

Moscow Journal

*of
Combinatorics
and
Number Theory*



Moscow Journal of Combinatorics and Number Theory. 2011. Vol. 1. Iss. 4. 112 p.

The journal was founded in 2010.

*Published by the Moscow Institute of Physics and Technology
with the support of Yandex and Microsoft.*

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

Website of our journal

<http://mjcnt.phystech.edu>

Address of the Editorial Board

Moscow institute of physics
and technology (state university)
Faculty of Innovations
and High Technology,
Laboratory Korpus, k. 209,
9, Institutskii pereulok,
Dolgoprudny,
Moscow Region,
Russia,
141700

E-mail

mjcnt@phystech.edu

Адрес редакции

Московский физико-технический
институт (государственный университет)
Факультет инноваций
и высоких технологий
Лабораторный корпус, к. 209,
Институтский переулок, д. 9,
г. Долгопрудный,
Московская область,
Российская Федерация,
141700

URSS Publishers

56, Nakhimovsky Prospekt,
Moscow,
Russia,
117335

Издательство «УРСС»

Нахимовский пр-т, 56
Москва,
Российская Федерация,
117335

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77–41900.

Формат 70 × 100/16. Печ. л. 7. Зак. № ЖТ-81.

Отпечатано в ООО «ЛЕНАНД».

117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.

ISSN 2220–5438

© УРСС, 2011

SCIENTIFIC LITERATURE
AND TEXTBOOKS

E-mail: URSS@URSS.ru
Our catalogue on the Internet:

<http://URSS.ru>

Phone/fax: +7(499) 724 25 45,
+34 (625) 37 87 73

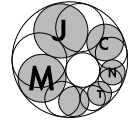


URSS

11762 ID 159732



All rights reserved. No part of this book may be used or reproduced in any manner whatsoever without written permission of the publisher.



Intrinsic approximation on Cantor-like sets, a problem of Mahler

Ryan Broderick (Evanston), Lior Fishman (Denton),
Asaf Reich (Waltham)

Abstract: In 1984, Kurt Mahler posed the following fundamental question: How well can irrationals in the Cantor set be approximated by rationals in the Cantor set? Towards development of such a theory, we prove a Dirichlet-type theorem for this intrinsic Diophantine approximation on Cantor-like sets, and discuss related possible theorems and conjectures. The resulting approximation function is analogous to that for \mathbb{R}^d , but with d being the Hausdorff dimension of the set, and logarithmic dependence on the denominator instead.

Keywords: Dirichlet's approximation theorem, intrinsic, approximation, Cantor sets

AMS Subject classification: 11K60, 11J83 (Primary), 28A80 (Secondary)

Received: 13.10.2011; **revised:** 29.11.2011

1. Introduction

The Diophantine approximation theory of the real line is classical, extensive, and essentially complete as far as characterizing how well real numbers can be approximated by rationals ([11] is a standard reference). The basic result on approximability of all reals is

THEOREM 1 (DIRICHLET'S APPROXIMATION THEOREM). *For each $x \in \mathbb{R}$, and for any $Q \in \mathbb{N}$, there exist $p/q \in \mathbb{Q}$ with $1 \leq q \leq Q$, such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ}.$$

COROLLARY 1. For every $x \in \mathbb{R}$,

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2} \quad \text{for infinitely many } \frac{p}{q} \in \mathbb{Q}. \quad (1)$$

That this rate cannot be improved significantly for all reals is evidenced by the existence of badly-approximable numbers (see definition 1). Furthermore, the rate cannot be improved for almost all irrationals, in the sense that the set of very well approximable numbers (see definition 2) is null. In addition, the subject of approximating points on fractals by rationals has been extensively studied in recent years; see for example [3], [7], and [8] for badly approximable numbers and [6] and [12] for very well approximable numbers.

