \mathbf{m}_\nu) - \text{angle}(\mathbf{w}, \mathbf{m}_{\nu-1}) \geqslant$$

$$\geqslant \text{angle}(\mathbf{m}_{\nu-1}, \mathbf{m}_\nu) - \text{angle}(\mathbf{w}_\nu, \mathbf{m}_{\nu-1}) - \text{angle}(\mathbf{w}, \mathbf{w}_\nu) \geqslant \frac{1}{4} - \varphi - \frac{1}{2^{10}} \geqslant \frac{5}{2^5}.$$

Now

$$\operatorname{dist}(\eta_\nu, \ell[\mathbf{w}] \cap \ell_\nu^0) \leqslant \frac{\left| \frac{\mathbf{w}}{|\mathbf{w}|} \cdot \eta_\nu \right|}{\sin(\operatorname{angle}(\mathbf{w}, \mathbf{m}_\nu))} \leqslant \frac{1}{2^7 \mathrm{M}_\nu^\omega \mathrm{M}_{\nu-1}}.$$

Lemma is proved. □

Lemma 5. *Any disk $\mathcal{D} \subset \mathfrak{L}_{\nu-1,\mu_*}$ of radius $2\mu_* \mathrm{M}_\nu$ contains an integer point $\mathbf{m}$ such that the lattice $\langle \mathbf{m}_\nu, \mathbf{m} \rangle_{\mathbb{Z}}$ is complete.*

Proof. We know that the vectors $\mathbf{m}_{\nu-1}, \mathbf{m}_\nu$ form the basis of $\mathcal{L}_{\nu-1,0} = \mathcal{L}_{\nu-1}$. So in any disk $\mathcal{D}' \subset \mathfrak{L}_{\nu-1,0}$ of radius $2\mathrm{M}_\nu$ there exists an integer point. A similar statement is true for the lattice $\mathcal{L}_{\nu-1,1}$ also: in any disk $\mathcal{D}' \subset \mathfrak{L}_{\nu-1,1}$ of radius $2\mathrm{M}_\nu$ there exists an integer point which belongs to the lattice $\mathcal{L}_{\nu-1,1}$.

Now we take an arbitrary integer point $\mathbf{n} \in \mathcal{L}_{\nu-1,1}$. As $\mathfrak{L}_{\nu-1,0}$ and $\mathfrak{L}_{\nu-1,1}$ are neighbouring affine subspaces, the triple $\mathbf{m}_{\nu-1}, \mathbf{m}_\nu, \mathbf{n}$ is a basis of the lattice $\mathbb{Z}^3$.

Consider an integer vector $\mathbf{m} = \mu_* \mathbf{n} + \mathbf{m}_{\nu-1} \in \mathcal{L}_{\nu-1,\mu_*}$. We shall show that the lattice $\langle \mathbf{m}_\nu, \mathbf{m} \rangle_{\mathbb{Z}}$ is complete. Indeed, we can split the lattice $\mathbb{Z}^3$ into a union of two-dimensional lattices

$$\Gamma_k = \{\mu \mathbf{n} + \lambda \mathbf{m}_\nu + k \mathbf{m}_{\nu-1} : \quad \mu, \lambda \in \mathbb{Z}\}.$$

Note that $\mathbf{m}_\nu \in \Gamma_0$ and for any μ one has $\mu \mathbf{n} + \mathbf{m}_{\nu-1} \in \Gamma_1$. Affine subspaces aff Γ_0 and aff Γ_1 are neighbouring. That is why the parallelogram generated by vectors $\mathbf{m}_\nu$ and $\mu \mathbf{n} + \mathbf{m}_{\nu-1}$ has no integer points inside. This means that the lattice $\langle \mathbf{m}_\nu, \mathbf{m} \rangle_{\mathbb{Z}}$ is complete.

To finish the proof of Lemma 5 we should note that for any disk $\mathcal{D} \subset \mathfrak{L}_{\nu-1,\mu_*}$ of radius $2\mu_* \mathrm{M}_\nu$ the shifted and contracted disk

$$\mathcal{D}' = \frac{1}{\mu_*}(\mathcal{D} - \mathbf{m}_{\nu-1}) \subset \mathfrak{L}_{\nu-1,1}$$

has radius $2\mathrm{M}_\nu$ and hence contains an integer point. Lemma 5 is proved. □

Note that for all $\nu \geqslant 1$ we have $\mathrm{M}_\nu \geqslant 2^{100}$ then by (29) and (26) for the radius of a disk $\mathcal{D}$ from Lemma 5 one has

$$2\mathrm{M}_\nu \mu_* \leqslant 16 d_{\nu-1} \mathrm{M}_{\nu-1}^{-1} \mathrm{M}_\nu^{1-\omega} H_\nu \leqslant 16 \mathrm{M}_\nu^{2-\omega} H_\nu \leqslant \frac{H_\nu}{2^{13}}. \tag{39}$$

