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On the asymptotic distribution of integer matrices

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Abstract: We give an asymptotic formula for the number of integer matrices M such that $M \in \Omega$, $\det M = N$, where Ω is a domain of some special form, and N is a given positive integer. Using this result, we obtain the formula

$$E_s(N) = \mathcal{C}(s) \cdot \ln^{s-1} N + O_s\big((\ln\ln N) \cdot \ln^{s-2} N\big) \quad \text{for any integer} \quad N > 2,$$

where $E_s(N)$ is the average number of the local minima of integer s-dimensional lattices Γ with $\det \Gamma = N$, and $\mathcal{C}(s)$ is a positive constant.

Keywords: distribution of integer matrices, local minimum, multidimensional continued fraction, average length of continued fraction

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1. Notation

We use the following notation:

1. $\#S$ is the number of elements in a finite set S;
2. $\sigma_\alpha(N) = \sum_{d|N} d^\alpha$ (d are divisors of N, $\alpha \in \mathbb{R}$);
3. $\mathrm{M}_{t,s}(Q)$ is the set of $t \times s$ matrices $((x_{ij}))$ (i is the row index, j is the column index) such that $x_{ij} \in Q$ ($Q = \mathbb{R}$ or $Q = \mathbb{Z}$);
4. $\mathrm{M}_s(Q) = \mathrm{M}_{s,s}(Q)$;

5. $\mathrm{M}_s(Q, N) = \{X \in M_s(Q) : \det X = N\}$;
6. $\mathrm{GL}_s(\mathbb{R}) = \{X \in \mathrm{M}_s(\mathbb{R}) : \det X \neq 0\}$;
7. $\mathrm{SL}_s(\mathbb{Z}) = \{M \in \mathrm{M}_s(\mathbb{Z}) : \det M = \pm 1\}$;
8. $\mathrm{TM}_s(\mathbb{Z}) = \big\{((t_{ij})) \in \mathrm{M}_s(\mathbb{Z}) : t_{ij} = 0 \text{ for } j > i,\ 0 \leqslant t_{ij} < t_{ii} \text{ for } j \leqslant i,\ i = \overline{1, s}\big\}$;
9. $\mathrm{TM}_s(\mathbb{Z}, N) = \mathrm{TM}_s(\mathbb{Z}) \cap \mathrm{M}_s(\mathbb{Z}, N)$;
10. $d(M)$ is the greatest common divisor of the tth-order minors of $M \in \mathrm{M}_{t,s}(\mathbb{Z})$;
11. $D(X)$ is the maximum absolute value of the tth-order minors of $X \in \mathrm{M}_{t,s}(\mathbb{R})$;
12. if $X \in \mathrm{M}_{t,s}(\mathbb{R})$, then

$$\mathcal{N}_i(X) = \max_{1 \leqslant j \leqslant s} |x_{ij}|, \quad \mathcal{N}(X) = \prod_{i=1}^{t} \mathcal{N}_i(X), \quad |X|_\infty = \max_{1 \leqslant i \leqslant t} \mathcal{N}_i(X);$$

13. $\mathcal{L}_s(\mathbb{Z})$ is the set of complete integer s-dimensional lattices;
14. $\mathcal{L}_s(\mathbb{Z}, D)$ is the set of lattices $\Gamma \in \mathcal{L}_s(\mathbb{Z})$ with $\det \Gamma = D$.

We write

$$f(x) \ll g(x) \quad (\text{or} \quad f(x) = O(g(x))) \quad \text{for} \quad x \in X,$$

if there exists an absolute constant $C > 0$ such that $|f(x)| \leqslant C \cdot g(x)$ for all $x \in X$. If C depends on a parameter θ, we write $f(x) \underset{\theta}{\ll} g(x)$ (or $f(x) = O_\theta(g(x))$). We write $f \asymp g$ if $f \ll g \ll f$.

We say that a hypersurface $S \subset \mathbb{R}^s$ is piecewise differentiable if S consists of a finite number of differentiable hypersurfaces.

A set $K \subset \mathbb{R}^s$ is called a cone with the vertex at the point $x = 0$ if $\lambda x \in K$ for any $\lambda \in \mathbb{R}_+ = (0, +\infty)$, $x \in K$.

2. Introduction

Suppose that the set $\Omega \subset \mathrm{GL}_s(\mathbb{R})$ satisfies the following conditions:

(A) $\Omega = \big\{((x_{ij})) : (x_{i1}, \ldots, x_{is}) \in V_i,\ i = \overline{1, s}\big\}$, where V_i $(i = \overline{1, s})$ is a connected cone in $\mathbb{R}^s$ with the vertex at the point $x = 0$, the boundary of V_i is piecewise differentiable;

(B) there exists a positive constant C such that

$$\prod_{i=1}^{s} \max_{1\leqslant j\leqslant s} |x_{ij}| \leqslant C\cdot \det X \quad \text{for any} \quad X=((x_{ij}))\in\Omega.$$

The main aim of this work is to obtain an asymptotic formula for the number of integer matrices $M\in\Omega$ with $\det M = N$. This problem arises when studying statistical properties of local minima and Klein polyhedra of integer lattices (see [5–8]).

We note that $\Omega = \mathcal{D}_s(\mathbb{R}_+)\cdot\Omega$, where $\mathcal{D}_s(\mathbb{R}_+)$ is the set of diagonal matrices $((x_{ij}))\in \mathrm{GL}_s(\mathbb{R})$ such that $x_{ii}\in\mathbb{R}_+$, $i=\overline{1,s}$.

Consider the $s(s-1)$-dimensional manifold $\mathrm{PGL}_s(\mathbb{R}) = \mathcal{D}_s(\mathbb{R}_+)\backslash \mathrm{GL}_s(\mathbb{R})$ (the set of right cosets). Let us define a measure on $\mathrm{PGL}_s(\mathbb{R})$. By $\mathcal{P}(\Omega)$ denote the image of $\Omega\subset \mathrm{GL}_s(\mathbb{R})$ under the projection $\mathrm{GL}_s(\mathbb{R})\to\mathrm{PGL}_s(\mathbb{R})$. Suppose that $k=(k_1,\ldots,k_s)$ is a permutation of $\{1,\ldots,s\}$, and $\theta=(\theta_1,\ldots,\theta_s)$, $\theta_i=\pm1$, then we define

$$\mathrm{GL}_s(\mathbb{R},k,\theta)=\left\{((x_{ij}))\in\mathrm{GL}_s(\mathbb{R}) : x_{ik_i}=\theta_i, \quad i=\overline{1,s}\right\}.$$

Let $\mathrm{PGL}_s(\mathbb{R},k,\theta)=\mathcal{P}(\mathrm{GL}_s(\mathbb{R},k,\theta))$ be a map on $\mathrm{PGL}_s(\mathbb{R})$. The set of all $\mathrm{PGL}_s(\mathbb{R},k,\theta)$ is an atlas on $\mathrm{PGL}_s(\mathbb{R})$. Define a measure μ on the map $\mathrm{PGL}_s(\mathbb{R},k,\theta)$ as

$$\mu(w)=\int\limits_W \frac{dW(X)}{|\det X|^s} \quad \text{for} \quad w\subset \mathrm{PGL}_s(\mathbb{R},k,\theta),$$

where W is the prototype of w under the projection $\mathrm{GL}_s(\mathbb{R},k,\theta)\to\mathrm{PGL}_s(\mathbb{R},k,\theta)$; here $dW(X)$ is the differential of an $s(s-1)$-dimensional Lebesgue measure on $\mathrm{GL}_s(\mathbb{R},k,\theta)\subset\mathbb{R}^{s(s-1)}$ taken at some point X.

It can be proved that the measure μ does not depend on the choice of the map. Therefore, the measure μ is well-defined on $\mathrm{PGL}_s(\mathbb{R})$.

For example, if

$$w=\mathcal{P}(W), \quad W=\left\{\begin{pmatrix}1 & x_1 & x_2\\ x_3 & 1 & x_4\\ x_5 & x_6 & 1\end{pmatrix} : x=(x_1,\ldots,x_6)\in W'\right\},$$

where W' is a Lebesgue measurable set in $\mathbb{R}^6$, then

$$\mu(w) = \int\limits_{W'} \left| \det \begin{pmatrix} 1 & x_1 & x_2 \\ x_3 & 1 & x_4 \\ x_5 & x_6 & 1 \end{pmatrix} \right|^{-3} dx.$$

It is easy to prove that $\mu(\mathcal{P}(\Omega)) < \infty$ if Ω satisfies the condition (B).

Remark. It can be proved that the measure μ is invariant under a right action of $\mathrm{GL}_s(\mathbb{R})$. However, this fact will not be used, and its explanation is omitted.

Define the function $\chi : \mathbb{N} \to \mathbb{R}_+$ as

$$\chi(N) = 1 + \sum_{p|N} \frac{\ln p}{p} \quad (p \text{ are prime divisors of N}).$$

It follows in the standard way that

$$\chi(N) \ll 1 + \ln \omega(N) \ll \ln \ln N \quad \text{for} \quad N > 2,$$

where $\omega(N)$ is the number of prime divisors of N.

