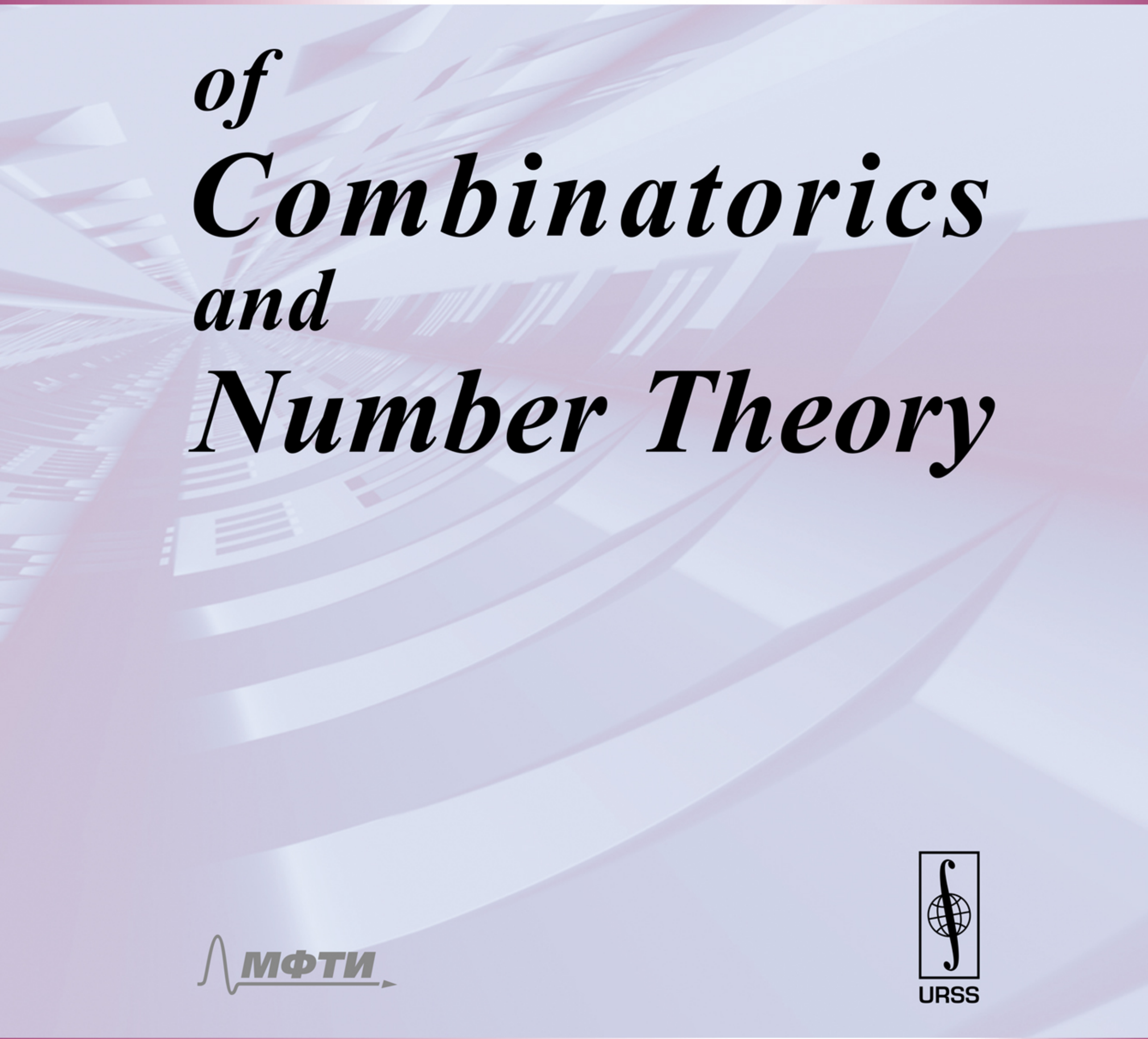
Reprint from

ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory



Volume 2 • Issue 1

2012

Moscow Journal of Combinatorics and Number Theory. 2012. Vol. 2. Iss. 1. 88 p.

The journal was founded in 2010.

Published by the Moscow Institute of Physics and Technology
with the support of Yandex and Microsoft.

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

Website of our journal
`http://mjcnt.phystech.edu`

E-mail
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URSS Publishers

`56, Nakhimovsky Prospekt,`
`Moscow,`
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`117335`

Издательство «УРСС»

`Нахимовский пр-т, 56`
`Москва,`
`Российская Федерация,`
`117335`

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77–41900.

Формат 70 × 100/16. Печ. л. 5.5. Зак. № ЖС-44.

Отпечатано в ООО «ЛЕНАНД».
117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.

ISSN 2220–5438

11995 ID 161043

9 785453 000296

Moscow Journal
of Combinatorics
and Number Theory
2012, vol. 2, iss. 1,
pp. 16–46

Holes or empty pseudo-triangles in planar point sets

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Abstract: Let $E(k,\ell)$ denote the smallest integer such that any set of at least $E(k,\ell)$ points in the plane, no three on a line, contains either an empty convex polygon with k vertices or an empty pseudo-triangle with ℓ vertices. The existence of $E(k,\ell)$ for positive integers $k,\ell \geqslant 3$, is the consequence of a result proved by Valtr [*Discrete and Computational Geometry* **37** (2007), 565–576]. In this paper, following a series of new results about the existence of empty pseudo-triangles in point sets with triangular convex hulls, we determine the exact values of $E(k,5)$ and $E(5,\ell)$, and prove bounds on $E(k,6)$ and $E(6,\ell)$, for $k,\ell \geqslant 3$. By dropping the emptiness condition, we define another related quantity $F(k,\ell)$, which is the smallest integer such that any set of at least $F(k,\ell)$ points in the plane, no three on a line, contains a convex polygon with k vertices or a pseudo-triangle with ℓ vertices. Extending a result of Bisztriczky and Tóth [*Discrete Geometry, Marcel Dekker*, 49–58, 2003], we obtain the exact values of $F(k,5)$ and $F(k,6)$, and obtain non-trivial bounds on $F(k,7)$.

Keywords: convex hull; discrete geometry; empty convex polygons; Erdős-Szekeres theorem; pseudo-triangles; Ramsey-type results

AMS Subject classification: 52C10, 52A10

Received: 15.04.2011; **revised:** 17.12.2011

1. Introduction

The famous Erdős-Szekeres theorem [9] states that for every positive integer m, there exists a smallest integer $ES(m)$, such that any set of at least $ES(m)$ points in the plane, no three on a line, contains m points which lie on the vertices of a convex polygon. Evaluating the exact value of $ES(m)$ is a long standing open problem. A construction due to Erdős [10] shows that $ES(m) \geqslant 2^{m-2}+1$, which is conjectured to be sharp. It is known that $ES(4)=5$ and $ES(5)=9$ [14]. Following a long computer search, Szekeres and Peters [22] recently proved that $ES(6)=17$. The value of $ES(m)$ is unknown for all $m>6$. The best known upper bound for $m \geqslant 7$ is due to Tóth and Valtr [23]:

$$ES(m) \leqslant \binom{2m-5}{m-3}+1.$$

In 1978 Erdős [8] asked whether for every positive integer k, there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, no three on a line, contains k points which lie on the vertices of a convex polygon whose interior contains no points of the set. Such a subset is called an *empty convex k-gon* or a *k-hole*. Esther Klein showed $H(4)=5$ and Harborth [12] proved that $H(5)=10$. Horton [13] showed that it is possible to construct arbitrarily large set of points without a 7-hole, thereby proving that $H(k)$ does not exist for $k \geqslant 7$. Recently, after a long wait, the existence of $H(6)$ has been proved by Gerken [11] and independently by Nicolás [19]. Later, Valtr [26] gave a simpler version of Gerken's proof.

These problems can be naturally generalized to polygons that are not necessarily convex. In particular, we are interested in pseudo-triangles, which are considered to be the natural counterpart of convex polygons. A pseudo-triangle is a simple polygon with exactly three vertices having interior angles less than 180°. A pseudo-triangle with ℓ vertices is called an ℓ-pseudo-triangle. A set of points is said to contain an empty ℓ-pseudo-triangle if there exists a subset of ℓ points forming an ℓ-pseudo-triangle which contains no point of the set in its interior. A pseudo-triangle with a, b, c as the convex vertices has three concave side chains between the vertices a, b and b, c, and c, a. Based on the length of the three side chains, a pseudo-triangle can be distinguished into three types: a *standard* pseudo-triangle, if each side chain has at least two edges, a *mountain*, if exactly one side chain has only one edge, and a *fan*, if exactly two side chains consist of only one edge (Fig. 1). The *apex* of a *fan* pseudo-triangle is the convex vertex having exactly one edge on both its incident side chains.

In spite of considerable research on the various combinatorial and algorithmic aspects of pseudo-triangles [21], little is known about the existence of empty pseudo-triangles in planar point sets. Kreveld and Speckmann [17] devised techniques

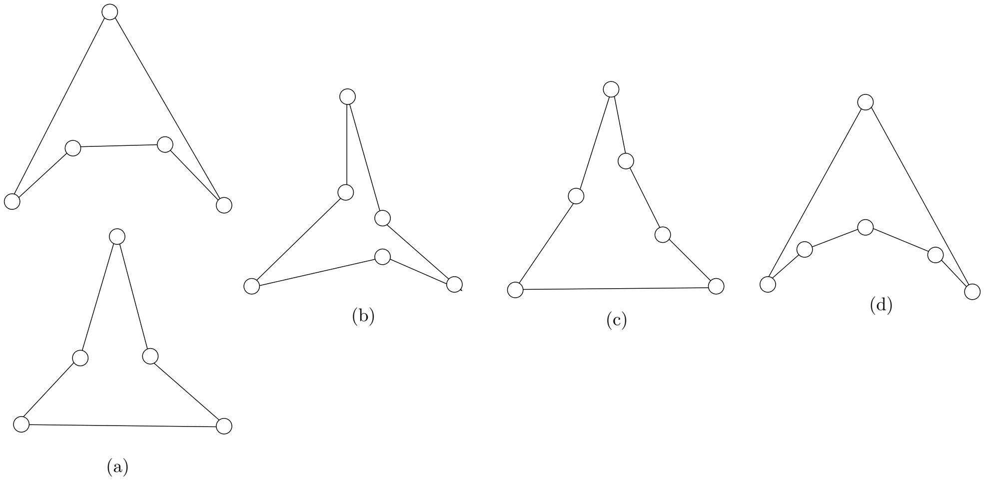


Fig. 1. Pseudo-triangles: (a) types of 5-pseudo-triangles, (b) standard 6-pseudo-triangle, (c) 6-mountain, (d) 6-fan

to analyze the maximum and minimum number of empty pseudo-triangles defined by any planar point set. Ahn et al. [1] considered the optimization problems of computing an empty pseudo-triangle with minimum perimeter, maximum area, and minimum longest concave chain.

In this paper, analogous to the quantity $H(k)$, we define the Ramsey-type quantity $E(k,\ell)$ as the smallest integer such that any set of at least $E(k,\ell)$ points in the plane, no three on a line, contains a k-hole or an empty ℓ-pseudo-triangle. The existence of $E(k,\ell)$ for all $k,\ell \geqslant 3$, is a consequence of a result proved by Valtr [25], and later by Cěrný [7].