In 1984, K. Mahler published a paper entitled “Some suggestions for further research” [10], in which he writes: “At the age of 80 I cannot expect to do much more mathematics. I may however, state a number of questions where perhaps further research might lead to interesting results.” One of these questions was regarding intrinsic and extrinsic approximation on the Cantor set. In Mahler’s words, “how close can irrational elements of Cantor’s set be approximated by rational numbers

1. In Cantor’s set, and
2. By rational numbers not in Cantor’s set?”

With the exception of [1], where explicit construction of elements of the middle third Cantor set with any prescribed irrationality exponent were investigated, there has been very little research on such intrinsic Diophantine approximation on fractals, though there is such for algebraic varieties (see for example [5]). It is also worth noting that while addressing non-intrinsic questions, the authors of [9] used intrinsic techniques. Studying the intersection of C with numbers approximable by rationals p/q with q of the specific form 3^n , they obtain lower bounds for general, non-intrinsic approximation. They point out that this type of rational approximation to a point of C will necessarily lie in C . Thus some of their results are *de facto* also results giving lower bounds on approximations on C by any rationals in C . However, these results are not likely to be optimal since only Cantor rationals with denominators of the form 3^n are considered, so we wish to consider intrinsic approximation as a topic in and of itself. We present some initial steps towards a theory determining the analogous optimal rate of intrinsic approximation for the Cantor set (and those constructed similarly).

The paper is organized as follows. In the following section we present our main result. Section 3 is devoted to formulating our conjecture regarding the distribution of rationals on the Cantor set, which if true would allow us to conclude that our main theorem is optimal. The last section is devoted to extrinsic approximation on the Cantor set, i. e., approximating irrationals on the Cantor by rationals not on the Cantor set.

2. Dirichlet type theorem for Cantor sets

Let C denote the Cantor-like set consisting of numbers in $I = [0, 1]$ which can be written in base $b > 2$ using only the digits in $S \subset \{0, 1, \dots, b-1\}$, where $|S| = a > 1$. Obviously this is equivalent to partitioning I into b equal subintervals, only keeping those indexed by S , and successively continuing this on each remaining subinterval. (The usual middle-third set is given by $b = 3$, $S = \{0, 2\}$.) We will assume that S is a proper subset of $\{0, 1, \dots, b-1\}$, i. e. that $C \neq I$, as the case $C = I$ is simply the classical one.

Throughout the paper we denote by $\{x\}$ the fractional part of x and by $[x]$ the integer part. To state the theorem we also associate to the set the number b_0 , where b_0 is the least integer greater than 1 such that C is invariant under $x \mapsto \{b_0 x\}$. As we shall see, this is simply equal to b except when the given definition of C is redundant in a certain sense, in which case we still have $b = b_0^r$ for some $r \in \mathbb{N}$.

THEOREM 2 (DIRICHLET FOR CANTOR SETS). *Let $d = \dim C$. For every $x \in C$ and every Q of the form b^n , there exists $p/q \in \mathbb{Q} \cap C$ with $q \leq b_0^{Q^d}$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ}.$$

It follows from the statement that for any $Q \in \mathbb{N}$, the same statement holds with the bound multiplied by b (by letting $b^n \leq Q < b^{n+1}$). As an immediate corollary to Theorem 2 we get:

COROLLARY 2. *For all $x \in C$, there exist infinitely many solutions $p \in \mathbb{N}$, $q \in \mathbb{N}$, $p/q \in C$ to*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q(\log_{b_0} q)^{1/d}}.$$

Before we begin the proof, we first need a characterization of the rationals in C :

LEMMA 1. *A rational number is in C if and only if it can be written either as a terminating base- b expansion (left end points in the construction) or as*

$$\frac{\left(\sum_{i=0}^{k+l-1} c_i b^{k+l-1-i} - \sum_{j=0}^{k-1} c_j b^{k-1-j} \right)}{b^{k+l} - b^k}, \tag{2}$$

where $l, k \in \mathbb{N}$, and $c_i \in S$. Equivalently (2) can also be expressed in terms of base- b expansions, i. e.,

$$\frac{((c_0 c_1 \dots c_{k+l-1})_b - (c_0 c_1 \dots c_{k-1})_b)}{b^{k+l} - b^k}. \tag{3}$$

PROOF OF LEMMA 1. A rational in C has either a terminating b -ary expansion (consisting of digits from S) or an eventually periodic one. If it is purely periodic of period l , it has the form

$$\sum_{n=1}^{n=\infty} \frac{b^{l-1}x_0 + b^{l-2}x_1 + \dots + x_{l-1}}{(b^l)^n} = \frac{b^{l-1}x_0 + b^{l-2}x_1 + \dots + x_{l-1}}{b^l - 1},$$

where the digits x_i are in S . If the rational is not purely periodic then one must insert some number k of initial zeros and then add the initial terminating expansion of length k , so we obtain the form

$$\frac{b^{l-1}x_0 + b^{l-2}x_1 + \dots + x_{l-1}}{(b^l - 1)b^k} + \frac{(b^l - 1)(b^{k-1}y_0 + b^{k-2}y_1 + \dots + y_{k-1})}{(b^l - 1)b^k}$$

for some digits $y_i \in S$. Rearranging this gives the result. □

PROOF OF THEOREM 2. Let $x \in C \setminus \mathbb{Q}$. Given any $n \in \mathbb{N}$, let $Q = b^n$. Denote by $M = (m_i)$ the semigroup of positive integer multiplication maps mod 1 leaving C invariant. We order the elements in increasing order from $1 = m_0$.