9. Fundamental Lemma: end of proof

By the inductive assumption we know that either $\mathbf{m}_{\nu-1}$ satisfies (24) and $\mathbf{m}_\nu$ satisfies (25) or $\mathbf{m}_{\nu-1}$ satisfies (25) and $\mathbf{m}_\nu$ satisfies (24), depending on the parity of ν. Let ν be even (without loss of generality). Then the angle between $\mathbf{m}_{\nu-1}$ and $\mathbf{y}^1$ is small. We define vector $\mathbf{y}^1_\nu$ as the orthogonal projection of the vector $\mathbf{y}^1$ onto the linear subspace $\mathfrak{L}_{\nu-1}$. Put

$$\mathbf{e} = (1, 0, 0) \in \mathfrak{S}.$$

As vectors ξ_j orthogonal to $\mathfrak{L}_j$ satisfy the conditions

$$\text{angle}(\xi_1, \mathbf{e}) \leqslant \frac{1}{2^{18}}, \quad \text{angle}(\xi_j, \xi_{j+1}) \leqslant \frac{2}{\mathrm{M}^{\omega}_{j+1}\mathrm{M}_j}, \quad 1 \leqslant j \leqslant \nu - 1$$

(we use (23) and

$$\xi_{j+1} \in \mathcal{C}_{j+1} \subset \mathcal{B}_j \subset \mathcal{C}_j,$$

which follows from (27)). So

$$\text{angle}(\xi_\nu, \mathbf{e}) \leqslant \frac{1}{2^{18}} + \sum_{j=2}^{\nu-1} \frac{2}{\mathrm{M}^{\omega}_{j+1}\mathrm{M}_j} \leqslant \frac{1}{2^{17}}.$$

Hence

$$\text{angle}(\mathbf{y}^1, \mathbf{y}^1_\nu) = \text{angle}(\xi_\nu, \mathbf{e}) \leqslant \frac{1}{2^{17}}. \tag{40}$$

For any vector $\mathbf{m} \in \mathcal{D}_\nu \subset \mathfrak{L}_{\nu-1,\mu_*}$ we consider its orthogonal projection $\mathbf{m}'$ onto $\mathfrak{L}_{\nu-1}$. From (32) we see that for any $\mathbf{m} \in \mathcal{D}_\nu$ one has

$$\text{angle}(\mathbf{m}, \mathbf{m}') \leqslant \frac{4}{\mathrm{M}^{\omega}_{\nu}\mathrm{M}_{\nu-1}} \leqslant \frac{1}{2^{17}}. \tag{41}$$

Let $\mathbf{u}_\nu$ be a vector parallel to π_ν with the length

$$|\mathbf{u}_\nu| = \frac{H_\nu}{2^{11}}.$$

Such a vector is defined uniquely up to its sign. We suppose in addition that

$$\mathbf{u}_\nu \cdot \mathbf{y}^1_\nu > 0.$$

This defines the sign of $\mathbf{u}_\nu$ and now the vector $\mathbf{u}_\nu$ is defined uniquely. Now we take the disk $\mathcal{D}_* \subset \mathfrak{L}_{\nu-1,\mu_*}$ of radius $2\mu_* \mathrm{M}_\nu$ from Lemma 5 such that the point $\mathbf{w}_\nu + \mathbf{u}_\nu$ is its center. From (39) we see that $\mathcal{D}_* \subset \mathcal{D}_\nu$. From (36) we see that for any $\mathbf{m} \in \mathcal{D}_*$ one has

$$\frac{1}{2^{16}} \leqslant \mathrm{angle}(\mathbf{m}', \mathbf{m}_{\nu-1}) \leqslant \frac{1}{2^{10}}. \tag{42}$$

All the vectors

$$\mathbf{m}_{\nu-1}, \quad \mathbf{m}', \quad \mathbf{y}^1_\nu, \quad \mathbf{u}_\nu$$

belong to linear subspace $\mathfrak{L}_{\nu-1}$. By the construction for any $\mathbf{m} \in \mathcal{D}_*$ one has

$$\mathrm{angle}(\mathbf{m}', \mathbf{y}^1_\nu) = \left|\mathrm{angle}(\mathbf{m}_{\nu-1}, \mathbf{y}^1_\nu) - \mathrm{angle}(\mathbf{m}', \mathbf{m}_{\nu-1})\right|.$$

That is why (42) leads to

$$\mathrm{angle}(\mathbf{m}', \mathbf{y}^1_\nu) \leqslant$$
$$\leqslant \begin{cases} \mathrm{angle}(\mathbf{m}_{\nu-1}, \mathbf{y}^1_\nu) - \dfrac{1}{2^{16}}, & \text{if} \quad \mathrm{angle}(\mathbf{m}_{\nu-1}, \mathbf{y}^1_\nu) \geqslant \mathrm{angle}(\mathbf{m}', \mathbf{m}_{\nu-1}), \\ \dfrac{1}{2^{10}}, & \text{if} \quad \mathrm{angle}(\mathbf{m}_{\nu-1}, \mathbf{y}^1_\nu) \leqslant \mathrm{angle}(\mathbf{m}', \mathbf{m}_{\nu-1}). \end{cases} \tag{43}$$