THEOREM 1 [7, 25]. *For any k, $\ell \leqslant 3$, there is a least integer $n(k,\ell)$ such that any point p in any set S of size at least $n(k,\ell)$, no three on a line, is the apex of an empty k-fan in S or it is one of the vertices of an ℓ-hole in S.*

Using Valtr's [25] upper bound on $N(k,l)$ it is easy to see that

$$E(k,\ell) \leqslant n(k,\ell) \leqslant 2^{\binom{k+\ell-2}{k+1}} + 1.$$

However, the upper bound $n(k,\ell)$ is double exponential in $k+\ell$. In this paper, following the long and illustrious history of the quantities $ES(k)$ and $H(k)$, we consider the problem of evaluating the exact values of $E(k,\ell)$ for small values of k and ℓ. Using a series of new results about the existence of empty pseudo-triangles in point sets with triangular convex hulls, we determine new bounds on $E(k,\ell)$ for small values of k and ℓ. We begin by proving that any set of points with three points on the convex hull and at least two, three, or five interior points always contains an empty 5-pseudo-triangle, an empty 6-pseudo-triangle, or an empty 7-pseudo-triangle, respectively. Using these three results and other results from the literature, we determine the exact values of $E(k,5)$ and $E(5,\ell)$, for all $k,\ell \geqslant 3$. We also obtain bounds on $E(k,6)$ and $E(\ell,6)$, for different values of k and ℓ and discuss other implications of our results.

If the condition of emptiness is dropped from $E(k,\ell)$ we get another related quantity $F(k,\ell)$. Let $F(k,\ell)$ be the smallest integer such that any set of at least $F(k,\ell)$ points in the plane, no three on a line, contains a convex polygon with k vertices or a ℓ-pseudo-triangle. From the Erdős-Szekeres theorem it follows that $F(k,\ell) \leqslant ES(k)$ for all $k,\ell \geqslant 3$. Evaluating non-trivial bounds of $F(k,\ell)$ is also an interesting problem. While addressing a different problem, Aichholzer et al. [3] showed that $F(6,6) = 12$. In this paper, using our results on empty-pseudo-triangles and extending a result of Bisztriczky and Fejes Tóth [6], we show that $F(k,5) = 2k-3$, $F(k,6) = 3k-6$. We also obtain non-trivial bounds on $F(k,7)$, for $k \geqslant 3$. Finally, we obtain the exact value of $F(5,\ell)$ and new bounds on $F(6,\ell)$, for $\ell \geqslant 3$.

The paper is organized as follows. In Section 2 we introduce the required notation and definitions. In Section 3 we prove two preliminary observations. The results regarding the existence of empty pseudo-triangles in point sets with triangular convex hulls are presented in Section 4. The bounds on $E(k,\ell)$ and $F(k,\ell)$ are presented in Section 5 and Section 6, respectively. In Section 7 we summarize our results and give directions for future works.

2. Notation and definitions

We first introduce the definitions and notation required for the remaining part of the paper. Let S be a finite set of points in the plane in general position, that is, no three on a line. Denote the *convex hull* of S by $CH(S)$. The boundary vertices of $CH(S)$, and the points of S in the interior of $CH(S)$ are denoted by $\mathcal{V}(CH(S))$ and $\widetilde{\mathcal{I}}(CH(S))$, respectively. A region R in the plane is said to be *empty* in S if R contains no elements of S in its interior. Moreover, for any set T, $|T|$ denotes the cardinality of T.

By $\mathcal{P} := p_1p_2 \dots p_m$ we denote the region bounded by the simple polygon with vertices $\{p_1, p_2, \dots, p_m\}$ ordered anti-clockwise. Let $\mathcal{V}(\mathcal{P})$ denote the set of vertices

$\{p_1, p_2, \ldots, p_m\}$ and $\mathcal{I}(\mathcal{P})$ the interior of $\mathcal{P}$. A simple polygon $\mathcal{P}_0$ is *contained* in a simple polygon $\mathcal{P}$ if $\mathcal{V}(\mathcal{P}_0) \subseteq \mathcal{V}(\mathcal{P})$ and $\mathcal{I}(\mathcal{P}_0) \subseteq \mathcal{I}(\mathcal{P})$.

For any three points $p, q, r \in S$, $\mathcal{H}(pq, r)$ denotes the open half plane bounded by the line pq containing the point r. Similarly, $\mathcal{H}_c(pq, r)$ denotes the closed half plane bounded by the line pq containing the point r. Similarly, $\overline{\mathcal{H}}(pq, r)$ is the open half plane bounded by the line pq not containing the point r.

The *j-th convex layer* of S, denoted by $L\{j, S\}$, is the subset of points of S that lie on the boundary of

$$CH\left(S \Big\backslash \left\{ \bigcup_{i=1}^{j-1} L\{i, S\} \right\}\right),$$

where

$$L\{1, S\} = \mathcal{V}(CH(S)).$$

Moreover, if $\angle rpq < \pi$, $Cone(rpq)$ denotes the interior of the angular domain $\angle rpq$. A point $s \in Cone(rpq) \cap S$ is called the *nearest angular neighbor* of $\overrightarrow{pq}$ in $Cone(rpq)$ if $Cone(spq)$ is empty in S. Similarly, for any convex region R a point $s \in R \cap S$ is called the *nearest angular neighbor* of $\overrightarrow{pq}$ in R if $Cone(spq) \cap R$ is empty in S. Also, for any convex region R, the point $s \in S$ which has the shortest perpendicular distance to the line segment pq, $p, q \in S$, is called the *nearest neighbor* of pq in R.

3. Empty pseudo-triangles: preliminary observations

A pseudo-triangle with vertices a, b, c of the convex hull has three concave side chains between the pairs of vertices a, b and b, c, and c, a. We denote the vertices of the pseudo-triangle lying on the concave side chain between a and b by $C(a, b)$. Similarly, we denote by $C(b, c)$ and $C(c, a)$, the vertices on the concave side chains between b, c and c, a, respectively.

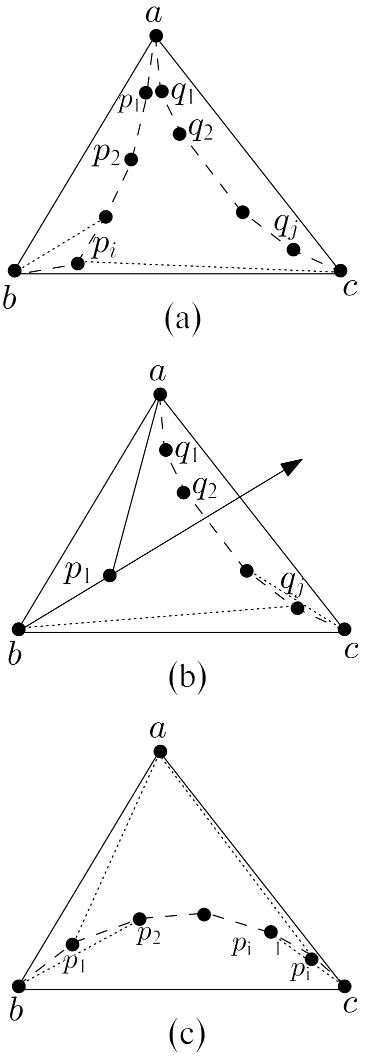
In this section, we prove two observations about transformation and reduction of pseudo-triangles.

OBSERVATION 1. *Any ℓ-pseudo-triangle can be transformed to a standard ℓ-pseudo-triangle, for every $\ell \geqslant 6$, by appropriate insertion and deletion of edges.*

PROOF. Let $\mathcal{P}$ be a non-standard ℓ-pseudo-triangle with $\ell \geqslant 6$, having convex vertices a, b, c. Then, we have the following two cases:

Case 1. Let $\mathcal{P}$ be an ℓ-mountain with convex chains $C(a,b) = \{a, p_1, p_2, \dots, p_i, b\}$, $C(b,c) = \{b, c\}$, and $C(a,c) = \{a, q_1, q_2, \dots, q_j, b\}$, such that $i + j + 3 = \ell$, arranged as shown in Fig. 2(a). Let s_α be the nearest neighbor of bc in $C(a,b) \cup C(a,c)$. Then, $\{b, s_\alpha, c\}$ are the vertices of a concave chain. If $i, j > 1$, then both $|C(a,b)\backslash\{s_\alpha\}| \geqslant 1$ and $|C(a,c)\backslash\{s_\alpha\}| \geqslant 1$, and w.l.o.g. we can assume that $s_\alpha \in C(a,b)$. In this case $s_\alpha = p_i$ and $\{a, p_1, p_2, \dots, p_{i-1}, b\}$, $\{b, p_i, c\}$, and $\{a, q_1, q_2, \dots, q_j, c\}$ are the vertices of the convex chains which form a standard ℓ-pseudo-triangle as shown in Fig. 2(a). So, w.l.o.g. it suffices to consider the case $i = 1$ (Fig. 2(b)). If $Cone(p_1bc)$ contains a point of $C(a,c)\backslash\{a,c\}$, then $\{a, p_1, b\}$, $\{b, q_j, c\}$, and $\{a, q_1, q_2, \dots, q_{j-1}, b\}$ are the vertices of the three concave chains of a standard ℓ-pseudo-triangle. Otherwise, all the points of $C(a,c)\backslash\{a,c\}$ are in $Cone(abp_1)$, and $\{a, q_1, b\}$, $\{b, p_1, c\}$, and $\{a, q_2, q_3, \dots, q_i, c\}$ are the vertices of the three concave chains of a standard ℓ-pseudo-triangle.

Case 2. Let $\mathcal{P}$ be an ℓ-fan with $C(a,b) = \{a, b\}$, $C(b,c) = \{b, p_1, p_2, \dots, p_i, c\}$ and $C(a,c) = \{a, b\}$, where $i + 3 = \ell$, as shown in Fig. 2(c). Then, the ℓ-pseudo-triangle with concave chains formed by the set of vertices $\{a, p_1, b\}$, $\{b, p_2, p_3, \dots, p_{i-1}, c\}$, and $\{a, p_i, b\}$ is standard (Fig. 2(c)). □



(a)

(b)

(c)

Fig. 2. Illustration for the proofs of Observation 1 and Observation 2

OBSERVATION 2. *An empty ℓ-mountain contains an empty m-mountain whenever $3 \leqslant m < \ell$.*

PROOF. We need to show that every empty ℓ-mountain contains an empty $(\ell - 1)$-mountain for any $\ell \geqslant 4$. Let $\mathcal{P}$ be an ℓ-mountain with $\ell \geqslant 4$, having convex vertices a, b, c. Let $C(a,b) = \{a, p_1, p_2, \dots, p_i, b\}$, $C(b,c) = \{b, c\}$, and $C(a,c) =$

$= \{a, q_1, q_2, \ldots, q_j, b\}$ be the vertices of the three concave chains of $\mathcal{P}$, such that $i + j + 3 = \ell$, as shown in Fig. 2(a). If both $i, j > 1$, then we can obtain an empty $(\ell - 1)$-mountain by taking the nearest neighbor of bc in $C(a, b) \cup C(a, c)$ and removing either b or c.