The main result of the present work is the following.

Theorem 1. *Suppose that the set $\Omega \subset \mathrm{GL}_s(\mathbb{R})$ satisfies the conditions (A) and (B). For any integer $N \geqslant 2$ the number of integer matrices M such that $M \in \Omega$ and* $\det M = N$ *is equal to*

$$\frac{\mathcal{R}_s(N)}{\zeta(2) \cdot \ldots \cdot \zeta(s)} \cdot \left(\frac{\mu(\mathcal{P}(\Omega))}{(s-1)!} \cdot \ln^{s-1} N + O_\Omega(\chi(N) \cdot \ln^{s-2} N) \right),$$

where $\mathcal{R}_s(N) = \#\mathrm{TM}_s(\mathbb{Z}, N)$, and ζ is the Riemann zeta function.

Remark. If $s = 2$, then Theorem 1 can easily be proved using the methods of [3] (two significant terms of the formula can be obtained by using [10]). The case $s = 3$ was considered in [8].

As an application of Theorem 1, we get an asymptotic formula for the average number of local minima of s-dimensional lattices Γ with $\det \Gamma = N$.

We prove Theorem 1 as follows. It is obvious that

$$\#\big(\Omega \cap \mathrm{M}_s(\mathbb{Z}, N)\big) = \sum_{d|N} \sum_{\substack{M \in \mathrm{M}_{s-1,s}(\mathbb{Z}),\\ d(M)=d}} K(M, N),$$

where $d(M)$ is the greatest common divisor of the $(s-1)$th-order minors of a matrix M, and $K(M, N)$ is the number of vectors $(n_1, \ldots, n_s) \in \mathbb{Z}^s$ such that

$$\begin{pmatrix} m_{11} & \ldots & m_{1s} \\ \vdots & & \vdots \\ m_{(s-1)1} & \ldots & m_{(s-1)s} \\ n_1 & \ldots & n_s \end{pmatrix} \in \Omega \cap \mathrm{M}_s(\mathbb{Z}, N).$$

To calculate $K(M, N)$, we must compute $\#(\Gamma \cap U)$, where Γ is a lattice and U is a domain in $\mathbb{R}^{s-1}$. This problem is quite simple. The main difficulty is to calculate the sum

$$\sum_{\substack{M \in \mathrm{M}_{s-1,s}(\mathbb{Z}),\\ d(M)=d}} K(M, N).$$

The rest of the paper is organized as follows.

Section 3 contains some auxiliary results.

In section 4 we obtain estimates for the number of integer matrices lying in a certain domain, as well as bounds for a sum of the form

$$\sum_{\substack{M \in \Theta,\\ d(M)=d}} f(M), \tag{1}$$

where

$$\Theta \subset \mathrm{M}_{t,s}(\mathbb{Z}), \quad f : \mathrm{M}_{t,s}(\mathbb{Z}) \to \mathbb{R}_+.$$

In section 5 we get an asymptotic formula for a sum of the form (1).

In section 6 we prove Theorem 1.

In the final section we obtain an asymptotic formula for the average number of local minima of s-dimensional lattices Γ with $\det \Gamma = N$.

3. Auxiliary results

We define the functions $\tau_k : \mathbb{N} \to \mathbb{R}_+$, $k = 0, 1, 2, \ldots$, as

$$\tau_0(N) = 1, \quad \tau_k(N) = \sum_{d|N} \frac{\tau_{k-1}(d)}{d} \quad \text{for} \quad k \geqslant 1. \tag{2}$$

Using induction, it is easy to prove that

$$\tau_k(N) = \sum_{\substack{m_1,\ldots,m_k \in \mathbb{N}, \\ m_1 \cdot \ldots \cdot m_k | N}} \frac{1}{m_1^k} \cdot \frac{1}{m_2^{k-1}} \cdot \ldots \cdot \frac{1}{m_k} \quad \text{for} \quad k, N \in \mathbb{N}. \tag{3}$$

Take $N \in \mathbb{N}$. Using (3), we get

$$\mathcal{R}_s(N) = \#\, \mathrm{TM}_s(\mathbb{Z}, N) = \sum_{\substack{m_1,\ldots,m_s \in \mathbb{N}, \\ m_1 \cdot \ldots \cdot m_s = N}} m_2 \cdot m_3^2 \cdot \ldots \cdot m_s^{s-1} =$$

$$= \sum_{\substack{m_1,\ldots,m_{s-1} \in \mathbb{N}, \\ m_1 \cdot \ldots \cdot m_{s-1} | N}} \frac{N^{s-1}}{m_1^{s-1} \cdot m_2^{s-2} \cdot \ldots \cdot m_{s-1}} = N^{s-1} \cdot \tau_{s-1}(N). \tag{4}$$

By (3) and (4), it follows that

$$\tau_{s-1}(N) \asymp \sigma_{-1}(N), \quad \mathcal{R}_s(N) \asymp N^{s-1} \cdot \sigma_{-1}(N). \tag{5}$$

The following result is well known (see [2, ch. 1]).

LEMMA 1. *Any nonsingular matrix $M \in \mathrm{M}_s(\mathbb{Z})$ can be uniquely represented in the form $M = T \cdot Q$, where $T \in \mathrm{TM}_s(\mathbb{Z})$, $Q \in \mathrm{SL}_s(\mathbb{Z})$.*

From Lemma 1 we have

$$\mathcal{R}_s(N) = \#\big(\mathrm{M}_s(\mathbb{Z}, N) / \mathrm{SL}_s(\mathbb{Z})\big),$$

($\mathrm{M}_s(\mathbb{Z}, N) / \mathrm{SL}_s(\mathbb{Z})$ is the set of left cosets).

LEMMA 2. *For any natural N, the following estimate holds*:

$$\sum_{d|N} \frac{\sigma_{-1}(d)}{d} \cdot \ln d \leqslant 4 \cdot \zeta(2) \cdot \sigma_{-1}(N) \cdot \chi(N). \tag{6}$$

PROOF. We have

$$\ln d = \sum_{r|d} \Lambda(r), \quad \Lambda(r) = \begin{cases} \ln p, & \text{if} \quad r = p^\alpha, \ p \text{ is a prime}, \\ 0, & \text{else}, \end{cases}$$

and therefore

$$\sum_{d|N} \frac{\sigma_{-1}(d)}{d} \cdot \ln d = \sum_{r|N} \Lambda(r) \sum_{\substack{d|N, \\ d \equiv 0 \pmod r}} \frac{\sigma_{-1}(d)}{d} \leqslant \sum_{r|N} \frac{\Lambda(r) \cdot \sigma_{-1}(r)}{r} \cdot \sum_{d|N} \frac{\sigma_{-1}(d)}{d}.$$

To conclude the proof, it remains to note that

$$\sum_{r|N} \frac{\Lambda(r) \cdot \sigma_{-1}(r)}{r} \leqslant \sum_{r|N} \frac{\Lambda(r) \cdot 2}{r} \leqslant 4 \cdot \chi(N),$$

$$\sum_{d|N} \frac{\sigma_{-1}(d)}{d} = \sum_{r \cdot r'|N} \frac{1}{r^2 r'} \leqslant \zeta(2) \cdot \sigma_{-1}(N). \qquad \square$$

LEMMA 3. *If* $M \in \mathrm{M}_{t,s}(\mathbb{Z})$, $t \leqslant s-1$, *then* $d(M) = d(M \cdot Q)$ *for any* $Q \in \mathrm{SL}_s(\mathbb{Z})$.

PROOF. Suppose that $Q = ((q_{ij})) \in \mathrm{SL}_s(\mathbb{Z})$, and there exist a permutation $k = (k_1, \ldots, k_s)$ of $\{1, \ldots, s\}$, integers $n, m \in [1, s]$, and an integer r such that

$$q_{ij} = \begin{cases} 1, & \text{if} \quad j = k_i, \\ r, & \text{if} \quad (i,j) = (n,m), \\ 0, & \text{else}. \end{cases} \tag{7}$$

Then the proof of the formula $d(M) = d(M \cdot Q)$ is trivial. It remains to note that any matrix $Q \in \mathrm{SL}_s(\mathbb{Z})$ can be represented as a product of matrices of the form (7). $\square$

4. Bounds for cardinalities of sets of matrices

LEMMA 4. *Suppose that* $X, Q \in \mathrm{GL}_s(\mathbb{R})$, *and*

$$\det Q = \pm 1, \quad |X|_\infty \leqslant 1, \quad |X \cdot Q|_\infty \leqslant 1.$$

Then $|Q|_\infty \leqslant s! \cdot |\det X|^{-1}$.

PROOF.

$$|Q|_\infty = |X^{-1} \cdot X \cdot Q|_\infty \leqslant s \cdot |X^{-1}|_\infty \cdot |X \cdot Q|_\infty \leqslant s \cdot |X^{-1}|_\infty \leqslant$$
$$\leqslant s \cdot \frac{(s-1)! \cdot |X|_\infty}{|\det X|} \leqslant \frac{s!}{|\det X|}. \qquad \square$$

By Lemma 4, we have

$$\#\{Q \in \mathrm{SL}_s(\mathbb{Z}) : |X \cdot Q|_\infty \leqslant 1\} \leqslant \left(1 + 2 \cdot \frac{s!}{|\det X|}\right)^{s^2-1} \tag{8}$$

for any $X \in \mathrm{GL}_s(\mathbb{R})$ such that $|X|_\infty \leqslant 1$.