There are a^n possibilities for the first n digits in the b -ary expansion of $\{qx\}$. Consider the elements of C given by $0, \{m_0x\}, \dots, \{m_{a^n-1}x\}$. By the pigeonhole principle either there exist $0 \leq q, q' < a^n$ such that $\{m_qx\}$ and $\{m_{q'}x\}$ have the same first n digits, or the same holds for some $0 \leq q < a^n$ and 0 . That is, they are in the same interval of the n -th stage of C 's construction. Assuming the former, it

follows that $|\{m_{q'}x\} - \{m_qx\}| < 1/Q$. Rewriting gives

$$|m_{q'}x - [m_{q'}x] - m_qx + [m_qx]| < \frac{1}{Q}.$$

Setting $p = [m_{q'}x] - [m_qx]$ we get

$$\left| x - \frac{p}{m_{q'} - m_q} \right| < \frac{1}{(m_{q'} - m_q)Q}.$$

If one of the above values is 0 rather than an m_i the calculation is trivial. Suppose M were generated by more than one element. Then by H. Furstenberg's seminal result in [4], stating that the only infinite closed subset of \mathbb{R}/\mathbb{Z} invariant under M is \mathbb{R}/\mathbb{Z} itself, we would have to have $C = I$. But we assumed $C \neq I$, so that cannot be true. (Notice however that we recover Dirichlet's original theorem in this case, since then $M = \mathbb{N}$, so $m_{q'} - m_q < a^n = Q$.) So M has a single generator, which we will denote b_0 , and then $b = b_0^r$ for some r , and $a = a_0^r$. C is also the set constructed in the corresponding way for b_0 and some S_0 (namely, the unique subset of $\{0, 1, \dots, b_0 - 1\}$ such that S is the set S_0^r of r -fold concatenations of elements of S_0). Then $m_{a^n-1} = b_0^{a^n-1}$. Since $m_{q'} - m_q = b_0^k(b_0^d - 1)$ for some integers k, d , following (3) it suffices to observe that

$$p = [b_0^{k+d}x] - [b_0^kx].$$

Thus

$$\frac{p}{m_{q'} - m_q} \in C, \quad \text{and} \quad m_{q'} - m_q < b_0^{a^n} = b_0^{Q^d}. \quad \square$$

It is worth noticing a similarity in form between Corollary 2 and the more general form of Dirichlet's theorem, stating that points in \mathbb{R}^d are approximable according to the function $1/qq^{1/d}$. However there is a "gap" between the approximation functions in these two theorems — the $q^{1/d}$ from the classic case is replaced by a $(\log_{b_0} q)^{1/d}$ in the fractal case, so that the intrinsic approximation function on C does not tend to q^{-2} even as $\dim C$ tends to 1. The reason for this dichotomy is evident from the proof, which hinges on the growth of m_k in terms of k : When M has two generators, $m_k = k$, corresponding to the classical case, and when M has a single generator, $m_k = b_0^k$, corresponding to the fractal case. These two possibilities for

the asymptotics of m_k — either exponential or linear — lead to the corresponding differences in the asymptotics of the approximation functions.

3. Further investigation, conjectures and consequences

The obvious next step would be to check whether this is the optimal rate of approximation for C . We first recall the following definitions:

DEFINITION 1. *The set of badly approximable numbers (**BA**) consists of all reals x such that for some $c(x) > 0$*

$$\left| x - \frac{p}{q} \right| > \frac{c(x)}{q^2} \quad \text{for all } \frac{p}{q} \in \mathbb{Q}.$$

We thus define, relative to the approximation function proven above, the *intrinsic* set of badly approximable numbers, denoted by \mathbf{BA}_C , as those in C satisfying

$$\left| x - \frac{p}{q} \right| > \frac{c(x)}{q(\log_{b_0} q)^{1/d}} \quad \text{for all } \frac{p}{q} \in \mathbb{Q} \cap C.$$

DEFINITION 2. *The set of very well approximable numbers (**VWA**) is the set of all reals x satisfying for some positive $\epsilon(x)$*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon(x)}}$$

for infinitely many rationals p/q .