Otherwise, w.l.o.g. assume that $i = 1$. If $Cone(p_1bc) \cap (C(a, c)\backslash\{a, c\})$ is non-empty, that is, $q_j \in Cone(p_1bc) \cap (C(a, c)\backslash\{a, c\})$, then $\{a, p_1, b\}$, $\{b, q_j\}$, and $\{a, q_1, q_2, \ldots, q_j\}$ forms an empty $(\ell - 1)$-mountain (Fig. 2(b)). Similarly, if $q_j \in Cone(abp_1) \cap (C(a, c)\backslash\{a, c\})$, then $\{b, p_1, q_1\}$, $\{b, c\}$, and $\{q_1, q_2, \ldots q_j, c\}$ form an empty $(\ell - 1)$-mountain. □

4. Empty pseudo-triangles in point sets with triangular convex hulls

In this section we prove three results about the existence of empty pseudo-triangles in point sets with triangular convex hulls. These results will be used later to obtain bounds on $E(k, \ell)$ and $F(k, \ell)$.

4.1. Empty 5-pseudo-triangle

Lemma 1. *Any set S of points in the plane in general position with*

$$|CH(S)| = 3 \quad \textit{and} \quad |\widetilde{\mathcal{I}}(CH(S))| \geqslant 2$$

contains an empty 5-pseudo-triangle.

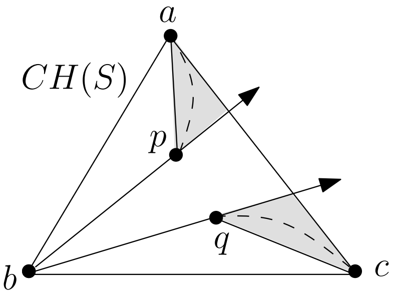


Fig. 3. Illustration for the proof of Lemma 1

Proof. Let $\mathcal{V}(CH(S)) = \{a, b, c\}$, where the vertices are taken in counter-clockwise order. Consider two points $p, q \in \widetilde{\mathcal{I}}(CH(S))$, which are consecutive in the radial order around the vertex b of $\mathcal{V}(CH(S))$, that is, $Cone(pbq)$ is empty in S. Let $C_p = \mathcal{V}(CH(\mathcal{H}_c(bp, a) \cap S))$ and $C_q = \mathcal{V}(CH(\mathcal{H}_c(bq, c) \cap S))$ (Fig. 3). Observe that $C_p \cup C_q$ forms an empty ℓ-mountain with $\ell \geqslant 5$. The existence of an empty 5-pseudo-triangle now follows from Observation 2. □

4.2. Empty 6-pseudo-triangle

Lemma 2. *Any set S of points in the plane in general position with*

$$|CH(S)| = 3 \quad \textit{and} \quad |\widetilde{\mathcal{I}}(CH(S))| \geqslant 3$$

contains an empty standard 6-pseudo-triangle.

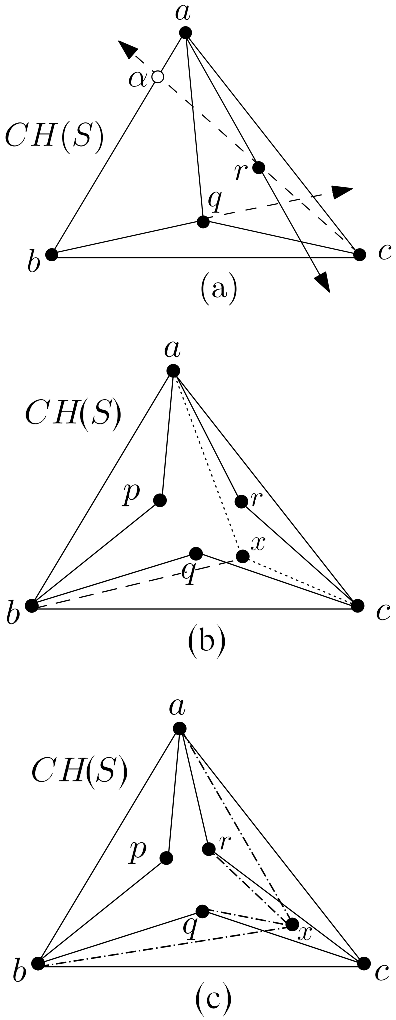


Fig. 4. Illustration for the proof of Lemma 2

Proof. Let $\mathcal{V}(CH(S)) = \{a, b, c\}$, with the vertices taken in counter-clockwise order. Suppose that $|\widetilde{\mathcal{I}}(CH(S))| = \{p, q, r\}$, and q be such that $\mathcal{I}(qbc)$ is empty in S (Fig. 4(a)). When both $\mathcal{I}(qab)$ and $\mathcal{I}(qac)$ are non-empty in S, either $apbqcr$ or $arbqcp$ forms an empty 6-pseudo-triangle. Therefore, w.l.o.g. assume that $\mathcal{I}(qab) \cap S$ is empty and $p, r \in \mathcal{I}(qac) \cap S$. Let r be the first angular neighbor of $\overrightarrow{ac}$ in $Cone(qac)$ and α be the point where $\overrightarrow{cr}$ intersects the boundary of $CH(S)$. If $p \in Cone(ar\alpha)$, then $aprcqb$ is an empty 6-pseudo-triangle. Otherwise, $Cone(ar\alpha)$ is empty and either $arcpbq$ or $arcqbp$ is an empty 6-pseudo-triangle. This empty pseudo-triangle can be transformed to an empty standard 6-pseudo-triangle by Observation 1.

Next, suppose that there are more than three points in $\widetilde{\mathcal{I}}(CH(S))$. It follows from the above that there are three points $p, q, r \in \widetilde{\mathcal{I}}(CH(S))$ such that $\mathcal{A}_1 = apbqcr$ is a standard 6-pseudo-triangle with minimal number of interior points.

If $\mathcal{A}_1$ is not empty, there exists a point $x \in S$ in the interior of $\mathcal{A}_1$. The three line segments xa, xb, and xc may or may not intersect the boundary of $\mathcal{A}_1$. If any two of these line segments, say xa and xc, do not intersect the edges of $\mathcal{A}_1$, then $\mathcal{A}_2 = apbqcx$ is a standard 6-pseudo-triangle which is contained in $\mathcal{A}_1$ (Fig. 4(b)). Otherwise, there are two segments, say xa and xb, which intersect the edges of $\mathcal{A}_1$.

In this case, $\mathcal{A}_2 = apbqxr$ is a standard 6-pseudo-triangle contained in $\mathcal{A}_1$, containing less interior points than $\mathcal{A}_1$ (Fig. 4(c)). This contradicts the minimality of $\mathcal{A}_1$ and implies that $\mathcal{A}_1$ is empty in S. $\square$

4.3. Empty 7-pseudo-triangles

Let S be a set of points in the plane in general position. An interior point $p \in S$ is called a (x, y, z) – *splitter* of $CH(S)$ if $|\mathcal{V}(CH(S))| = 3$ and the three triangles formed inside $CH(S)$ by the three line segments pa, pb, and pc contain $x \geqslant y \geqslant z$ interior points of S.

We use this definition to establish a sufficient condition for the existence of an empty 7-pseudo-triangle in sets having triangular convex hull.

Theorem 2. *Any set S of points in the plane in general position with $|CH(S)| = 3$ and $|\widetilde{\mathcal{I}}(CH(S))| \geqslant 5$ contains an empty 7-pseudo-triangle. Moreover, there exists a set S with $|CH(S)| = 3$ and $|\widetilde{\mathcal{I}}(CH(S))| = 4$, that does not contain a 7-pseudo-triangle.*

Proof of Theorem 2. We begin the proof of Theorem 2 with the following lemma:

Lemma 3. *Any set S of points in the plane, in general position, with*

$$|CH(S)| = 3 \quad \textit{and} \quad |\widetilde{\mathcal{I}}(CH(S))| \geqslant 5,$$

contains a 7-pseudo-triangle.

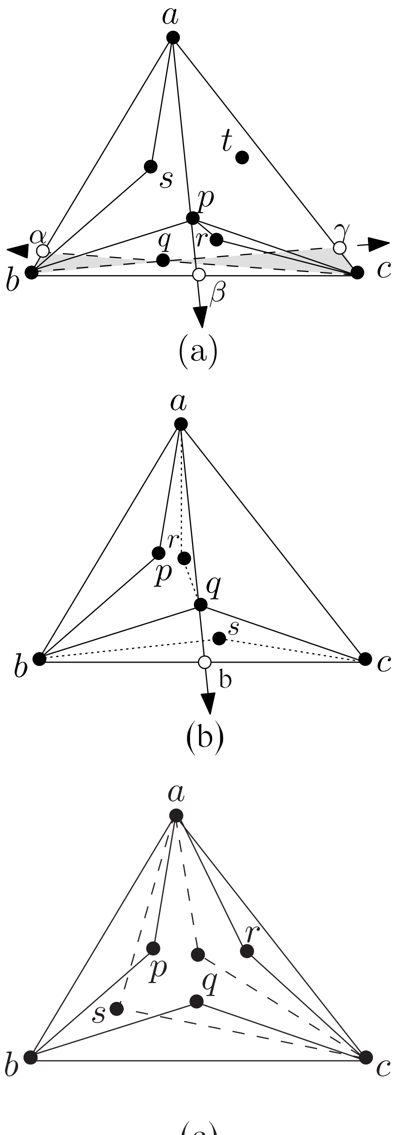
Proof. Let $\mathcal{V}(CH(S)) = \{a, b, c\}$ with the vertices taken in counter-clockwise order. Since we have to find a 7-pseudo-triangle, which is not necessarily empty, it suffices to assume that $|\widetilde{\mathcal{I}}(CH(S))| = 5$.

Assume that $p \in \widetilde{\mathcal{I}}(CH(S))$ is such that $\mathcal{I}(pab)$, $\mathcal{I}(pbc)$, and $\mathcal{I}(pca)$ are all non-empty in S. Therefore, p must be a $(2, 1, 1)$-splitter of $CH(S)$. Without loss of generality, let $q, r \in \mathcal{I}(pbc) \cap S$, $s \in \mathcal{I}(pab) \cap S$, and $t \in \mathcal{I}(pac) \cap S$ be such that q is the nearest angular neighbor of $\overrightarrow{bc}$ in $\mathcal{I}(pbc)$. Let α, β, γ be the points where $\overrightarrow{cq}$, $\overrightarrow{ap}$, $\overrightarrow{bq}$ intersect the boundary of $CH(S)$, respectively. Let $R_1 = \mathcal{I}(bq\alpha) \cap \mathcal{I}(bpc)$ and $R_2 = \mathcal{I}(cq\gamma) \cap \mathcal{I}(bpc)$ (see Fig. 5(a)). If $r \in R_1 \cup R_2$, then $asbqrcp$ or $asbrqcp$ is a 7-pseudo-triangle. Thus, assume that $(R_1 \cup R_2) \cap S$ is empty. If $r \in \mathcal{I}(\beta pc) \cap S$, then $asbqcrp$ is a 7-pseudo-triangle. Otherwise, $r \in \mathcal{I}(\beta pb) \cap S$, and $aprbqct$ is a 7-pseudo-triangle.