COROLLARY 1. *Suppose that* $C \in \mathbb{R}_+$, $Y \in \mathrm{GL}_s(\mathbb{R})$, $(P_1, \ldots, P_s) \in \mathbb{R}_+^s$. *Then the number of matrices* $X \in Y \cdot \mathrm{SL}_s(\mathbb{Z})$ *such that*

$$\mathcal{N}_i(X) \leqslant P_i, \quad i = \overline{1, s}, \quad P_1 \cdot \ldots \cdot P_s \leqslant C \cdot |\det X|, \tag{9}$$

is at most $C_1 = (1 + 2 \cdot s! \cdot C)^{s^2-1}$.

PROOF. Let $F(Y, P)$ be the set of matrices $X \in Y \cdot \mathrm{SL}_s(\mathbb{Z})$ satisfying (9). In the case $F(Y, P) = \varnothing$ there is nothing to prove. Suppose that there is a matrix $X_0 \in F(Y, P)$. Then

$$P_1 \cdot \ldots \cdot P_s \leqslant C \cdot |\det X_0|, \quad Y \cdot \mathrm{SL}_s(\mathbb{Z}) = X_0 \cdot \mathrm{SL}_s(\mathbb{Z}).$$

Divide every ith row of X_0 by P_i, and denote the resulting matrix as X_0'. Then

$$\#F(Y,P) = \#\{X \in X_0 \cdot \mathrm{SL}_s(\mathbb{Z}) : \mathcal{N}_i(X) \leqslant P_i, \ \ i = \overline{1,s}\} =$$
$$= \#\{X' \in X_0' \cdot \mathrm{SL}_s(\mathbb{Z}) : |X'|_\infty \leqslant 1\} = \#\{Q \in \mathrm{SL}_s(\mathbb{Z}) : |X_0' \cdot Q)|_\infty \leqslant 1\}.$$

Using the estimate $|X_0'|_\infty \leqslant 1$ and (8), we have

$$\#F(Y, P) \leqslant \left(1 + 2 \cdot \frac{s!}{|\det X_0'|}\right)^{s^2-1}.$$

It remains to note that $|\det X_0'| = |\det X_0| \cdot (P_1 \cdot \ldots \cdot P_s)^{-1} \geqslant C^{-1}$. $\square$

Corollary 2. *Suppose that* $N\in\mathbb{N}$, $C\in\mathbb{R}_+$, $(P_1,\dots,P_s)\in\mathbb{R}_+^s$, *and* $P_1\cdot\ldots\cdot P_s\leqslant$ $\leqslant C\cdot N$. *Then the number of matrices* $M\in \mathrm{M}_s(\mathbb{Z},N)$ *satisfying the inequalities* $\mathcal{N}_i(M)\leqslant P_i$, $i=\overline{1,s}$, *is at most* $C_1\cdot\mathcal{R}_s(N)$, *where* $C_1=(1+2\cdot s!\cdot C)^{s^2-1}$.

Proof. By Lemma 1, we have $\mathrm{M}_s(\mathbb{Z},N)=\mathrm{TM}_s(\mathbb{Z},N)\cdot\mathrm{SL}_s(\mathbb{Z})$. This equality, together with Cor 1, yields

$$\#\{M\in\mathrm{M}_s(\mathbb{Z},N):\mathcal{N}_i(M)\leqslant P_i,\ i=\overline{1,s}\}\leqslant$$
$$\leqslant\sum_{T\in\mathrm{TM}_s(\mathbb{Z},N)}\#\{M\in T\cdot\mathrm{SL}_s(\mathbb{Z}):\mathcal{N}_i(M)\leqslant P_i,\ i=\overline{1,s}\}\leqslant$$
$$\leqslant\sum_{T\in\mathrm{TM}_s(\mathbb{Z},N)}C_1=C_1\cdot\mathcal{R}_s(N).\qquad\square$$

Corollary 3. *Suppose that* $N\in\mathbb{N}\cap[2,+\infty)$, $C\in[1,+\infty)$. *Then*

a) *the number of matrices* $M\in\mathrm{M}_s(\mathbb{Z},N)$ *such that* $\mathcal{N}(M)\leqslant C\cdot N$ *is at most* $O_{s,C}(\mathcal{R}_s(N)\cdot\ln^{s-1}N)$;

b) *the number of matrices* $M\in\mathrm{M}_s(\mathbb{Z},N)$ *such that* $\mathcal{N}(M)\leqslant C\cdot N$ *and*

$$\exists\, n,l\in\{1,\dots,s\}:\frac{1}{C}\cdot\mathcal{N}_l(M)\leqslant\mathcal{N}_n(M)\leqslant C\cdot\mathcal{N}_l(M),\quad n\neq l,\tag{10}$$

is at most $O_{s,C}\big(\mathcal{R}_s(N)\cdot\ln^{s-2}N\big)$.

Proof. If $M\in\mathrm{M}_s(\mathbb{Z},N)$, and $\mathcal{N}(M)\leqslant C\cdot N$, then there exists a collection $k=(k_1,\dots,k_s)\in\mathbb{N}^s$ satisfying the inequalities

$$2^{k_i-1}\leqslant\mathcal{N}_i(M)\leqslant 2^{k_i},\quad i=\overline{1,s},\tag{11}$$

$$\log_2 N-C'\leqslant\sum_{i=1}^{s}k_i\leqslant\log_2 N+C',\tag{12}$$

where $C'=\max\{s!,\log_2 C+s\}$. If, in addition, M satisfies (10), then

$$k_l-C''\leqslant k_n\leqslant k_l+C'',\quad C''=\log_2 C+1.\tag{13}$$

We have $2^{k_1}\cdot\ldots\cdot 2^{k_s}\leqslant 2^{C'}\cdot N$. Combining this with Cor 2, we can see that the number of $M\in\mathrm{M}_s(\mathbb{Z},N)$ satisfying (11) is at most $O_{s,C}(\mathcal{R}_s(N))$. It remains to note that

$$\#\{k \in \mathbb{N}^s : k \text{ satisfies } (12)\} \underset{s,C'}{\ll} \ln^{s-1} N,$$

$$\#\{k \in \mathbb{N}^s : k \text{ satisfies } (12), (13)\} \underset{s,C'}{\ll} \ln^{s-2} N. \qquad \square$$

By $d_t(M)$ denote the greatest common divisor of the tth-order minors composed from the first t rows of a matrix $M \in \mathrm{M}_s(\mathbb{Z})$.

LEMMA 5. *Suppose that* $t \in \{1, \dots, s-1\}$, $C \in \mathbb{R}_+$, $d \in \mathbb{N}$, $(P_1, \dots, P_s) \in \mathbb{R}_+^s$. *Then the number of matrices* $M \in \mathrm{M}_s(\mathbb{Z})$ *such that*

$$d_t(M) \equiv 0 \pmod d, \tag{14}$$

$$\mathcal{N}_i(M) \leqslant P_i,\ i = \overline{1, s}, \quad \prod_{i=1}^{s} P_i \leqslant C \cdot |\det M|, \tag{15}$$

is at most

$$O_{s,t,C}\left(\frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \Big(P_1 \cdot \ldots \cdot P_s\Big)^s\right).$$

PROOF. If the matrix $M \in \mathrm{M}_s(C)$ satisfies (15), then

$$|\det M| \leqslant R = s! \cdot \prod_{i=1}^{s} P_i.$$

Using this equality, Lemmas 1 and 3, as well as Cor 1, we can see that the number of matrices $M \in \mathrm{M}_s(\mathbb{Z})$ satisfying (14) and (15) is at most

$$\sum_{\substack{T \in \mathrm{TM}_s(\mathbb{Z}), \det T \leqslant R,\\ d_t(T) \equiv 0 \pmod d}} \#\{M \in T \cdot \mathrm{SL}_s(\mathbb{Z}) : M \text{ satisfies } (15)\} \underset{s,C}{\ll} f_t(R, d),$$

where $f_t(R, d)$ is the number of matrices $T \in \mathrm{TM}_s(\mathbb{Z})$ such that $\det T \leqslant R$ and $d_t(T) \equiv 0 \pmod d$. By $\mathbb{N}^s(R)$ denote the set of vectors $m \in \mathbb{N}^s$ such that $m_1 \cdot \ldots \cdot m_s \leqslant R$. Then

$$f_t(R, d) = \sum_{\substack{m \in \mathbb{N}^s(R),\\ m_1 \cdot \ldots \cdot m_t \equiv 0 \pmod d}} m_2 \cdot m_3^2 \cdot \ldots \cdot m_s^{s-1} = \sum_{n=1}^{\infty} \sum_{\substack{m \in \mathbb{N}^s(R),\\ m_1 \cdot \ldots \cdot m_t = nd}} m_2 \cdot m_3^2 \cdot \ldots \cdot m_s^{s-1} \underset{s,C}{\ll}$$