As $\mathbf{y}^1_\nu$ is the orthogonal projection onto $\mathfrak{L}_{\nu-1} \ni \mathbf{m}_{\nu-1}$ we see that

$$\mathrm{angle}(\mathbf{m}_{\nu-1}, \mathbf{y}^1_\nu) \leqslant \mathrm{angle}(\mathbf{m}_{\nu-1}, \mathbf{y}^1). \tag{44}$$

By the triangle inequality for any $\mathbf{m} \in \mathcal{D}_*$ one has

$$\mathrm{angle}(\mathbf{m}, \mathbf{y}^1) \leqslant \mathrm{angle}(\mathbf{m}, \mathbf{m}') + \mathrm{angle}(\mathbf{m}', \mathbf{y}^1_\nu) + \mathrm{angle}(\mathbf{y}^1_\nu, \mathbf{y}^1).$$

In the last inequality we substitite (40), (41), (43), (44) to get for each $\mathbf{m} \in \mathcal{D}_*$ the inequatity

$$\mathrm{angle}(\mathbf{m}, \mathbf{y}^1) \leqslant \frac{1}{2^{16}} + \mathrm{angle}(\mathbf{m}', \mathbf{y}^1_\nu) \leqslant \max\left(\mathrm{angle}(\mathbf{m}_{\nu-1}, \mathbf{y}^1), \frac{1}{2^8}\right). \tag{45}$$

Now we take $\mathbf{m}_{\nu+1} \in \mathcal{D}_*$ such that the lattice $\langle \mathbf{m}_\nu, \mathbf{m}_{\nu+1} \rangle_\mathbb{Z}$ is complete. It is possible to do it by Lemma 5. Obviously (45) and the inductive assumption leads to (24) or (25) with $j = \nu + 1$. Of course vectors $\mathbf{m}_{\nu-1}$, $\mathbf{m}_\nu$, $\mathbf{m}_{\nu+1}$ are linearly independent by the construction.

Now vector ξ_ν, subspaces Ξ_ν, ℓ^1_ν, $\ell^0_{\nu+1}$ and the set $\mathcal{G}_\nu$ are defined automatically.

From Lemma 4 we see that

$$\xi_\nu = \ell^0_\nu \cap \ell^0_{\nu+1} \cap \mathfrak{S} \in \mathcal{B}'_\nu.$$

This ensures the right embedding from (27): $\mathcal{G}_\nu \subset \mathcal{B}_\nu$. Obviously we can take the next point $\eta_{\nu+1}$ and the ball $\mathcal{B}_{\nu+1}$ satisfying all necessary conditions of the $(\nu+1)$-th step (it follows from the inequality for the angle between $\mathbf{m}_\nu$ and $\mathbf{m}_{\nu+1}$ and that $\mathrm{M}_{\nu+1}$ is much larger than M_ν).

By Lemma 3 we have

$$H_\nu \leqslant M_{\nu+1} \leqslant 2^5 H_\nu.$$

So the inductive procedure for constructing objects defined in 1)–6) is described.

We should give a comment on how to ensure the condition that the limit point $(1, \alpha_1, \alpha_2)$ consists of numbers linearly independent over $\mathbb{Z}$. As we have certain choice for the vector $\mathbf{m}_\nu$ at each step of the inductive construction we can get (α_1, α_2) satisfying linearly independence condition: we can easily enforce $\mathcal{B}_\nu$ to go away from all two-dimensional linear rational subspaces in $\mathbb{R}^3$. So the point (α_1, α_2) constructed satisfies all the conditions of Fundamental Lemma. □

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Bibliography

1. ***V. Jarník,*** *Contribution à la théorie des approximations diophantiennes linéaires et homogènes*, Czechoslovak Math. J. **4** (1954), 330–353 (in Russian, French summary).
2. ***E. V. Kolomeikina and N. G. Moshchevitin,*** *Nonrecurrence in mean of sums along the Kronecker sequence*, Math. Notes **73**, № 1 (2003), 132–135.

3. ***M. Laurent,*** *Exponents of Diophantine approximations in dimension two*, Canad. J. Math. **61**, № 1 (2009), 165–189.

4. ***N. G. Moshchevitin,*** *Diophantine approximations with positive integers*: *a remark to W. M. Schmidt's theorem*, preprint available at arXiv:0904.1906 (2009).

5. ***N. G. Moshchevitin,*** *Khintchine's singular Diophantine systems and their applications*, Russian Mathematical Surveys **65**, № 3 (2010) 43–126.

6. ***W. M. Schmidt,*** *Two questions in Diophantine approximations*, Monatshefte für Mathematik **82** (1976), 237–245.

7. ***W. M. Schmidt,*** *Open problems in Diophantine approximations*, in «Approximations Diophantiennes et nombres transcendants» Luminy, 1982, Progress in Mathematics, Birkhäuser, 271–289 (1983).

8. ***P. Thurneer,*** *On Dirichlet's theorem concerning diophantine approximations*, Acta Arithmetica **54** (1990), 241–250.

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