Therefore, suppose that none of the interior points of $CH(S)$ is a $(2, 1, 1)$-splitter of $CH(S)$. From the proof of Lemma 2 it is clear that the three vertices of $CH(S)$ along with any three points $p, q, r \in \widetilde{\mathcal{I}}(CH(S))$ form a standard 6-pseudo-triangle $\mathcal{P} = apbqcr$. This 6-pseudo-triangle has the vertices of $CH(S)$ as the three convex vertices, and it is not necessarily empty. Now, there are two cases:

Case 1. $\mathcal{P}$ is empty in S. The remaining two points s and t in $\widetilde{\mathcal{I}}(CH(S))$, must be in some of the three triangles — pab, qbc, and rca. W. l. o. g., assume that $s \in \mathcal{I}(qbc) \cap S$. Since q is not a $(2, 1, 1)$-splitter, either $\mathcal{I}(qab) \cap S$ or $\mathcal{I}(qac) \cap S$ is empty in S. If $\mathcal{I}(qac) \cap S$ is empty, $apbscqr$ is a 7-pseudo-triangle (Fig. 5(b)). Otherwise, $\mathcal{I}(qab)$ is empty in S then $apqbscr$ is a 7-pseudo-triangle.

Case 2. $\mathcal{P}$ is non-empty in S. Let $s \in \mathcal{I}(\mathcal{P}) \cap S$. If any one of three line segments sa, sb, or sc intersects the boundary of $\mathcal{P}$ we get a 7-pseudo-triangle. Otherwise, two of these three segments go directly, and we have a smaller 6-pseudo-triangle with a, b, c as its convex vertices (Fig. 5(c)). Continuing in this way, we finally get a 7-pseudo-triangle or an empty 6-pseudo-triangle with a, b, c as its convex vertices, which then reduces to *Case* 1. □



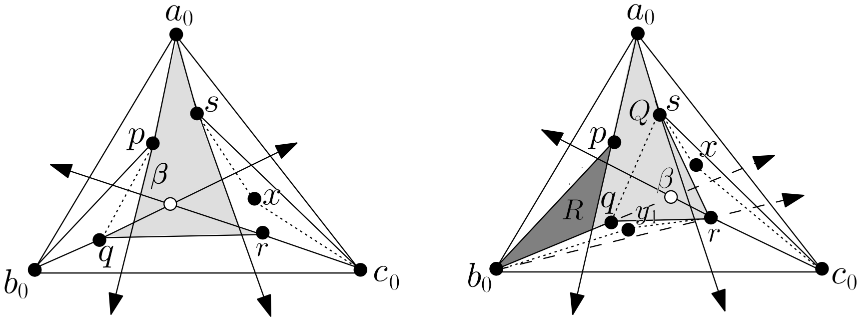
(a)

(b)

(c)

Fig. 5. Illustration for the proof of Lemma 3

Lemma 3 implies that any triangle with more than 4 interior points contains a standard 7-pseudo-triangle. Now we will obtain an empty 7-pseudo-triangle. Let S be a set of points with $|CH(S)| = 3$ and $|\widetilde{\mathcal{I}}(CH(S))| \geqslant 5$. Let $\mathcal{P}_0 = a_0pb_0qrc_0s$ be a standard 7-pseudo-triangle contained in S with convex vertices a_0, b_0, c_0, and the minimal number of interior points among all the standard 7-pseudo-triangles contained in S.



(a) (b)

Fig. 6. Existence of an empty 7-pseudo-triangle: (a) $q, r \notin \mathcal{I}(Cone(pa_0s)) \cap S$, (b) $q \in \mathcal{I}(Cone(pa_0s)) \cap S$ and $r \notin \mathcal{I}(Cone(pa_0s)) \cap S$, and (c) $q, r \in \mathcal{I}(Cone(pa_0s)) \cap S$

Note that the points a_0, b_0, c_0 may not be the vertices of $CH(S)$. Now, we have the following three cases:

Case 1. $q, r \notin Cone(pa_0s) \cap S$. Let β be the point of intersection of $\overrightarrow{b_0q}$ and $\overrightarrow{c_0r}$, and $x \in \mathcal{I}(\mathcal{P}_0) \cap S$. If $x \in \mathcal{I}(qr\beta) \cap S$, then $\mathcal{P}_1 = a_0pqxrc_0s$ is a smaller 7-pseudo-triangle contained in $\mathcal{P}_0$. Therefore, $\mathcal{I}(qr\beta) \cap S$ can be assumed to be empty. Observe that, if (i) the line segment xa_0, and either of the line segments xb_0 or xc_0 do not intersect the boundary of $\mathcal{P}_0$, or (ii) both the line segments xb_0 and xc_0 intersect the boundary of $\mathcal{P}_0$, then we can easily construct a 7-pseudo-triangle with fewer interior points than $\mathcal{P}_0$. Therefore, the shaded region Q inside $\mathcal{P}_0$, shown in Fig. 6(a), must be empty. Thus, x lies outside this shaded region and either a_0pqrc_0xs or a_0pxb_0qrs is a 7-pseudo-triangle with fewer interior points than $\mathcal{P}_0$ (Fig. 6(a)).

Case 2. $q \in Cone(pa_0s) \cap S$ and $r \notin \mathcal{I}(Cone(pa_0s)) \cap S$. By similar arguments as in *Case* 1, the lightly shaded region Q inside $\mathcal{P}_0$ shown in Fig. 6(b) is empty in S. Let R be the deeply shaded region inside $\mathcal{P}_0$ as shown in Fig. 6(b). If $x \in R$, then a_0pxb_0qrs is a 7-pseudo-triangle with fewer interior points than $\mathcal{P}_0$. Therefore, $Q \cup R$ can be assumed to be empty in S. Let $x \in \mathcal{I}(\mathcal{P}_0) \backslash (Q \cup R) \cap S$. The following cases may arise:

Case 2.1. x lies below the line $\overrightarrow{b_0r}$. Then both xa_0 and xb_0 intersect the boundary of $\mathcal{P}_0$ and a_0pb_0qrxs is a 7-pseudo-triangle with fewer interior points.

Case 2.2. x lies above the line $\overrightarrow{b_0r}$ but below $\overrightarrow{b_0q}$. Then $a_0pb_0qxc_0s$ is a 7-pseudo-triangle with fewer interior points.

Case 2.3. All the interior points of $\mathcal{P}_0$ must be above the line $\overrightarrow{b_0q}$. If $\mathcal{I}(b_0qr) \cap S$ is empty, $a_0qb_0rc_0xs$ is a 7-pseudo-triangle with fewer interior points. Otherwise, $\mathcal{I}(b_0qr) \cap S$ is non-empty. Let $Z = (\mathcal{I}(b_0qr) \cap S) \cup \{b_0, r\} = \{b_0, y_1, y_2, \ldots, y_m, r\}$, where $m \geqslant 1$.

Case 2.3.1: $|CH(Z)| \geqslant 4$, that is, $m \geqslant 2$. Then $b_0qx_0c_0ry_1 \ldots y_m$ forms an empty k-mountain, with $k \geqslant 7$, where x_0 is the nearest angular neighbor of $\overrightarrow{b_0q}$ in $\mathcal{H}(b_0q, a_0) \cap \mathcal{I}(\mathcal{P}_0)$. Hence, $\mathcal{P}_0$ contains an empty 7-pseudo-triangle from Observation 2.

Case 2.3.2. $|CH(Z)| = 3$. In this case $\mathcal{V}(CH(S)) = \{b_0, y_1, r\}$, and $\mathcal{P}_1 = b_0y_1rc_0xsq$ is a 7-pseudo-triangle having fewer interior points than $\mathcal{P}_0$.

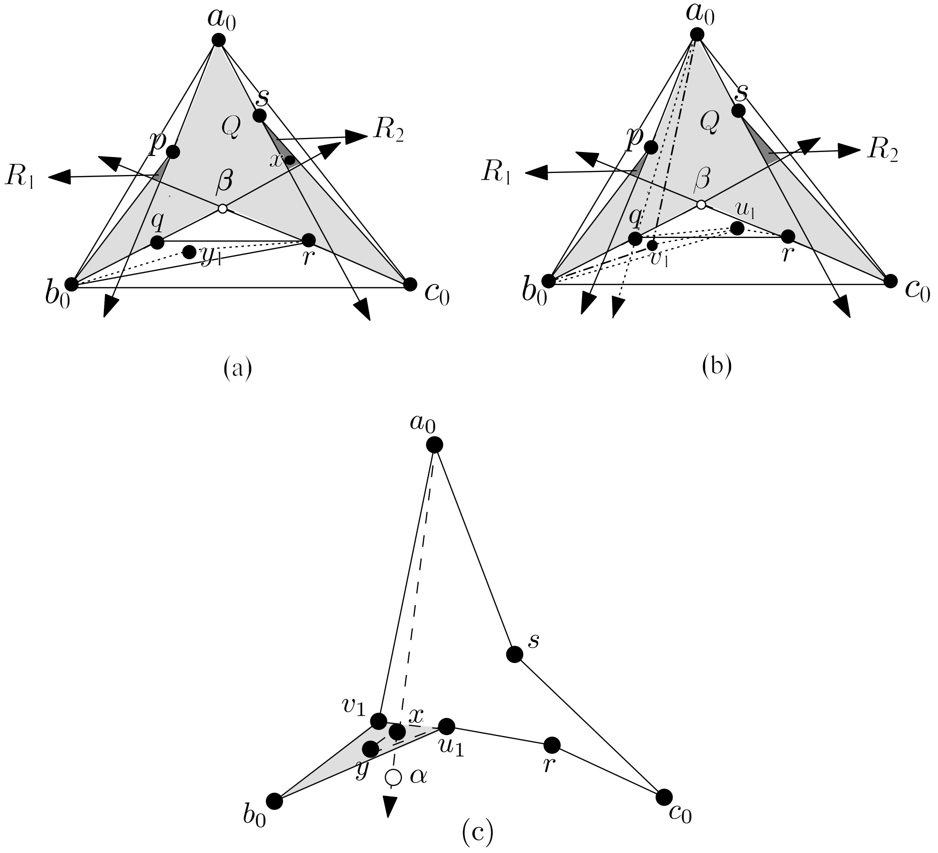
Case 3. $q, r \in Cone(pa_0s) \cap S$. By similar arguments as in *Case* 1 and *Case* 2, the lightly shaded region Q inside $\mathcal{P}_0$, shown in Fig. 7(a), can be assumed to be empty. Let $x \in (\mathcal{I}(\mathcal{P}_0) \backslash Q) \cap S = (R_1 \cup R_2) \cap S$ (see Fig. 7(a)). W.l.o.g. assume that $x \in R_2 \cap S$. The following cases may arise:

Case 3.1. $\mathcal{I}(qr\beta) \cap S$ is empty. If $\mathcal{I}(b_0qr) \cap S$ is empty, $a_0qb_0rc_0xs$ is a 7-pseudo-triangle with fewer interior points. Otherwise, $\mathcal{I}(b_0qr) \cap S$ is non-empty. Let $Z = (\mathcal{I}(b_0qr) \cap S) \cup \{b_0, r\} = \{b_0, y_1, y_2, \ldots, y_m, r\}$, where $m \geqslant 1$.

Case 3.1.1. $|CH(Z)| \geqslant 4$, that is, $m \geqslant 2$. Then $b_0qx_0cry_1 \ldots y_m$ forms an empty k-mountain, with $k \geqslant 7$, where x_0 is the nearest angular neighbor of $\overrightarrow{b_0q}$ in $\mathcal{H}(b_0q, a_0) \cap \mathcal{I}(\mathcal{P}_0)$. Hence, $\mathcal{P}_0$ contains an empty 7-pseudo-triangle from Observation 2.