$$\ll_{s,C} \sum_{n=1}^{\infty} \sum_{\substack{m\in\mathbb{N}^t,\\ m_1\cdot\ldots\cdot m_t=nd}} m_2\cdot m_3^2\cdot\ldots\cdot m_t^{t-1}\cdot\left(\frac{R}{m_1\cdot\ldots\cdot m_t}\right)^s =$$

$$= R^s\cdot\sum_{n=1}^{\infty} \sum_{\substack{m\in\mathbb{N}^t,\\ m_1\cdot\ldots\cdot m_t=nd}} \frac{1}{m_1^s}\cdot\frac{1}{m_2^{s-1}}\cdot\ldots\cdot\frac{1}{m_t^{s-t+1}} =$$

$$= R^s\cdot\sum_{n=1}^{\infty}\frac{1}{(n\cdot d)^{s-t+1}} \sum_{\substack{m\in\mathbb{N}^t,\\ m_1\cdot\ldots\cdot m_t=nd}} \frac{1}{m_1^{t-1}}\cdot\frac{1}{m_2^{t-2}}\cdot\ldots\cdot\frac{1}{m_{t-1}} \ll_t$$

$$\ll_t \frac{R^s}{d^{s-t+1}}\sum_{n=1}^{\infty}\frac{1}{n^{s-t+1}}\cdot\sigma_{-1}(nd) \ll \frac{R^s}{d^{s-t+1}}\cdot\sigma_{-1}(d) \leqslant$$

$$\leqslant (s!)^s\cdot\frac{\sigma_{-1}(d)}{d^{s-t+1}}\cdot(P_1\cdot\ldots\cdot P_s)^s. \qquad \square$$

Corollary 4. *Suppose that* $t\in\{1,\ldots,s-1\}$, $C\in\mathbb{R}_+$, $d\in\mathbb{N}$, $(P_1,\ldots,P_t)\in\mathbb{R}_+^t$. *Then the number of matrices* $M\in \mathrm{M}_{t,s}(\mathbb{Z})$ *such that*

$$d(M)\equiv 0 \pmod d,\quad \mathcal{N}_i(M)\leqslant P_i,\quad i=\overline{1,t},\quad \prod_{i=1}^{t} P_i\leqslant C\cdot D(M),$$

is at most

$$O_{s,t,c}\left(\frac{\sigma_{-1}(d)}{d^{s-t+1}}\cdot(P_1\cdot\ldots\cdot P_t)^s\right).$$

Proof. Without loss of generality we can assume that

$$D(M)=\left|\det\begin{pmatrix} m_{11} & \ldots & m_{1t}\\ \vdots & & \vdots\\ m_{t1} & \ldots & m_{tt}\end{pmatrix}\right|.$$

Let $\widetilde{M}$ be the matrix obtained from M by adding the following rows:

$$(\underbrace{0,\ldots,0}_{t},1,0\ldots,0),\quad (\underbrace{0,\ldots,0}_{t+1},1,0\ldots,0),\quad \ldots,\quad (0,\ldots,0,1).$$

Then $|\det \widetilde{M}| = D(M)$, and $\widetilde{M}$ satisfies the conditions (14) and (15) with $P_i = 1$ for $i \geqslant t+1$. It remains to apply Lemma 5. □

Corollary 5. *Let $t \in \{1, \dots, s-1\}$, $C \in [1; +\infty)$, $d \in \mathbb{N}$, $N \in \mathbb{N} \cap [2; \infty)$, $\alpha \in \mathbb{R}_+$. Suppose that sets $\omega_t = \omega_t(N, d, C)$, $\omega_t' = \omega_t(N, d, C, \alpha)$, $\omega_t'' = \omega_t(N, d, C)$ are defined by the formulas*

$$\omega_t = \big\{M \in \mathrm{M}_{t,s}(\mathbb{Z}) : d(M) \equiv 0 \pmod d,\ 1 \leqslant \mathcal{N}(M) \leqslant C \cdot D(M) \leqslant C^2 \cdot N\big\},$$
$$\omega_t' = \big\{M \in \omega_t : \mathcal{N}_t(M) \leqslant C \cdot d^{\alpha}\big\},$$
$$\omega_t'' = \big\{M \in \omega_t : |M|_\infty \cdot \mathcal{N}(M) \leqslant C \cdot N\big\}.$$

Then

$$\sum_{M \in \omega_t} \big(\mathcal{N}(M)\big)^{-s} \ll_{s,C} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \ln^t N, \tag{16}$$

$$\sum_{M \in \omega_t'} \big(\mathcal{N}(M)\big)^{-s} \ll_{s,C,\alpha} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \ln(d+1) \cdot \ln^{t-1} N, \tag{17}$$

$$\sum_{M \in \omega_t''} |M|_\infty \cdot \big(\mathcal{N}(M)\big)^{-s+1} \ll_{s,C} N \cdot \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \ln^{t-1} N. \tag{18}$$

Proof. Suppose that $k = (k_1, \dots, k_t) \in \mathbb{N}^t$; then by $\omega_{t,k}$ we denote the set of matrices $M \in M_{t,s}(\mathbb{Z})$ satisfying the conditions

$$d(M) \equiv 0 \pmod d, \quad \mathcal{N}(M) \leqslant C \cdot D(M), \quad 2^{k_i - 1} \leqslant \mathcal{N}_i(M) \leqslant 2^{k_i}, \quad i = \overline{1, t}.$$

By Cor 4, we have

$$\#\omega_{t,k} \ll_{s,t,C} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \left(\prod_{i=1}^{t} 2^{k_i}\right)^s. \tag{19}$$

It is obvious that

$$\omega_t \subset \bigcup_{k \in K} \omega_{t,k}, \quad K = \left\{k \in \mathbb{N}^t : \sum_{i=1}^{t} k_i \leqslant \log_2 N + C'\right\},$$
$$\omega_t' \subset \bigcup_{k \in K'} \omega_{t,k}, \quad K' = \left\{k \in \mathbb{N}^t : \sum_{i=1}^{t} k_i \leqslant \log_2 N + C', \ \ k_t \leqslant \log_2 d + C_\alpha\right\},$$

$$\omega_t'' \subset \bigcup_{k\in K''} \omega_{t,k}, \quad K'' = \left\{k \in \mathbb{N}^t : \max_{1\leqslant j\leqslant t} k_j + \sum_{i=1}^{t} k_i \leqslant \log_2 N + C'\right\},$$

where the constant C' depends only on s, t, and C, and the constant C_α depends only on C and α. Using the above representation together with (19), we can write

$$\sum_{M\in\omega_t} \big(\mathcal{N}(M)\big)^{-s} \ll_{s,C} \sum_{k\in K} \frac{\#\omega_{t,k}}{(2^{k_1}\cdot\ldots\cdot 2^{k_t})^s} \ll_{s,C,t} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \sum_{k\in K} 1 \ll_{C',t} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \ln^t N,$$

$$\sum_{M\in\omega_t'} \big(\mathcal{N}(M)\big)^{-s} \ll_{s,C} \sum_{k\in K'} \frac{\#\omega_{t,k}}{(2^{k_1}\cdot\ldots\cdot 2^{k_t})^s} \ll_{C',t,C_\alpha} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \sum_{k\in K'} 1 \ll_{s,C,t}$$

$$\ll_{s,C,t} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \ln(d+1)\cdot \ln^{t-1} N,$$

$$\sum_{M\in\omega_t''} \frac{|M|_\infty}{\big(\mathcal{N}(M)\big)^{s-1}} \ll_{s,C} \sum_{k\in K''} \max_{1\leqslant i\leqslant t} 2^{k_i}\cdot \left(\prod_{j=1}^{t} 2^{k_j}\right)^{-s+1} \cdot \#\omega_{t,k} \ll_{s,C,t}$$

$$\ll_{s,C,t} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot \sum_{k\in K''} \big(\max_{1\leqslant i\leqslant t} 2^{k_i}\big)\cdot \prod_{j=1}^{t} 2^{k_j} \ll_{C,t}$$

$$\ll_{C,t} \frac{\sigma_{-1}(d)}{d^{s-t+1}} \cdot N \cdot \ln^{t-1} N. \qquad \square$$

From (18) and (4) we obtain the following estimate:

$$N^{s-2}\sum_{d|N} d \sum_{M\in\omega_{s-1}''} \frac{|M|_\infty}{\big(\mathcal{N}(M)\big)^{s-1}} \ll_{s,C} N^{s-1}\cdot \ln^{s-2} N \cdot \sum_{d|N} \frac{\sigma_{-1}(d)}{d} \leqslant$$
$$\leqslant \mathcal{R}_s(N)\cdot \ln^{s-2} N. \tag{20}$$

Suppose that $k, n \in \{1, \ldots, s-1\}$, $k \neq n$. Let $\omega = \omega_{s-1}(N, d, C, k, n)$ be the set of matrices $M \in \omega_{s-1}(N, d, C)$ such that $\mathcal{N}_k(M) \underset{C}{\asymp} \mathcal{N}_n(M)$. Using the same method as in the proof of Cor 5, we obtain

$$\sum_{M\in\omega} \big(\mathcal{N}(M)\big)^{-s} \ll_{s,C} \frac{\sigma_{-1}(d)}{d^2} \cdot \ln^{s-2} N.$$

Hence, we have

$$N^{s-1}\sum_{d|N} d\sum_{M\in\omega}\big(\mathcal{N}(M)\big)^{-s} \underset{s,C}{\ll} \mathcal{R}_s(N)\cdot\ln^{s-2}N. \qquad (21)$$

5. Calculation of certain sums

We are going to use the following notation.