We thus define the *intrinsic VWA* numbers, denoted \mathbf{VWA}_C , as those in C satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{1+\epsilon(x)}} \quad \text{for infinitely many } \frac{p}{q} \in \mathbb{Q} \cap C. \quad (4)$$

One could definitively say that our approximation function is optimal if one proved that intrinsic \mathbf{BA}_C is nonempty, and one could conclude that the power of q in the denominator is correct if \mathbf{VWA}_C has zero d -dimensional Hausdorff measure.

Even restricting ourselves to the case where C is the classical Cantor ternary set, these results appear difficult to achieve, however, without having a deeper understanding of the rationals in C . For the real line, there exists a large arsenal of useful information regarding distribution properties of rationals, their quantity within

bounds on q , etc., whereas for C nothing is even known about which denominators can appear (in reduced form, of course; the expression (3) is not useful here since it is not reduced). In fact Mahler points out basically the same fundamental difficulty. To obtain the desired results, we believe a major new piece of information will be necessary, such as a sharp bound either on how many rationals appear in C with (reduced) denominators in a given range or on how these rationals “repel” each other as a function of q . We nonetheless make the following conjecture for the standard Cantor set supported by computer calculations (see table 1 below).

CONJECTURE 1. Let

$$N(s, t) = \left| \left\{ \frac{p}{q} \in C, \gcd(p, q) = 1, s \leq q \leq t \right\} \right|.$$

Then

$$N(3^n, 3^{n+1}) \in O(2^{(1+\epsilon)n})$$

for all $\epsilon > 0$.

Notice that defining

$$\phi(n) = \frac{1}{2} \frac{N(3^n, 3^{n+1})}{N(3^{n-1}, 3^n)},$$

the conjecture is equivalent to the convergence of $\phi(n)$ to 1. The table below sums up our computer based results:

Table 1

n	4	5	6	7	8	9	10	11
$\phi(n)$	1.808	1.064	1.32	1.18	1.258	1.057	1.176	1.063

The conjecture is significant, implying that the power of q in our theorem cannot be improved.

COROLLARY OF CONJECTURE 1. $\mu(\mathbf{VWA}_C) = 0$, where μ is the d -dimensional Hausdorff measure restricted to C .

PROOF. Let V_ϵ denote the subset of \mathbf{VWA}_C satisfying (4) for a particular $\epsilon(x) = \epsilon$, and observe that V_ϵ is the lim sup of the sets

$$S_n = \bigcup_{\substack{p/q \in C, \\ 3^n \leq q < 3^{n+1}}} \left(\frac{p}{q} - \frac{1}{q^{1+\epsilon}}, \frac{p}{q} + \frac{1}{q^{1+\epsilon}} \right). \tag{5}$$

By the Borel–Cantelli lemma, $\mu(V_\epsilon) = 0$ if $\sum_{n=0}^\infty \mu(S_n) < \infty$. The number of intervals in the union S_n is by assumption in $O(2^{(1+\epsilon')n})$ for all $\epsilon' > 0$. The radius of each interval is $O\left(\frac{1}{3^{(1+\epsilon)n}}\right)$, and μ satisfies a d -power law, so we have that $\mu(B(x, r))$ is $O(r^d)$ for $x \in C$ and small r . So the measure of each interval is $O\left(\frac{1}{2^{(1+\epsilon)n}}\right)$, and

$$\mu(S_n) \in O\left(\frac{2^{(1+\epsilon')n}}{2^{(1+\epsilon)n}}\right) = O(2^{(\epsilon'-\epsilon)n}) \quad \text{for all } \epsilon' > 0. \tag{6}$$

Thus we take $\epsilon' < \epsilon$, and obtain that $\sum_{n=0}^\infty \mu(S_n) < \infty$ as desired. Finally observe $\mathbf{VWA}_C = \bigcup_{m=1}^\infty V_{1/m}$, so $\mu(\mathbf{VWA}_C) = 0$. □

At this point the reader might be thinking that a stronger bound on N would allow one to prove that Theorem 2 is even sharper: specifically, if $N(3^n, 3^{n+1}) \in O(2^n)$, then a proof of similar type would show that the superset

$$\left\{ x : \left| \frac{p}{q} - x \right| < \frac{1}{q(\log_{b_0} q)^{1/d+\epsilon(x)}} \quad \text{for infinitely many } \frac{p}{q} \in C \right\} \supset \mathbf{VWA}_C$$

is μ -null. However this avenue is not possible, as a forthcoming followup paper by Noam Solomon and Barak Weiss discussing Conjecture 1 and partial progress towards it will prove that this stronger bound on N does not hold.