Case 3.1.2. $|CH(Z)| = 3$. In this case $\mathcal{V}(CH(S)) = \{b_0, y_1, r\}$, and $\mathcal{P}_1 = b_0y_1rc_0xsq$ is a 7-pseudo-triangle having fewer interior points than $\mathcal{P}_0$.

Case 3.2. $\mathcal{I}(qr\beta) \cap S$ is non-empty. Let $z \in \mathcal{I}(qr\beta) \cap S$. If there exists another point $x \in R_1 \cup R_2$ (where R_1 and R_2 are as shown in Fig. 7(b)), then either $\mathcal{P}_1 = a_0pxb_0qzr$ (if $x \in R_1$) or $\mathcal{P}_1 = a_0qzrc_0xs$ (if $x \in R_2$) is a 7-pseudo-triangle with $|\mathcal{I}(\mathcal{P}_1) \cap S| < |\mathcal{I}(\mathcal{P}_0) \cap S|$, where z is any point in $\mathcal{I}(qr\beta)$. Therefore, assume that $R_1 \cup R_2$ is empty in S. Let $Z' = \mathcal{V}(CH((\mathcal{I}(qr\beta) \cap S) \cup \{q, r\})) = \{q, u_1, u_2, \ldots, u_w, r\}$, with $w \geqslant 1$.



(a)

(b)

(c)

Fig. 7. Existence of an empty 7-pseudo-triangle: $q, r \in \mathcal{I}(Cone(pa_0s)) \cap S$

Case 3.2.1. $|Z'| \geqslant 4$. This means $w \geqslant 2$, and $a_0pb_0qu_1 \dots u_wr$ is an empty k-mountain, with $k \geqslant 7$. This can be shortened to obtain an empty 7-mountain by Observation 2.

Case 3.2.2. $|Z'| = 3$. Then $Z' = \{q, u_1, r\}$. Now, if $|\mathcal{I}(qb_0u_1) \cap S| = 0$, then $a_0qb_0u_1rc_0s$ is 7-pseudo-triangle contained in $\mathcal{P}_0$ with less interior points. Therefore, assume that $|\mathcal{I}(qb_0u_1) \cap S| \geqslant 1$. Let

$$\begin{aligned} Z_1 &= \mathcal{V}(CH((\mathcal{I}(b_0\beta r) \cap S) \cup \{b_0, r\})) \\ &= \{b_0, v_1, v_2, \dots v_h, r\}, \end{aligned}$$

with $h \geqslant 1$. As $|\mathcal{I}(qb_0u_1) \cap S| \geqslant 1$, we have $|Z_1| \geqslant 4$, that is, $h \geqslant 2$. Consider the following two cases:

$|Z_1| \geqslant 5$: Then $a_0qb_0v_1 \dots v_hr$ is an empty k-mountain, with $k \geqslant 7$, Hence, $\mathcal{P}_0$ contains an empty 7-pseudo-triangle from Observation 2.

$|Z_1| = 4$: This implies that $Z_1 = \{b_0, u_1, v_1, r\}$. If $v_1 \in \mathcal{H}(a_0q, b_0) \cap$ $\cap S$, then $a_0qv_1u_1rc_0s$ is an empty 7-pseudo-triangle. Otherwise, $v_1 \in \mathcal{H}(a_0q, c_0) \cap S$, and $\mathcal{P}_1 := a_0v_1b_0u_1rc_0s$ is a 7-pseudo-triangle, where $\mathcal{I}(\mathcal{P}_1) \backslash \mathcal{I}(b_0u_1v_1)$ is empty in S. Now, if $\mathcal{I}(b_0u_1v_1) \cap S$ is non-empty, then $\mathcal{P}_1$ can be easily reduced to an empty 7-pseudo-triangle $a_0xyu_1rc_0s$, where x is the nearest angular neighbor of $\overrightarrow{a_0u_1}$ in $\mathcal{I}(b_0u_1v_1)$, and y is the nearest angular neighbor of $\overrightarrow{x\alpha}$ in $\mathcal{I}(b_0u_1v_1)$ (see Fig. 7(c)). Note that y may be equal to b_0.

From Lemma 3 and the three cases discussed above we obtain that any set of points S in the plane in general position with $|CH(S)| = 3$ and $|\widetilde{\mathcal{I}}(CH(S))| \geqslant 5$ contains an empty 7-pseudo-triangle.

To show that this is tight, observe that one of the side chains of a 7-pseudo-triangle must have at least three edges. Therefore, any set S with $|CH(S)| = 3$ and $|\widetilde{\mathcal{I}}(CH(S))| = 4$ containing a 7-pseudo-triangle must contain a 4-hole with exactly two consecutive vertices belonging to the vertices of $\mathcal{V}(CH(S))$. It is easy to see that this condition is violated in the point set shown in Fig. 8(a), and the result follows.

5. *E*(*k*, *ℓ*)

As mentioned earlier, $E(k, \ell)$ is the smallest integer such that any set of at least $E(k, \ell)$ points in the plane, no three on a line, contains a k-hole or an empty ℓ-pseudo-triangle. The existence of $E(k, \ell)$ for all $k, \ell \geqslant 3$, is a consequence of Theorem 1 from above [7, 25]. However, the general upper bound on $E(k, \ell)$ obtained from Valtr's [25] result is double exponential in $k + \ell$. In this section we obtain new bounds on $E(k, \ell)$ for small values of k and ℓ.

It is clear that $E(k, 3) = E(3, \ell) = 3$, for all $k, \ell \geqslant 3$. Also, $E(k, 4) = k$ for $k \geqslant 4$ and $E(4, \ell) = 5$, $\ell \geqslant 5$, since $H(4) = 5$. Using the results from the previous sections we will obtain bounds on $E(k, \ell)$ for small values of k and ℓ.

We introduce the notion of λ-convexity, where λ is a non-negative integer. A set S of points in the plane in general position is said to be λ-*convex* if every triangle determined by S contains at most λ points of S. Valtr [24, 25] and Kun and Lippner [18] proved that for any $\lambda \geqslant 1$ and $\nu \geqslant 3$, there is a least integer $N(\lambda, \nu)$ such that any λ-convex point set of size at least $N(\lambda, \nu)$ contains a ν-hole. The best known upper bound on $N(\lambda, \nu)$ for general λ and ν, due to Valtr [25], is

$$N(\lambda, \nu) \leqslant 2^{\binom{\lambda+\nu}{\lambda+2}-1} + 1,$$

which is double-exponential in $\lambda + \nu$. All known lower bounds on $N(\lambda, \nu)$ are exponential in $\lambda + \nu$.

5.1. $E(k, 5)$

In this section we determine the exact value of $E(k, 5)$. We will use Lemma 1 and a result of Károlyi et al. [15].

Although, in general, there is a gap of exponential of $\lambda + \nu$ between the best known upper and lower bounds of $N(\lambda, \nu)$, in the case when $\lambda = 1$ much more can be said. Kun and Lippner [18] proved the general upper bound

$$N(1, \nu) \leqslant 2^{\lceil (2\nu+5)/3 \rceil}.$$

Károlyi et al. [16] proved that $N(1, \nu) \geqslant M_\nu$ for odd values of ν, where

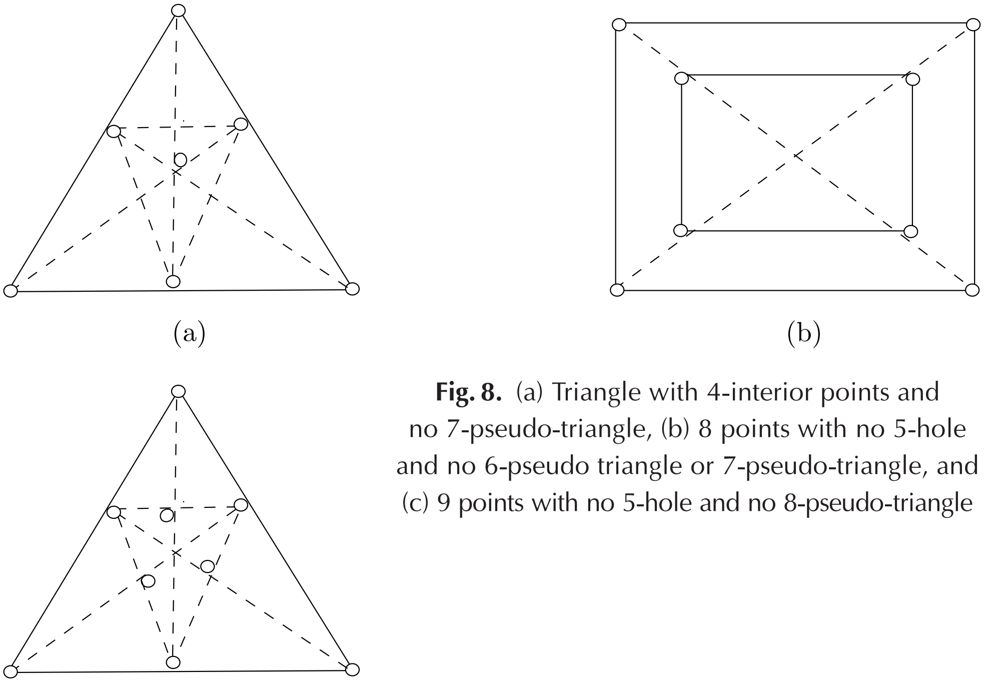
$$M_\nu := \begin{cases} 2^{(\nu+1)/2} - 1, & \text{for } \nu \geqslant 3 \text{ odd;} \\ \frac{3}{2} 2^{\nu/2} - 1, & \text{for } \nu \geqslant 4 \text{ even.} \end{cases}$$

Finally, Károlyi et al. [15] proved that for any $\nu \geqslant 3$, $N(1, \nu) = M_\nu$.

Using this result, we prove the following theorem:

THEOREM 3. *For every positive integer* $k \geqslant 3$, $E(k, 5) = M_k$.

PROOF. Let S be a set of M_k points in the plane, in general position. If there are three points in S such that the triangle determined by them contains more than 1 point of S in its interior, then by Lemma 1 S contains an empty 5-pseudo-triangle. Therefore, S contains an empty 5-pseudo-triangle unless S is 1-convex. However, the maximum size of a 1-convex set not containing a 5-hole is $N(1, k) - 1 = M_k - 1$. Therefore, if S is 1-convex, it always contains a 5-hole. This implies that $E(k, 5) \leqslant M_k$.



(a)

(b)

(c)

Fig. 8. (a) Triangle with 4-interior points and no 7-pseudo-triangle, (b) 8 points with no 5-hole and no 6-pseudo triangle or 7-pseudo-triangle, and (c) 9 points with no 5-hole and no 8-pseudo-triangle

Moreover, if a set is 1-convex, it does not contain any empty 5-pseudo-triangle. This implies that

$$E(k, 5) > N(1, k) - 1 = M_k - 1,$$

and it completes the proof that for every $k \geqslant 3$, $E(k, 5) = M_k$. $\square$

5.2. $E(5, \ell)$

It is obvious that $E(5, 3) = 3$ and $E(5, 4) = 5$. It follows from Theorem 3 that $E(5, 5) = 7$. In this section we will determine the values of $E(5, \ell)$, for $\ell \geqslant 6$. We will use the following:

THEOREM 4 [5]. *Any set Z of 9 points in the plane in general position, with $|CH(Z)| \geqslant 4$, contains a 5-hole.*

Using Lemma 1 and the above theorem, we will determine the exact values of $E(5, \ell)$ for $\ell \geqslant 6$.