1. ∂U is the boundary of a set $U\subset\mathbb{R}^s$.
2. $\overline{U}$ is the closure of a set $U\subset\mathbb{R}^s$.
3. $C^1(U)$ is the set of all continuously differentiable functions $f: U\to\mathbb{R}$.
4. mes is the Lebesgue measure.
5. If $x\in\mathbb{R}^s$, $V\subset\mathbb{R}^s$, $\epsilon>0$, then

$$|x| = (x_1^2+\ldots+x_s^2)^{1/2}, \quad |x|_\infty = \max_{1\leqslant i\leqslant s}|x_i|,$$

$$\operatorname{dist}_\infty(x,\partial V) = \inf_{y\in\partial V}|x-y|_\infty,$$

$$U(V,\epsilon) = \big\{x\in\mathbb{R}^s : \operatorname{dist}_\infty(x,V)\leqslant\epsilon\big\},$$

$$B(V,\epsilon) = \big\{x\in V : \operatorname{dist}_\infty(x,\partial V)\leqslant s\cdot\epsilon\big\}.$$

6. $|\nabla f|$ is the absolute value of the gradient of a scalar function f.

LEMMA 6. *Suppose that the following conditions hold*:

a) V *is a bounded and connected Lebesgue measurable set in* $\mathbb{R}^s$;

b) f *is a nonnegative function from* $C^1(\overline{V})$, *and there exists a function* $g:\overline{V}\to\mathbb{R}_+$ *such that*

$$|\nabla f(x)|\leqslant g(x) \text{ for } x\in V, \quad g(x)\leqslant g(y) \text{ for } x,y\in V, \quad (x-y)\in[0,+\infty)^s;$$

c) $D\in\mathbb{N}$, $\Gamma\in\mathcal{L}_s(\mathbb{Z},D)$.

Then

$$\sum_{\gamma\in V\cap\Gamma} f(\gamma) = \frac{1}{D}\int_V f(x)\,dx + O_s(\xi_1+\xi_2+\xi_3),$$

$$\xi_1 = \frac{1}{D}\int_{B(V,D)} f(x)\,dx, \quad \xi_2 = \sum_{\gamma\in\Gamma\cap B(V,D)} f(\gamma), \quad \xi_3 = D\cdot\sum_{\gamma\in V\cap\Gamma} g(\gamma).$$

PROOF. There exists a basis $e^{(1)}, \dots, e^{(s)}$ of the lattice Γ such that (see [2, ch. 1] or Lemma 1)

$$e_i^{(j)} = 0 \quad \text{for} \quad j > i; \quad 0 \leqslant e_i^{(j)} < e_i^{(i)} \quad \text{for} \quad j < i; \quad i = \overline{1, s}. \tag{22}$$

We define the set

$$\Pi(\gamma) = \left\{ \gamma + \sum_{i=1}^{s} t_i e^{(i)} : t_i \in [0, 1), \quad i = \overline{1, s} \right\}.$$

Then we have

$$|x - y|_\infty \leqslant s \cdot D \quad \text{for} \quad x, y \in \Pi(\gamma). \tag{23}$$

Let

$$\Gamma' = \left\{ \gamma \in \Gamma \cap V : \Pi(\Gamma) \cap \partial V = \varnothing \right\}, \quad V' = \bigcup_{\gamma \in \Gamma'} \Pi(\gamma).$$

Using (23), we have $(V \setminus V') \subset B(\partial V, D)$. Therefore,

$$\sum_{\gamma \in V \cap \Gamma} f(\gamma) = \sum_{\gamma \in \Gamma'} f(\gamma) + O(\xi_2), \tag{24}$$

$$\frac{1}{D} \int\limits_{V'} f(x)\, dx = \frac{1}{D} \int\limits_{V} f(x)\, dx + O(\xi_1). \tag{25}$$

Take a point $\gamma \in \Gamma'$. From mean value theorems it follows that there exist points $x_\gamma, y_\gamma \in \Pi(\gamma)$ such that

$$\left| \frac{1}{D} \int\limits_{\Pi(\gamma)} f(x)\, dx - f(\gamma) \right| \leqslant \left| \nabla f(y_\gamma) \right| \cdot |x_\gamma - \gamma|.$$

From b) and (23) we can obtain the following formula:

$$\left| \frac{1}{D} \int\limits_{\Pi(\gamma)} f(x)\, dx - f(\gamma) \right| \ll_s g(y_\gamma) \cdot D \leqslant D \cdot g(\gamma).$$

Hence,

$$f(\gamma) = \frac{1}{D} \int\limits_{\Pi(\gamma)} f(x)\, dx + O_s(D \cdot g(\gamma)).$$

Using the above together with (25), we obtain

$$\sum_{\gamma\in\Gamma'} f(\gamma) = \frac{1}{D}\sum_{\gamma\in\Gamma'}\int_{\Pi(\gamma)} f(x)\,dx + O_s\Big(D\cdot\sum_{\gamma\in\Gamma'} g(\gamma)\Big) =$$
$$= \frac{1}{D}\int_{V'} f(x)\,dx + O_s(\xi_3) = \frac{1}{D}\int_{V} f(x)\,dx + O_s(\xi_1+\xi_3).$$

It remains to substitute the last relation into (24). □

LEMMA 7. *Let S be a closed piecewise differentiable hypersurface in $\mathbb{R}^s$. Suppose that $r=(r_1,\dots,r_s)\in\mathbb{R}^s_+$, $\epsilon\in\mathbb{R}_+$,*

$$U(S,\epsilon,r) = \big\{x\in U(S,\epsilon) : |x_i| \leqslant r_i,\ i=\overline{1,s}\big\}.$$

Then

$$\operatorname{mes} U(S,\epsilon,r) \ll_S \epsilon\cdot\sum_{j=1}^{s}\prod_{i\neq j} r_i.$$

The proof is trivial.

COROLLARY 6. *Suppose that the following conditions hold*:

a) *V is a connected set in $\mathbb{R}^s$, its boundary ∂V is piecewise differentiable, and there exists a positive constant C such that*

$$|x_i| \leqslant C\cdot x_s, \quad 1\leqslant x_s \quad \text{for} \quad x=(x_1,\dots,x_s)\in V, \quad i=\overline{1,s-1};$$

b) *f is a nonnegative function from the class $C^1\big(\overline{V(l,L)}\big)$, and there exists a positive constant R such that*

$$|f(x)| \leqslant \frac{R}{x_s^s}, \quad |\nabla f(x)| \leqslant \frac{R}{x_s^{s+1}} \quad \text{for} \quad x\in V(l,L), \tag{26}$$

where $V(l,L)$ is defined as the set of points $x\in V$ such that $l\leqslant x_s\leqslant L$ for given positive real constants l and L, $l\leqslant L$;

c) $D\in\mathbb{N}$, $\Gamma\in\mathcal{L}_s(\mathbb{Z},D)$.

Then

$$\sum_{\gamma\in\Gamma\cap V(l,L)} f(\gamma)=\frac{1}{D}\int\limits_{V(l,L)} f(x)\,dx+O_V\left(R\cdot\left(\xi+\frac{1+\ln D}{D}\right)\right),\quad \xi=\sum_{\substack{\gamma\in\Gamma\cap V(l,L),\\ \gamma_s\leqslant D^2}}\frac{1}{\gamma_s^s}.$$

PROOF. We can assume that $R=1$. If $D^2\leqslant l$ or $D^2\geqslant L$, then the proof is trivial. Let

$$l\leqslant D^2\leqslant L.$$

Let us define the following sets

$$V(l,D^2)=\{x\in V: l\leqslant x_s\leqslant D^2\},\quad V(D^2,\ L)=\{x\in V: D^2\leqslant x_s\leqslant L\}.$$

Then

$$\sum_{\gamma\in\Gamma\cap V(l,L)} f(\gamma)=\sum_{\gamma\in\Gamma\cap V(D^2,\,L)} f(\gamma)+O(\xi). \tag{27}$$

From Lemma 6 and the conditions (26) we can obtain the following formula:

$$\begin{gathered}\sum_{\gamma\in\Gamma\cap V(D^2,\,L)} f(\gamma)=\frac{1}{D}\int\limits_{V(D^2,\,L)} f(x)\,dx+O_{C,s}(\xi_1+\xi_2+\xi_3),\\ \xi_1=\frac{1}{D}\int\limits_{B(V(D^2,\,L),D)}\frac{1}{x_s^s}\,dx,\quad \xi_2=\sum_{\gamma\in\Gamma\cap B(V(D^2,\,L),D)}\frac{1}{\gamma_s^s},\quad \xi_3=\sum_{\gamma\in\Gamma\cap V(D^2,\,L)}\frac{D}{\gamma_s^{s+1}}.\end{gathered} \tag{28}$$