4. Extrinsic approximation

As mentioned in the introduction, Mahler also raised the complementary *extrinsic* question: how well can a rational in C be approximated by rationals *outside* of C ? To this end, consider a weaker variant of Conjecture 1:

CONJECTURE 2. $N(3^n, 3^{n+1}) \in O(2^{\gamma n})$ for some $\gamma < 2$.

Note that a confirmation of this conjecture (for which our calculations are quite strong evidence) would be sufficient to show, by an argument similar to the one above, that the set

$$\left\{ x \in C : \left| \frac{p}{q} - x \right| < \frac{1}{q^2} \text{ for infinitely many } \frac{p}{q} \in C \right\} \quad (7)$$

is μ -null. In turn, consider the set of $x \in C$ such that

$$\text{For all } \epsilon > 0, \text{ there exist infinitely many } \frac{p}{q} \notin C \text{ satisfying } \left| \frac{p}{q} - x \right| < \frac{\epsilon}{q^2}. \quad (8)$$

We denote this extrinsic analogue of **WA**, the complement of **BA**, by \mathbf{WA}_{C^c} and its complement by \mathbf{BA}_{C^c} . Thus (7), combined with [2, Corollary 1.10] stating that $\mathbf{BA} \cap C$ is μ -null, show the following:

COROLLARY OF CONJECTURE 2. \mathbf{WA}_{C^c} is of μ -full measure.

We note that the sharpness in either direction of this statement is already known by [7] and [12].

Bibliography

1. **Y. Bugeaud**, *Diophantine approximation and Cantor sets*, *Mathematische Annalen* **341** (2008), № 3, 677–684.
2. **M. Einsiedler, L. Fishman, U. Shapira**, *Diophantine approximations on fractals*, *Geometric and Functional Analysis* **21** (2011), № 1, 14–35.
3. **L. Fishman**, *Schmidt's game on fractals*, *Israel J. Math.* **171** (2009), № 1, 77–92.
4. **H. Furstenberg**, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, *Math. Systems Theory* **1** (1967), 1–49.
5. **A. Ghosh, A. Gorodnik, A. Nevo**, *Diophantine approximation and automorphic spectrum*, Preprint, arXiv:1007.0593v1.
6. **D. Kleinbock, E. Lindenstrauss, B. Weiss**, *On fractal measures and Diophantine approximation*, *Selecta Math.* **10** (2004), 479–523.
7. **D. Kleinbock, B. Weiss**, *Badly approximable vectors on fractals*, *Israel Journal of Mathematics* **149** (2005), 137–170.
8. **S. Kristensen, R. Thorn, L. Velani**, *Diophantine approximation and badly approximable sets*, *Advances in Math.* **203** (2006), 132–169.

9. **J. Levesley, C. Salp, S. Velani**, *On a problem of K. Mahler: Diophantine approximation and Cantor sets*, Math. Ann. **338** (2007), 97–118.
10. **K. Mahler**, *Some suggestions for further research*, Bulletin of the Australian Mathematical Society **29** (1984), 101–108.
11. **W. M. Schmidt**, *Diophantine approximation*, Lecture Notes in Mathematics **785** (1980), Springer-Verlag, Berlin.
12. **B. Weiss**, *Almost no points on a Cantor set are very well approximable*, Proc. R. Soc. Lond. A **457** (2001), 949–952.

RYAN BRODERICK

Department of Mathematics,
Northwestern University,
2033 Sheridan Road,
Evanston IL 60208–2370, USA
ryan.broderick@northwestern.edu

LIOR FISHMAN

Department of Mathematics,
University of North Texas,
1155 Union Circle 311430,
Denton TX 76203–5017, USA
lfishman@unt.edu

ASAF REICH

Department of Mathematics,
Brandeis University,
415 South Street,
Waltham MA 02454–9110, USA
azmreich@brandeis.edu