THEOREM 5. *$E(5,6)=E(5,7)=9$, and $E(5,\ell)=10$, for $\ell \geqslant 8$.*

PROOF. The set of 8 points shown in Fig. 8(b) contains no 5-hole and no empty 6- or 7-pseudo-triangle. This implies that $E(5,6)>8$ and $E(5,7)>8$.

Now, consider a set S of 9 points in general position. It follows from Theorem 4 that S contains a 5-hole whenever $|CH(S)| \geqslant 4$. Now, if $|CH(S)|=3$, then $|\widetilde{\mathcal{I}}(CH(S))| \geqslant 6$, and the existence of an empty 6-pseudo-triangle and an empty 7-pseudo-triangle in S follows from Lemma 2 and Lemma 3, respectively. Therefore, $E(5,6) \leqslant 9$ and $E(5,7) \leqslant 9$, and together with the lower bound it implies that $E(5,6)=E(5,7)=9$.

We know that for $\ell \geqslant 3$, $E(5,\ell) \leqslant H(5)=10$, since every set of 10 points in general position contains a 5-hole. The set of 9 points shown in Fig. 8(c) contains no 5-hole and no empty ℓ-pseudo-triangle for $l \geqslant 8$. This implies that for $\ell \geqslant 8$, $E(5,\ell)=10$. □

5.3. $E(k, 6)$

In Lemma 2 it was proved that any set S of points in the plane in general position with $|CH(S)|=3$ and $|\widetilde{\mathcal{I}}(CH(S))| \geqslant 3$ contains an empty standard 6-pseudo-triangle. This implies that

$$E(k,6)=N(2,k) \leqslant 2^{\binom{k+2}{4}-1}+1,$$

since any 2-convex point set cannot contain a 6-pseudo-triangle.

In the special case when $k=6$ we can obtain better bounds. For this reason, we need the following technical lemma:

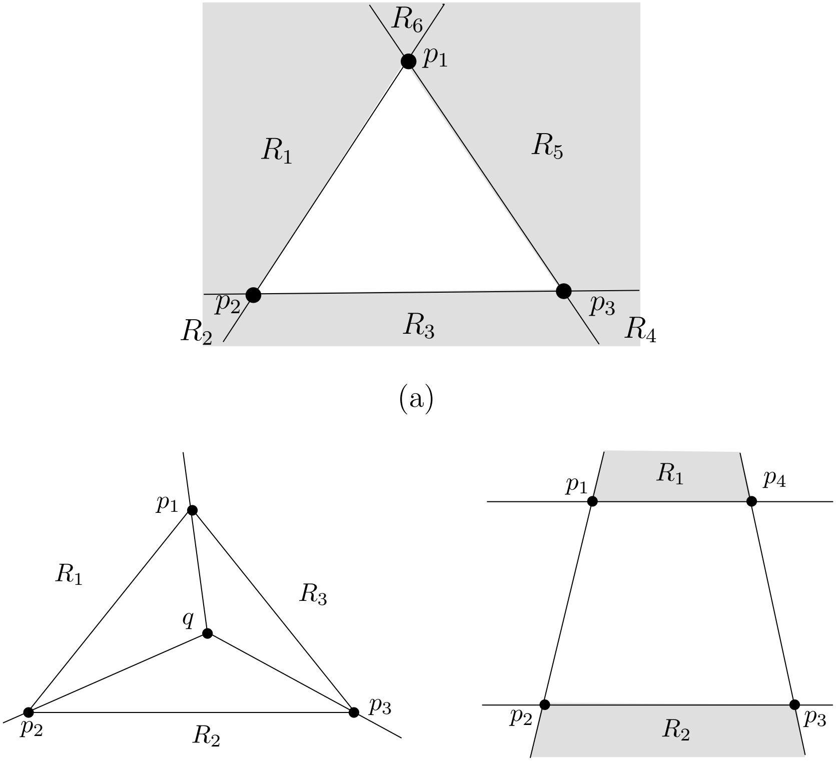
LEMMA 4. *If Z is a set of points in the plane in general position, with*

$$|CH(Z)| \geqslant 8 \quad \textit{and} \quad |\widetilde{\mathcal{I}}(CH(Z))| \leqslant 4,$$

then Z contains a 6-hole.

PROOF. To prove the lemma it is sufficient to prove the theorem for $|CH(Z)|=8$, since every convex 9-gon can be reduced to a convex 8-gon with at most as many interior points.

If $|\widetilde{\mathcal{I}}(CH(Z))|=1$, then a 6-hole can be obtained easily. Now, if $|\widetilde{\mathcal{I}}(CH(Z))|=$ $=2$, then the line joining these two points divides the plane into two half planes one



(a)

(b)

(c)

Fig. 9. Illustration for the proof of Lemma 4: (a) $|\widetilde{\mathcal{I}}(CH(Z))| = 3$, (b) $|\widetilde{\mathcal{I}}(CH(Z))| = 4$ and $|L\{2, Z\}| = 3$, and (c) $|\widetilde{\mathcal{I}}(CH(Z))| = 4$ and $|L\{2, Z\}| = 4$

of which must contain at least four points of $\mathcal{V}(CH(Z))$. These 4 points together with the two points in $\widetilde{\mathcal{I}}(CH(Z))$ form a 6-hole.

The remaining two cases are considered separately as follows:

Case 1. $|\widetilde{\mathcal{I}}(CH(Z))| = 3$. Consider the partition of the exterior of the triangle formed in the second convex layer into disjoint regions R_i as shown in Fig. 9(a). Clearly, Z contains 6-hole, unless the following inequalities hold:

$$|R_1| \leqslant 2, \quad |R_3| \leqslant 2, \quad |R_5| \leqslant 2, \tag{1}$$

$$\begin{gathered} |R_6| + |R_1| + |R_2| \leqslant 3, \\ |R_2| + |R_3| + |R_4| \leqslant 3, \\ |R_4| + |R_5| + |R_6| \leqslant 3. \end{gathered} \tag{2}$$

Summing the inequalities of (2) and using the fact $|\mathcal{V}(CH(Z))| = 8$ we get $|R_2| + |R_4| + |R_6| \leqslant 1$. Adding this inequality to those from (1) we get

$$\sum_{i=1}^{6} |R_i| \leqslant 7 < 8 = |\mathcal{V}(CH(Z))|,$$

a contradiction.

Case 2. $|\widetilde{\mathcal{I}}(CH(Z))| = 4$. We have the following two subcases based on the size of the second layer.

Case 2.1. $|L\{2, Z\}| = 3$. Then $|L\{3, Z\}| = 1$. Consider the partition of the exterior of $CH(L\{2, Z\})$ into three disjoint regions R_i as shown in Fig. 9(b). Clearly, S contains a 6-hole whenever $|R_i| \geqslant 3$, for $i \in \{1, 2, 3\}$. Otherwise,

$$|R_1| + |R_2| + |R_3| \leqslant 6 < 8 = |\mathcal{V}(CH(Z))|,$$

a contradiction.

Case 2.2. $|L\{2, Z\}| = 4$. Let $L\{2, Z\} = \{p_1, p_2, p_3, p_4\}$ be the vertices of the second layer taken in counter-clockwise order. Let R_1 and R_2 be the shaded regions as shown in Fig. 9(c). It is easy to see that S contains a 6-hole unless

$$|R_1| + |R_2| \leqslant 1, |\overline{\mathcal{H}}(p_1p_2, p_3) \cap S| \leqslant 3,$$

and $|\overline{\mathcal{H}}(p_3p_4, p_1) \cap S| \leqslant 3$. Summing these three inequalities, we get $|\mathcal{V}(CH(Z))| \leqslant 7 < 8$, a contradiction. ☐

Using this lemma we prove the following theorem:

THEOREM 6. $12 \leqslant E(6, 6) \leqslant 18$.

PROOF. Using the order-type database, Aichholzer et al. [3] obtained a set of 11 points that contains neither a convex hexagon nor a 6-pseudo-triangle. This implies that $E(6, 6) \geqslant 12$.

Consider a set S of 18 points in general position. Suppose $|CH(S)| = k \leqslant 7$ and partition $CH(S)$ into $k - 2$ triangles whose vertex set is $\mathcal{V}(CH(S))$. Since there are $18 - k$ points inside $CH(S)$, there exists a triangle which has at least

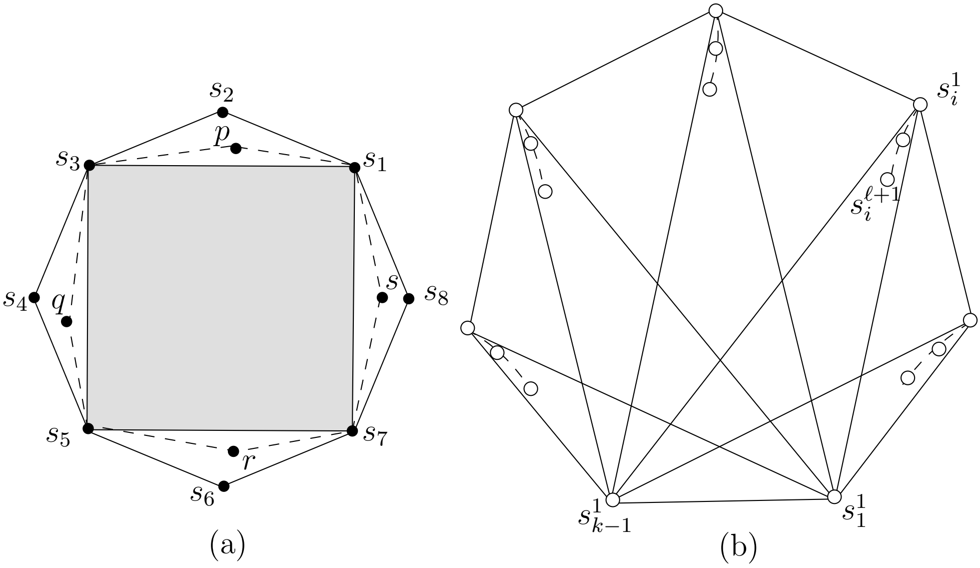


Fig. 10. (a) illustration for the proof of Theorem 6 and (b) illustration for the proof of Lemma 5

$\lceil(18-k)/(k-2)\rceil$ points of S inside it. Observe that $\lceil(18-k)/(k-2)\rceil \geqslant 3$, since $k \leqslant 7$. Therefore, if $|CH(S)| \leqslant 7$, then it is possible to find a triangle with at least three interior points and, according to Lemma 2, there exists an empty 6-pseudo-triangle.

Next, suppose that $|CH(S)| = 8$. Let

$$\mathcal{V}(CH(S)) = \{s_1, s_2, \dots, s_8\},$$

where the vertices are taken in counter-clockwise order. If $|\mathcal{I}(s_1s_3s_5s_7) \cap S| \geqslant 5$, one can find a triangle with at least three interior points and, according to Lemma 2, there is an empty 6-pseudo-triangle. Therefore, suppose that $|\mathcal{I}(s_1s_3s_5s_7) \cap S| \leqslant 4$. Let p be the nearest neighbor of the line segment s_1s_3 in $\mathcal{H}(s_1s_3, s_2) \cap S$. Note that p can be the same as s_2, if $\mathcal{I}(s_1s_2s_3) \cap S$ is empty. Similarly, let $q, r, s \in S$ be the nearest neighbors of the line segments s_3s_5, s_5s_7, and s_7s_1, respectively (see Fig. 10). Observe that the convex octagon $s_1ps_3qs_5rs_7s$ can have at most four points of S inside it. By Lemma 4 this convex octagon always contains a 6-hole.