By (26), we have

$$\left|\int\limits_{V(D^2,\,L)} f(x)\,dx-\int\limits_{V(l,L)} f(x)\,dx\right|\leqslant\int\limits_{V(l,D^2)}\frac{1}{x_s^s}\,dx\leqslant C^{s-1}\int\limits_1^{D^2}\frac{dx_s}{x_s}=C^{s-1}\cdot\ln D^2,$$

$$\xi_3\leqslant\sum_{\substack{n\in\mathbb{Z}^s,\\ |n_i|\leqslant C\cdot n_s,D^2\leqslant n_s}}\frac{D}{n_s^{s+1}}\ll_{s,C}\sum_{D^2\leqslant n_s}\frac{D}{n_s^2}\ll\frac{1}{D}.$$

Combining this with (27) and (28), we obtain

$$\sum_{\gamma\in\Gamma\cap V(l,L)} f(\gamma)=\frac{1}{D}\int\limits_{V(l,L)} f(x)\,dx+O_{s,C}\left(\xi+\xi_1+\xi_2+\frac{1+\ln D}{D}\right). \tag{29}$$

It remains to estimate ξ_1, ξ_2. For any $k = (k_1, \dots, k_s) \in \mathbb{N}^s$ we define

$$\Pi_k = \left\{x \in \mathbb{R}^s : 2^{k_i - 1} \leqslant |x_i| \leqslant 2^{k_i}, \quad i = \overline{1, s}\right\}.$$

Then

$$V(D^2,\ L) \subset \bigcup_{k \in K(D^2,\ L)} \Pi_k,$$

$$K(D^2,\ L) = \left\{k \in \mathbb{N}^s : k_i \leqslant k_s + 1 + \log_2 C, \quad \log_2 D^2 \leqslant k_s \leqslant \log_2 L + 1\right\},$$

Put $\Pi_k(V, D^2,\ L) = \Pi_k \cap B(V(D^2,\ L), D)$. By Lemma 7, we have

$$\operatorname{mes} \Pi_k(V, D^2,\ L) \underset{V}{\ll} D \cdot \sum_{j=1}^{s} \prod_{i \neq j} 2^{k_i}.$$

It is easy to prove that

$$\#\left(\mathbb{Z}^s \cap \Pi_k(V, D^2,\ L)\right) \underset{V}{\ll} D \cdot \sum_{j=1}^{s} \prod_{i \neq j} 2^{k_i}.$$

Hence, we have

$$\xi_1 \leqslant \frac{1}{D} \sum_{k \in K(D^2,\ L)} \int\limits_{\Pi_k(V, D^2,\ L)} \frac{1}{x_s^s}\, dx \leqslant \frac{1}{D} \cdot \sum_{k \in K(D^2,\ L)} \frac{1}{(2^{k_s - 1})^s} \cdot \operatorname{mes} \Pi_k(V, D^2,\ L) \underset{V}{\ll}$$

$$\underset{V}{\ll} \sum_{k \in K(D^2,\ L)} \left(\frac{1}{(2^{k_s})^s} \sum_{j=1}^{s} \prod_{i \neq j} 2^{k_i}\right) \underset{s,C}{\ll} \sum_{\log_2 D^2 \leqslant k_s} \frac{1}{2^{k_s}} \ll \frac{1}{D^2},$$

$$\xi_2 \leqslant \sum_{k \in K(D^2,\ L)} \left(\sum_{n \in \mathbb{Z}^s \cap \Pi_k(V, D^2,\ L)} \frac{1}{n_s^s}\right) \underset{s}{\ll} \sum_{k \in K(D^2,\ L)} \frac{1}{(2^{k_s})^s} \sum_{n \in \mathbb{Z}^s \cap \Pi_k(V, D^2,\ L)} 1 \underset{V}{\ll}$$

$$\underset{V}{\ll} \sum_{k \in K(D^2,\ L)} \left(\frac{D}{(2^{k_s})^s} \sum_{j=1}^{s} \prod_{i \neq j} 2^{k_i}\right) \underset{s,C}{\ll} \sum_{\log_2 D^2 \leqslant k_s} \frac{D}{2^{k_s}} \ll \frac{1}{D}.$$

It remains to substitute these estimates into (29). □

For $Y = ((y_{ij})) \in \mathrm{M}_{t-1,s}(\mathbb{R})$, $x = (x_1, \dots, x_s) \in \mathbb{R}^s$, we define

$$\begin{pmatrix} Y \\ x \end{pmatrix} = \begin{pmatrix} y_{11} & \cdots & y_{1s} \\ \vdots & & \vdots \\ y_{(t-1)1} & \cdots & y_{(t-1)s} \\ x_1 & \cdots & x_s \end{pmatrix} \in \mathrm{M}_{t,s}(\mathbb{R}).$$

LEMMA 8. *Let* $M \in \mathrm{M}_{t-1,s}(\mathbb{Z})$, $2 \leqslant t \leqslant s$, $r = d(M)$, $d \in \mathbb{N}$, *and* $r \mid d$. *Suppose that the lattice* $\Gamma = \Gamma(M, d)$ *consists of solutions* $n \in \mathbb{Z}^s$ *of the following congruence*

$$d\left(\begin{pmatrix} M \\ n \end{pmatrix}\right) \equiv 0 \pmod{d}.$$

Then $\det \Gamma = (d/r)^{s-t+1}$.

PROOF. If

$$m_{ij} = 0 \quad \text{for} \quad j > i, \tag{30}$$

then the proof is trivial. Lemmas 1 and 3 imply that M can always be reduced to the form (30). □

LEMMA 9. *Suppose that a set* Θ_t *and a function* f *satisfy the following conditions*:

a) Θ_t *is a subset of* $\mathrm{M}_{t,s}(\mathbb{R})$, *and* Θ_t *can be represented as*

$$\Theta_t = \left\{((x_{ij})) : (x_{i1}, \dots, x_{is}) \in V_i,\ i = \overline{1,t}\right\},$$

where V_i, $i = 1, \dots, t$, *are connected sets in* $\mathbb{R}^s$, *and the boundaries* ∂V_i *are piecewise differentiable*;

b) *there exists a constant* $C \in [1; +\infty)$ *such that*

$$\mathcal{N}_i(X) \leqslant C \cdot x_{ii}, \quad i = \overline{1,t}, \tag{31}$$

$$\mathcal{N}(X) \leqslant C \cdot D(X) \quad \textit{for} \quad X \in \Theta_t; \tag{32}$$

c) f *is a nonnegative function from* $C^1(\overline{\Theta_t(N)})$, *where for* $N \in (1; +\infty)$ *and the sets* $\Theta_t(N)$ *defined as*

$$\Theta_t(N) = \left\{((x_{ij})) \in \Theta_t : 1 \leqslant x_{11} \leqslant x_{22} \leqslant \dots \leqslant x_{tt}, \quad x_{tt}^{s-t+1} \prod_{i=1}^{t-1} x_{ii} \leqslant N\right\}$$

there exists a positive constant R *such that*

$$f(X) \leqslant \frac{R}{(\mathcal{N}(X))^s}, \tag{33}$$

$$\left|\frac{\partial f(X)}{\partial x_{kl}}\right| \leqslant \frac{R}{\mathcal{N}_k(X)\cdot(\mathcal{N}(X))^s} \tag{34}$$

for $k=\overline{1,t},\ l=\overline{1,s},\ X=((x_{ij}))\in\Theta_t(N)$.

Then

$$S_t(N,d,f)=\frac{\tau_{t-1}(d)}{d^{s-t+1}}\cdot\left(\prod_{k=s-t+1}^{s}\zeta(k)\right)^{-1}\times$$
$$\times\int\limits_{\Theta_t(N)} f(X)\,dX+O_{\Theta_t}\left(R\cdot\ln^{t-1}N\cdot\frac{1+\ln d}{d^{s-t+1}}\cdot\sigma_{-1}(d)\right)$$

for $d\in\mathbb{N}$, *where*

$$S_t(N,d,f)=\sum_{\substack{M\in\mathrm{M}_{t,s}(\mathbb{Z})\cap\Theta_t(N),\\ d(M)=d}} f(M).$$

Proof. The proof is by induction on t.