Finally, if $|CH(S)| \geqslant 9$, then $CH(S)$ can be reduced to a convex octagon with at most as many interior points, and we can apply the same argument as before. Therefore, we have $E(6,6) \leqslant 18$. □

Remark 1. Using the order type data-base Aichholzer et al. [3] observed that there exist precisely 9 out of over 2.33 billion realizable order types of 11 points which contain neither a convex hexagon nor a pseudo-triangle with 6 vertices. Experimenting with Overmars' *empty 6-gon program* [20] we were unable to find a set of 12 points which contains no 6-hole and empty 6-pseudo-triangle. In fact, it follows from Lemma 2 and the proof of Theorem 6 that a set S of 12 points contains an empty 6-pseudo-triangle or a 6-hole whenever $|CH(S)| \leqslant 5$ or $|CH(S)| \geqslant 8$. Therefore, a set of 12 points without a 6-hole or an empty 6-pseudo-triangle must have $|CH(S)| = 6$ or $|CH(S)| = 7$. Although we were unable to geometrically show the existence of a 6-hole or an empty 6-pseudo-triangle in these two cases, experimental evidence motivates us to conjecture that $E(6,6)=12$. We believe that a very detailed analysis for the different cases that arise when $|CH(S)|$ is either 6 or 7, or some computer-aided enumeration method might be useful in settling the conjecture.

5.4. Other improvements and remarks

We now turn our attention to $E(6,\ell)$. Clearly, $E(6,\ell) \leqslant H(6)$ and $E(6,\ell) \geqslant N(\ell-4,6)$, since an $(\ell-4)$-convex set cannot contain an ℓ-pseudo-triangle. However, when $\ell = 7$ we can obtain a better upper bound $E(6,7) \leqslant 33$ using Theorem 2 and a result of Gerken [11] which says that any set which contains a 9-gon contains a 6-hole. Consider a set S of 33 points in the plane in general position. Then if $|CH(S)| \geqslant 9$, S contains a 6-hole and so we can assume that $|CH(S)| = k \leqslant 8$. $CH(S)$ can be partitioned into $k-2$ triangles whose vertex set is exactly $\mathcal{V}(CH(S))$. Since $|\widetilde{\mathcal{I}}(CH(S)| = 33-k$, one of these $k-2$ triangles contains at least $\lceil (33-k)/(k-2) \rceil$ interior points. As $k \leqslant 8$, we have $\lceil (33-k)/(k-2) \rceil \geqslant 5$, and the existence of an empty 7-pseudo-triangle in S follows from Theorem 2.

Remark 2. Note that Theorem 2 gives a proof of the existence of $E(7,7)$, which does not use Theorem 1. Valtr's result [24, 25] implies that any 4-convex set without a 7-hole has at most $N(4,7)-1$ points. So using Theorem 2 we obtain $E(7,7) \leqslant N(4,7)$. Moreover, a 3-convex set cannot contain a 7-pseudo-triangle, which implies that $E(7,7) \geqslant N(3,7)$.

If one can show that for every integer $\ell \geqslant 3$ there exists a smallest integer $\Delta(\ell)$ such that any triangle with more than $\Delta(\ell)$ interior points contains an empty ℓ-pseudo-triangle, then from Valtr's $\Delta(\ell)$-convexity result it will follow that $E(k,\ell) \leqslant N(\Delta(\ell),k)$.

The bounds obtained on the values $E(k,5)$, $E(5,\ell)$, $E(k,6)$, and $E(6,\ell)$ for different values of k and ℓ are summarized in Table 1.

Table 1

Bounds on $E(k,\ell)$

$$E(k,5) = M_k := \begin{cases} 2^{(k+1)/2} - 1, & \text{for } k \geqslant 3 \text{ odd;} \\ \frac{3}{2}2^{k/2} - 1, & \text{for } k \geqslant 4 \text{ even.} \end{cases}$$

$$E(5,\ell) = \begin{cases} 3 & \text{for } \ell = 3, \\ 4 & \text{for } \ell = 4, \\ 7 & \text{for } \ell = 5, \\ 9 & \text{for } \ell = 6, \\ 9 & \text{for } \ell = 7, \\ 10 & \text{for } \ell \geqslant 8. \end{cases}$$

$$E(k,6) = N(2,k) = \begin{cases} 3 & \text{for } k = 3, \\ 5 & \text{for } k = 4, \\ 9 & \text{for } k = 5, \\ [12, 18] & \text{for } k = 6. \end{cases}$$

$$E(6,\ell) = \begin{cases} 3 & \text{for } \ell = 3, \\ 4 & \text{for } \ell = 4, \\ 7 & \text{for } \ell = 5, \\ [12, 18] & \text{for } \ell = 6, \\ [N(3,6), 33] & \text{for } \ell = 7, \\ [N(\ell-4,6), H(6)] & \text{for } \ell \geqslant 8. \end{cases}$$

6. $F(k, \ell)$

In the previous sections we have discussed the existence of *empty* convex polygons or pseudo-triangles in point sets. If the empty condition is dropped, we get another related quantity $F(k, \ell)$, which we define as the smallest integer such that any set of at least $F(k, \ell)$ points in the plane, in general position, contains a convex k-gon or an ℓ-pseudo-triangle. From the Erdős-Szekeres theorem it follows that $F(k, \ell) \leqslant ES(k)$ for all $k, \ell \geqslant 3$. Obtaining bounds on $F(k, \ell)$ is also an interesting problem. Aichholzer et al. [3] showed that $F(6, 6) = 12$. Moreover, Aichholzer et al. [3] claim that $21 \leqslant F(7, 7) \leqslant 23$. In this section, we extend a result of Bisztriczky and Fejes Tóth [6], and obtain the exact values of $F(k, 5)$ and $F(k, 6)$, and non-trivial bounds on $F(k, 7)$.

Bisztriczky and Fejes Tóth [6] proved that any ℓ-convex point set with at least $(k-3)(\ell+1)+3$ points, not necessarily in general position, contains a convex k-gon and the bound is tight. This means that there exists a set of $(k-3)(\ell+1)+2$ points, not necessarily in general position, which is ℓ-convex but has no convex k-gon.

In the following lemma we generalize the construction of Bisztriczky and Fejes Tóth [6] to obtain a set of $(k-3)(\ell+1)+2$ points *in general position* which is ℓ-convex but has no convex k-gon, if $k < \ell/2$.

Lemma 5. *Let k, ℓ denote natural numbers such that $k \geqslant 3$ and $\ell < k/2$. Any ℓ-convex set of at least $(k-3)(\ell+1)+3$ points in the plane in general position contains k points in convex position, and this bound is tight.*

Proof. Consider an ℓ-convex set S of $(k-3)(\ell+1)+3$ points in the plane in general position. Assume that $|CH(S)| = b \leqslant k-1$. Consider a triangulation of $CH(S)$ into $b-2$ triangles. Since S is ℓ-convex, this implies that

$$|S| \leqslant b + (b-2)\ell \leqslant (k-3)(\ell+1)+2,$$

which is a contradiction.

Now we construct an ℓ-convex set Z of $(k-3)(\ell+1)+2$ points in general position, which contains no convex k-gon. Refer to Fig. 10(b). Let $s_1^1, s_2^1, \ldots, s_{k-1}^1$ be a set of $k-1$ points forming a convex $k-1$-gon ordered in counter-clockwise direction. Consider

$$Z = \left\{ s_i^j \middle| i = 2, 3, \ldots, k-2; \quad j = 1, 2, \ldots, \ell+1 \right\},$$

where for every fixed

$$i \in \{2, 3, \ldots, k-2\} \quad \text{and} \quad j \in \{2, 3, \ldots, \ell+1\}$$

the point s_i^j is inside the triangles $s_{i-1}^1 s_i^1 s_{i+1}^1$ and $s_1^1 s_i^1 s_{k-1}^1$. Moreover, depending on whether $k-1$ is even or odd the points in Z satisfy the following properties.

Case A. $k-1 = 2m$ is even. The set of points $\{s_i^j | j = 2, 3, \ldots, \ell+1\}$ lies on a concave chain $C(s_i^1, s_1^1)$ from s_i^1 to s_1^1, for $i = 2, 3, \ldots, m$. Similarly, the set of points $\{s_i^j | j = 2, 3, \ldots, \ell+1\}$ lies on a concave chain $C(s_i^1, s_{k-1}^1)$ from s_i^1 to s_{k-1}^1, for

$$i = m+1, m+2, \ldots, 2m-1 \quad (= k-2).$$

Case B. $k-1 = 2m+1$ is odd. The set of points $\{s_i^j | j = 2, 3, \ldots, \ell+1\}$ lies on a concave chain $C(s_i^1, s_1^1)$ from s_i^1 to s_1^1, for $i = 2, 3, \ldots, m$. Similarly, the set of points $\{s_i^j | j = 2, 3, \ldots, \ell+1\}$ lies on a concave chain $C(s_i^1, s_{k-1}^1)$ from s_i^1 to s_{k-1}^1, for

$$i = m+1, m+2, \ldots, 2m \quad (= k-2).$$

Clearly, $|Z| = (k-3)(\ell+1)+2$. We shall now show that the set Z constructed above is ℓ-convex. Consider three distinct points s_i^p, s_j^q, and s_k^r in S. Let $p < q < r$. We will consider three cases:

Case 1. $i = j = k$. Then $\mathcal{I}(s_i^p s_j^q s_k^r)$ is empty in Z.

Case 2. $i = j \neq k$. Then the points of Z contained in $\mathcal{I}(s_i^p s_j^q s_k^r)$ are $s_i^{p+1}, s_i^{p+2}, \ldots, s_i^{q-1}$. Therefore,

$$|\mathcal{I}(s_i^p s_j^q s_k^r) \cap S| = q - p - 1 \leqslant \ell - 1.$$

Case 3. $i \neq j \neq k$. This implies, the points of S contained in $\mathcal{I}(s_i^p s_j^q s_k^r)$ are $s_j^{q+1}, s_j^{q+2}, \ldots, s_j^{\ell+1}$. Hence,

$$|\mathcal{I}(s_i^p s_j^q s_k^r) \cap S| = \ell - q + 1 \leqslant \ell.$$

From the above three cases, we conclude that the set Z is ℓ-convex. It remains to show that it contains no convex k-gon. Let $\mathcal{P} \subset Z$ be a set of points that form a convex polygon. Let $\mathcal{P}_i \subset \mathcal{P}$ be the set of points in $\mathcal{P}$ which has subscript i, for $i \in \{2, 3, \ldots, k-2\}$.