Base of induction. Let $t=1$. Then

$$\Theta_1\subset\mathbb{R}^s,\quad \Theta_1(N)=\left\{x=(x_1,\dots,x_s)\in\Theta_1:|x|_\infty\leqslant C\cdot x_1,\quad 1\leqslant x_1\leqslant N^{1/s}\right\},$$
$$|f(x)|\leqslant\frac{R}{x_1^s},\quad |\nabla f(x)|\leqslant\frac{s\cdot R}{x_1^{s+1}}\quad\text{for}\quad x=(x_1,\dots,x_s)\in\Theta_1(N), \tag{35}$$
$$d(m)=\gcd(m_1,\dots,m_s)\quad\text{for}\quad m=(m_1,\dots,m_s)\in\mathbb{Z}^s,$$

where $\gcd(m_1,\dots,m_s)$ is the greatest common divisor of $m_1,\dots,m_s$. Using the Möbius function μ, we obtain

$$S_1(N,d,f)=\sum_{n\in\mathbb{N}}\mu(n)\cdot s(n),\quad s(n)=\sum_{\substack{m\in\Theta_1(N),\\ m\equiv 0 \pmod{nd}}} f(m).$$

The relations (35) imply that we can use Cor 6 to calculate the sum $s(n)$ by swapping x_1 and x_s. We have

$$s(n) = \frac{1}{(nd)^s} \int\limits_{\Theta_1(N)} f(x)\,dx + O\left(R \cdot \left(\xi + \frac{1+\ln(nd)^s}{(nd)^s}\right)\right),$$

$$\xi = \sum_{\substack{m\in\mathbb{N}\times\mathbb{Z}^{s-1},\ |m|_\infty \leqslant C\cdot m_1 \leqslant C\cdot(nd)^{2s},\\ m\equiv 0 \pmod{nd}}} \frac{1}{m_1^s}.$$

In this proof, the constants in asymptotic estimates $O(\ldots)$ and $\ll$ depend only on Θ_t. It is easy to prove that

$$\xi \ll \frac{1}{(nd)^s} \sum_{1\leqslant m_1 \leqslant (nd)^{2s-1}} \frac{1}{m_1} \ll \frac{1+\ln(nd)}{(nd)^s}.$$

Therefore,

$$S_1(N,d,f) = \frac{1}{d^s}\left(\sum_{n\in\mathbb{N}} \frac{\mu(n)}{n^s} \int\limits_{\Theta_1(N)} f(x)\,dx\right) + O\left(R\cdot \sum_{n\in\mathbb{N}} \left(\frac{1+\ln(nd)^s}{(nd)^s}\right)\right) =$$

$$= \frac{1}{d^s\cdot\zeta(s)} \int\limits_{\Theta_1(N)} f(x)\,dx + O\left(R\cdot \frac{1+\ln d}{d^s}\right).$$

This completes the proof in the case $t = 1$.

Induction step from $(t-1)$ to t. We define the sets

$$\Theta_{t-1} = \left\{((x'_{ij})) : (x'_{i1},\ldots,x'_{is}) \in V_i, \quad i = \overline{1,t-1}\right\},$$

$$\Theta_{t-1}(N) = \Big\{((x'_{ij})) \in \Theta_{t-1} : 1 \leqslant x'_{11} \leqslant x'_{22} \leqslant \ldots \leqslant x'_{(t-1)(t-1)},$$

$$(x'_{(t-1)(t-1)})^{s-t+2} \prod_{i=1}^{t-2} x'_{ii} \leqslant N\Big\},$$

$$V_t(X',N) = \left\{x \in V_t :\ x'_{(t-1)(t-1)} \leqslant x_t \leqslant N^{1/(s-t+1)} \cdot \left(\prod_{i=1}^{t-1} x'_{ii}\right)^{-1/(s-t+1)}\right\}$$

for $X' = ((x'_{ij})) \in \Theta_{t-1}(N)$.

Then $\Theta_t(N)$ can be represented in the following form:

$$\Theta_t(N) = \left\{ \begin{pmatrix} X' \\ x \end{pmatrix} : X' \in \Theta_{t-1}(N), \quad x \in V_t(X', N) \right\}.$$

It is obvious that

$$S_t(N, d, f) = \sum_{r|d} \sum_{\substack{M' \in \Theta_{t-1}(N), \\ d(M')=r}} \Phi(M'), \tag{36}$$

where

$$\begin{gathered} \Phi(M') = \sum_{\substack{m \in V_t(M',N), \\ d(M)=d}} f\left(\begin{pmatrix} M' \\ m \end{pmatrix} \right) = \sum_{n \in \mathbb{N}} \mu(n) \cdot F(M', n), \quad M = \begin{pmatrix} M' \\ m \end{pmatrix}, \\ F(M', n) = \sum_{m \in V_t(M',N) \cap \Gamma(M',nd)} f\left(\begin{pmatrix} M' \\ m \end{pmatrix} \right), \end{gathered} \tag{37}$$

and the lattice $\Gamma(M', nd)$ is defined as in Lemma 8.

Applying Lemma 8, we obtain

$$\det \Gamma(M', nd) = \left(\frac{nd}{r} \right)^{s-t+1}.$$

By the condition c), we have

$$f\left(\begin{pmatrix} M' \\ x \end{pmatrix} \right) \leqslant \frac{R}{x_t^s \cdot (\mathcal{N}(M'))^s}, \qquad \left| \frac{\partial}{\partial x_k} f\left(\begin{pmatrix} M' \\ x \end{pmatrix} \right) \right| \leqslant \frac{R}{x_t^{s+1} \cdot (\mathcal{N}(M'))^s}.$$

Therefore, we can apply Cor 6 to compute the sum $F(M', n)$, obtaining

$$\begin{aligned} F(M', n) &= \left(\frac{r}{nd} \right)^{s-t+1} F_0(M') + O\left(\frac{R}{(\mathcal{N}(M'))^s} \cdot \left(\xi(M', n) + \frac{1 + \ln(nd)}{\left(\frac{nd}{r}\right)^{s-t+1}} \right) \right), \\ F_0(M') &= \int\limits_{V_t(M',N)} f\left(\begin{pmatrix} M' \\ x \end{pmatrix} \right) dx, \\ \xi(M', n) &= \sum_{\substack{m \in V_t(M',N) \cap \Gamma(M',nd), \\ |m|_\infty \leqslant (nd)^{(s-t+1)2}}} \frac{1}{|m|_\infty^s}. \end{aligned}$$

Substituting this equality in (37) yields

$$\Phi(M') = \left(\frac{r}{d}\right)^{s-t+1} \frac{1}{\zeta(s-t+1)} \cdot F_0(M') + \\ + O\left(\frac{R}{(\mathcal{N}(M'))^s} \cdot \left(\sum_{n\in\mathbb{N}} \xi(M',n) + \frac{1+\ln d}{\left(\frac{d}{r}\right)^{s-t+1}}\right)\right). \tag{38}$$

Taking into account (36), we have

$$S_t(N,d,f) = \frac{1}{d^{s-t+1}\cdot\zeta(s-t+1)} \sum_{r|d} r^{s-t+1} \cdot S_{t-1}(N,r,F_0) + O(R\cdot(\eta_1+\eta_2)), \tag{39}$$

where

$$S_{t-1}(N,r,F_0) = \sum_{\substack{M'\in\Theta_{t-1}(N),\\ d(M')=r}} F_0(M'), \quad \eta_1 = \sum_{n\in\mathbb{N}} \sum_{\substack{M\in\Theta_t(N),\\ \mathcal{N}_t(M)\leqslant(nd)^{2(s-t+1)},\ d(M)\equiv 0 \pmod{nd}}} (\mathcal{N}(M))^{-s},$$

$$\eta_2 = \frac{1+\ln d}{d^{s-t+1}} \sum_{r|d} r^{s-t+1} \sum_{\substack{M'\in\Theta_{t-1}(N),\\ d(M')=r}} (\mathcal{N}(M'))^{-s}.$$

From (16), (17), and (31), (32) we obtain the estimates

$$\eta_1 \ll \sum_{n\in\mathbb{N}} \frac{1+\ln(nd)}{(nd)^{s-t+1}} \cdot \sigma_{-1}(nd)\cdot \ln^{t-1} N \ll \frac{1+\ln d}{d^{s-t+1}} \cdot \ln^{t-1} N \cdot \sigma_{-1}(d), \tag{40}$$

$$\eta_2 \ll \frac{1+\ln d}{d^{s-t+1}} \sum_{r|d} r^{s-t+1} \cdot \frac{\sigma_{-1}(r)}{r^{s-t+2}} \ln^{t-1} N \ll \frac{1+\ln d}{d^{s-t+1}} \cdot \ln^{t-1} N \cdot \sigma_{-1}(d). \tag{41}$$

Using the conditions of the Lemma, it can be proved that the function

$$F_0(X') = \int\limits_{V_t(X',N)} f\left(\begin{pmatrix} X' \\ x \end{pmatrix}\right)\, dx$$

is nonnegative, continuously differentiable on $\overline{\Theta_{t-1}(N)}$, and

$$F_0(X') \ll \frac{R\cdot\ln N}{(\mathcal{N}(X'))^s}, \qquad \left|\frac{\partial F_0(X')}{\partial x'_{kl}}\right| \ll \frac{R\cdot \ln N}{\mathcal{N}_k(X')\cdot(\mathcal{N}(X'))^s}.$$

If $D(X') < \mathcal{N}(X')/(C \cdot t)$ for any $X' \in \Theta_{t-1}$, then

$$D\left(\begin{pmatrix} X' \\ x \end{pmatrix}\right) \leqslant t \cdot D(X') \cdot \|x\|_\infty < \frac{1}{C} \cdot \mathcal{N}(X') \cdot \|x\|_\infty = \\ = \frac{1}{C} \cdot \mathcal{N}\left(\begin{pmatrix} X' \\ x \end{pmatrix}\right) \quad \text{for} \quad \begin{pmatrix} X' \\ x \end{pmatrix} \in \Theta_t.$$