If for all $i \in \{2, 3, \ldots, k-2\}$, $|\mathcal{P}_i| \leqslant 1$, then clearly $|\mathcal{P}| \leqslant k-1 < k$. Otherwise assume that $|\mathcal{P}_i| \geqslant 2$, for at least some $i \in \{2, \ldots, k-2\}$. Note that because of the orientations of the arrangements of the points in $\mathcal{P}_i$ along concave chains as described above, there can be at most one subscript i for which $|\mathcal{P}_i| \geqslant 3$. Next, observe that there cannot be more than 3 subscripts i such that $|\mathcal{P}_i| \geqslant 2$, since the set $\mathcal{P}_i$ is contained in triangles $s_{i-1}^1 s_i^1 s_{i+1}^1$ and $s_1^1 s_i^1 s_{k-1}^1$. If there are two subscripts $i < j$ such that

$$|\mathcal{P}_i|, |\mathcal{P}_j| \geqslant 2, \quad \text{then} \quad \mathcal{P} \subset \bigcup_{z=i}^{j} \mathcal{P}_z.$$

Therefore, if both $|\mathcal{P}_i|, |\mathcal{P}_j| \geqslant 2$, then none of the points s_1^1 and s_{k-1}^1 can be in $\mathcal{P}$. Similarly, if

$$|\mathcal{P}_i| \geqslant 2, \quad \text{then either} \quad \mathcal{P} \subset \bigcup_{z=i}^{k-1} \mathcal{P}_z \quad \text{or} \quad \mathcal{P} \subset \bigcup_{z=0}^{i} \mathcal{P}_z,$$

and only the point s_{k-1}^1 or s_1^1 can be in $\mathcal{P}$, respectively.

With these observations, we have the following two cases:

Case 1. Let $|\mathcal{P}_{i_0}| \geqslant 3$, for some i_0. We now have the following two cases:

Case 1.1. For all $i \neq i_0$, $|\mathcal{P}_i| \leqslant 1$. In this case the largest size of a convex polygon in Z can be obtained by taking all the points in $\mathcal{P}_{i_0}$, where $i_0 = (k-1)/2$ or $i_0 = k/2$, depending on whether $k-1$ is even or odd, and one point from each $\mathcal{P}_i$ on one side of P_{i_0}, depending on the curvature of the concave chain at $\mathcal{P}_{i_0}$. Therefore, the largest possible size of a convex polygon is $|\mathcal{P}| \leqslant (k-1)/2 + \ell$ for $k-1$ even, and $|\mathcal{P}| \leqslant k/2 + \ell$ for $k-1$ odd. Since $\ell < k/2$, it follows by assumption that $|\mathcal{P}| < k$.

Case 1.2. There exists some $j_0 \neq i_0$ such that $|\mathcal{P}_{j_0}| = 2$. In this case the largest size of a convex polygon can be obtained by taking i_0 as in *Case* 1.1, $j_0 = 2$ or $j_0 = k-2$, and one point from each $\mathcal{P}_i$ between $\mathcal{P}_{i_0}$ and $\mathcal{P}_{j_0}$. As none of the points s_1^1 or s_{k-1}^1 can be in $\mathcal{P}$, it follows that $|\mathcal{P}| \leqslant (k-1)/2 + \ell$ for $k-1$ even and $|\mathcal{P}| \leqslant k/2 + \ell$ for $k-1$ odd.

Case 2. Let $|\mathcal{P}_{i_0}| = 2$, for some i_0, and $|P_{j_0}| \leqslant 2$. If there exits some other $j_0 \neq i_0$ such that $|\mathcal{P}_{j_0}| = 2$, then the size of a convex polygon that can be found in Z is obtained by taking $i_0 = 2$ and $j_0 = k-2$ (or vice versa) and one point from each $\mathcal{P}_i$ between $\mathcal{P}_{i_0}$ and $\mathcal{P}_{j_0}$. Clearly, the size of the largest convex polygon that can be obtained in this way is $|\mathcal{P}| \leqslant k-1$. Otherwise, for all $i \neq i_0$, $|P_{i_0}| = 1$, and it is easy to see that $|\mathcal{P}| \leqslant k-1$. □

Using this lemma, we now obtain the exact values of $F(k,5)$ and $F(k,6)$ in the following theorem:

THEOREM 7. *For any positive integer* $k \geqslant 3$, *we have*
(*i*) $F(k,5) = 2k-3$ *for* $k \geqslant 3$,
(*ii*) $F(k,6) = 3k-6$ *for* $k \geqslant 3$.

PROOF. Lemma 1 implies that any set which has a triangle with 2 interior points has a 5-pseudo-triangle. Moreover, a 1-convex set cannot contain a 5-pseudo-triangle. Therefore, part (i) follows from Lemma 5 by with $\ell = 1$.

Similarly, Lemma 2 implies that any set which has a triangle with 3 interior points has a 6-pseudo-triangle. Moreover, a 2-convex set cannot contain a 5-pseudo-triangle. Therefore, part (ii) follows from Lemma 5 by with $\ell = 2$. □

In the following theorem, using Lemma 5 and the results on 7-pseudo-triangles, we obtain new bounds on $F(k,7)$.

THEOREM 8.

$$F(k,7) = \begin{cases} 3 & \text{for } k = 3, \\ 5 & \text{for } k = 4, \\ 9 & \text{for } k = 5, \\ [16, 17] & \text{for } k = 6, \\ [21, 23] & \text{for } k = 7, \\ [4k-9, 5k-12] & \text{for } k \geqslant 8. \end{cases}$$

PROOF. Using the fact that $ES(4) = 5$ and $ES(5) = 9$, it is easy to obtain $F(4,7) = 5$ and $F(5,7) = 9$, respectively. For $k = 6$ we slightly modify the construction in Lemma 5 to obtain a set of 15 points, shown in Fig. 11(a) which contains no 6-gon or 7-pseudo-triangle. This example and the fact that $ES(6) = 17$ [22], implies $16 \leqslant F(6,7) \leqslant 17$.

Theorem 2 implies that any triangle with 5 or more points in its interior contains a 7-pseudo-triangle. Lemma 5 with $\ell = 4$ implies that any 4-convex set

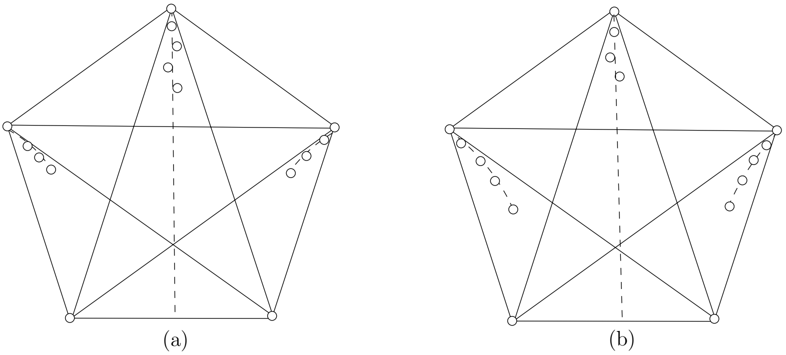


Fig. 11. (a) a set of 15 points not containing a 6-gon or a 7-pseudo-triangle, (b) a set of 16 points not containing a 6-gon or an ℓ-pseudo-triangle for $\ell \geqslant 8$

of $5k-12$ points contains a k-hole, thus proving that $F(k,7) \leqslant 5k-12$. Moreover, any 3-convex point set cannot contain a 7-pseudo-triangle. The lower bound on $F(k,7)$ now follows from the tightness part of Lemma 5, with $\ell = 3$ and $k \geqslant 7$. Therefore, for $k \geqslant 7$ we have

$$4k-9 \leqslant F(k,7) \leqslant 5k-12.$$

For $k = 7$, the above inequalities give $19 \leqslant F(7,7) \leqslant 23$. As mentioned earlier, the improved lower bound of 21 on $F(7,7)$ follows from a claim of Aichholzer et al. [2]. □

Remark 3. The set of 16 points shown in Fig. 11(b) is clearly 4-convex. This implies that it cannot contain any ℓ-pseudo-triangle, for $\ell \geqslant 8$. Moreover, arguing as in Lemma 5, it is easy to see that it contains no convex 6-gon. Since $ES(6) = 17$, we have

$$F(6,\ell) = 17, \quad \text{for} \quad \ell \geqslant 8.$$

Remark 4. Since an ℓ-convex point set cannot not contain any $(\ell+4)$-pseudo-triangle, it follows from Lemma 5 that

$$F(k,\ell+4) \geqslant (k-3)(\ell+1)+3,$$

whenever $\ell < k/2$.

The bounds obtained on the values $F(k,5)$, $F(5,\ell)$, $F(k,6)$, $F(6,\ell)$, and $F(k,7)$ for different values of k and ℓ are summarized in Table 2.

Table 2

Summary of the results

$F(k,5) = 2k-3$
$F(5,\ell) = \begin{cases} 3 & \text{for } \ell = 3, \\ 4 & \text{for } \ell = 4, \\ 7 & \text{for } \ell = 5, \\ 9 & \text{for } \ell \geqslant 6. \end{cases}$
$F(k,6) = 3k-6$
$F(6,\ell) = \begin{cases} 3 & \text{for } \ell = 3, \\ 4 & \text{for } \ell = 4, \\ 7 & \text{for } \ell = 5, \\ 12 & \text{for } \ell = 6, \\ [16, 17] & \text{for } \ell = 7, \\ 17 & \text{for } \ell \geqslant 8. \end{cases}$
$F(k,7) = \begin{cases} 3 & \text{for } k = 3, \\ 5 & \text{for } k = 4, \\ 9 & \text{for } k = 5, \\ [16, 17] & \text{for } k = 6, \\ [21, 23] & \text{for } k = 7, \\ [4k-9, 5k-12] & \text{for } k \geqslant 8. \end{cases}$

7. Conclusions

In this paper we have introduced the quantity $E(k,\ell)$, which denotes the smallest integer such that any set of at least $E(k,\ell)$ points in the plane, no three on a line, contains either an empty convex polygon with k vertices or an empty pseudo-triangle with ℓ vertices. The existence of $E(k,\ell)$ for positive integers $k,\ell \geqslant 3$, is the consequence of a result proved by Valtr [25]. However, the general upper bound on $E(k,\ell)$ is double-exponential in $k+\ell$. In this paper we prove a series of results regarding the existence of empty pseudo-triangles in point sets with triangular convex hulls. Using them we determine the exact values of $E(k,5)$ and $E(5,\ell)$, and prove improved bounds on $E(k,6)$ and $E(6,\ell)$, for $k,\ell \geqslant 3$. In particular, we show that $12 \leqslant E(6,6) \leqslant 18$ and conjecture the lower bound is, in fact, an equality. Verifying this conjecture and improving the bounds on $E(6,\ell)$, for $\ell \geqslant 7$ are interesting problems. Proving the existence of $E(k,\ell)$, for $k,\ell \geqslant 3$, without using Valtr's result [25], to obtain a better upper bound in general is also worth investigating.

We have also introduced the quantity $F(k,\ell)$, which is the smallest integer such that any set of at least $F(k,\ell)$ points in the plane, no three on a line, contains a convex polygon with k vertices or an ℓ-pseudo-triangle. We extend a result of Bisztriczky and Tóth [6] and obtain exact values of $F(k,5)$ and $F(k,6)$, and prove bounds on $F(k,7)$. Obtaining exact values of $F(k,7)$ for $k \geqslant 6$, and better general upper bounds on $F(k,\ell)$ for $k,\ell \geqslant 3$ remains open.

Acknowledgements. The authors wish to thank Bettina Speckmann for her insightful comments on the various properties of pseudo-triangles which led to simplified proofs of some of the results. The authors are indebted to an anonymous referee for valuable comments which has improved both the quality and the presentation of the paper. The authors also thank Tibor Bisztriczky for providing them copies of some of his papers.

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