This contradicts (32), and therefore we must have

$$\mathcal{N}(X') \leqslant C \cdot t \cdot D(X') \quad \text{for} \quad X' \in \Theta_{t-1}.$$

Hence, we can apply the induction assumption to calculate $S_{t-1}(N, r, F_0)$. We have

$$S_{t-1}(N, r, F_0) = \frac{\tau_{t-2}(r)}{r^{s-t+2}} \cdot \left(\prod_{k=s-t+2}^{s} \zeta(k)\right)^{-1} \cdot \int\limits_{\Theta_{t-1}(N)} F_0(X')\, dX' + \\ + O\left(R \cdot \ln^{t-1} N \cdot \frac{1 + \ln r}{r^{s-t+2}} \cdot \sigma_{-1}(r)\right).$$

Therefore, we can write

$$\sum_{r|d} r^{s-t+1} \cdot S_{t-1}(N, r, F_0) = \\ = \left(\sum_{r|d} \frac{\tau_{t-2}(r)}{r}\right) \cdot \left(\prod_{k=s-t+2}^{s} \zeta(k)\right)^{-1} \cdot \int\limits_{\Theta_{t-1}(N)} F_0(X')\, dX' + \\ + O\left(R \cdot \ln^{t-1} N \cdot \sum_{r|d} \frac{1 + \ln r}{r} \cdot \sigma_{-1}(r)\right) = \\ = \tau_{t-1}(d) \cdot \left(\prod_{k=s-t+2}^{s} \zeta(k)\right)^{-1} \cdot \int\limits_{\Theta_t(N)} f(X)\, dX + O\big(R \cdot \ln^{t-1} N \cdot (1 + \ln d) \cdot \sigma_{-1}(d)\big). \tag{42}$$

It remains to substitute (40), (41), and (42) into (39). □

Corollary 7. *Suppose that the set $\Theta = \Theta_{s-1}$ satisfies the conditions a) and b) of Lemma 9 for $t = s-1$. Let $C_1 \in \mathbb{R}_+$, $N \in \mathbb{N} \cap [2; +\infty)$, and let f be a nonnegative* *function from* $C^1(\overline{\Theta(N)})$, *where*

$$\Theta(N)=\Big\{((x_{ij}))\in\Theta : 1\leqslant x_{ii},\quad i=\overline{1,s-1},\quad \big(\max_{1\leqslant j\leqslant s-1}x_{jj}\big)\cdot\prod_{i=1}^{s-1}x_{ii}\leqslant C\cdot N\Big\},$$

and there exists a constant $R\in\mathbb{R}_+$ *such that the estimates* (33) *and* (34) *hold.*

Then

$$N^{s-1}\sum_{d|N}d\sum_{\substack{M\in\mathrm{M}_{s-1,s}(\mathbb{Z})\cap\Theta(N),\\ d(M)=d}}f(M)=\frac{\mathcal{R}_s(N)}{\zeta(2)\cdot\ldots\cdot\zeta(s)}\times$$
$$\times\bigg(\int\limits_{\Theta(N)}f(X)\,dX+O_{\Theta,C}\Big(\chi(N)\cdot\ln^{s-2}N\Big)\bigg).$$

Proof. Let $P(s-1)$ be the set of permutations of $\{1,\ldots,s-1\}$. For any $k=(k_1,\ldots,k_{s-1})\in P(s-1)$ we put

$$\Theta(N,k)=\{X\in\Theta(N): x_{k_1k_1}\leqslant x_{k_2k_2}\leqslant\ldots\leqslant x_{k_{s-1}k_{s-1}}\}.$$

Using (21) and (33), we obtain the estimate

$$N^{s-1}\sum_{d|N}d\sum_{\substack{M\in\Theta(N,k)\cap\Theta(N,n),\\ d(M)=d}}f(M)\underset{s,C_1}{\ll}\mathcal{R}_s(N)\cdot\ln^{s-2}N$$

for $k,n\in P(s-1)$, $k\neq n$. Hence,

$$N^{s-1}\sum_{d|N}d\sum_{\substack{M\in\mathrm{M}_{s-1,s}(\mathbb{Z})\cap\Theta(N),\\ d(M)=d}}f(M)=\sum_{k\in P(s-1)}N^{s-1}\sum_{d|N}d\sum_{\substack{M\in\mathrm{M}_{s-1,s}(\mathbb{Z})\cap\Theta(N,k),\\ d(M)=d}}f(M)+$$
$$+O_{s,C}\big(\mathcal{R}_s(N)\cdot\ln^{s-2}N\big).\tag{43}$$

It remains to prove that

$$N^{s-1}\sum_{d|N}d\sum_{\substack{M\in\mathrm{M}_{s-1,s}(\mathbb{Z})\cap\Theta(N,k),\\ d(M)=d}}f(M)=\frac{\mathcal{R}_s(N)}{\zeta(2)\cdot\ldots\cdot\zeta(s)}\cdot\int\limits_{\Theta(N,k)}f(X)\,dX+$$
$$+O_{\Theta,C,k}\big(\mathcal{R}_s(N)\cdot\chi(N)\cdot\ln^{s-2}N\big)\quad\text{for any}\quad k\in P(s-1).\tag{44}$$

Without loss of generality, we consider only the case $k=(1,\dots,s)$. Using Lemma 9, we obtain that

$$N^{s-1}\sum_{d|N} d \sum_{\substack{M\in \mathrm{M}_{s-1,s}(\mathbb{Z})\cap\Theta(N,k),\\ d(M)=d}} f(M) =$$

$$= N^{s-1}\cdot\sum_{d|N}\frac{\tau_{s-2}(d)}{d}\cdot\left(\prod_{k=2}^{s}\zeta(k)\right)^{-1}\cdot\int_{\Theta(N,k)} f(X)\,dX +$$

$$+ O_{\Theta_{s-1},C}\left(N^{s-1}\cdot\ln^{s-2}N\cdot\sum_{d|N}\left(\frac{1+\ln d}{d}\cdot\sigma_{-1}(d)\right)\right).$$

Combining this expression with (2), (4), and (6) yields (44).

To conclude the proof, it remains to substitute (44) into (43) and note that

$$\sum_{k\in P(s)}\int_{\Theta(N,k)} f(X)\,dX = \int_{\Theta(N)} f(X)\,dX. \qquad \square$$

6. Proof of Theorem 1

LEMMA 10. *Let G be a connected Lebesgue measurable set in $\mathbb{R}^s$. Then*

$$\#(G\cap\mathbb{Z}^s) = \operatorname{mes} G + O_s\big(\operatorname{mes} U(\partial G, 1)\big).$$

The proof is given in [7].

For any lattice $\Gamma\in\mathcal{L}_s(\mathbb{Z})$ we define

$$\lambda(\Gamma)=\min\Big(\max_{1\leqslant i\leqslant s}|a^{(i)}|_\infty\Big),$$

where the minimum is taken over all systems of vectors $\{a^{(i)}\}_{i=1}^s$ such that $\{a^{(i)}\}_{i=1}^s$ is a basis of Γ. Using well-known results [2, ch. 8], we can write

$$\lambda(\Gamma)\leqslant C_0\cdot\max_{1\leqslant i\leqslant s}|b^{(i)}|_\infty \quad \big(C_0=C_0(s)>0\big) \tag{45}$$

for any linearly independent system $\{b^{(i)}\}_{i=1}^s\subset\Gamma$.

Corollary 8. *Suppose that* $\Gamma \in \mathcal{L}_s(\mathbb{Z}, D)$, $x_0 \in \mathbb{R}^s$, $\lambda = s \cdot \lambda(\Gamma)$. *Let* G *be a connected Lebesgue measurable set in* $\mathbb{R}^s$. *Then*

$$\#\big(G \cap (\Gamma + x_0)\big) = \frac{1}{D}\operatorname{mes} G + O_s\left(\frac{1}{D}\operatorname{mes} U(\partial G, \lambda)\right).$$

Proof. We can assume that $x_0 = 0$.

Let $\{a^{(i)}\}_{i=1}^{s}$ be a basis of Γ such that

$$\max_{1 \leqslant i \leqslant s} |a^{(i)}|_\infty = \lambda(\Gamma).$$

By M denote the matrix with the columns ${a^{(1)}}^T, \dots, {a^{(s)}}^T$. Then

$$\#(\Gamma \cap G) = \#(\mathbb{Z}^s \cap (M^{-1} \cdot G)).$$

Using Lemma 10, we obtain

$$\#(\Gamma \cap G) = \operatorname{mes}(M^{-1} \cdot G) + O_s\big(\operatorname{mes} U(\partial(M^{-1} \cdot G), 1)\big). \tag{46}$$

Clearly, we have

$$\operatorname{mes}(M^{-1} \cdot G) = |\det M^{-1}| \cdot \operatorname{mes} G = \frac{1}{D}\operatorname{mes} G. \tag{47}$$

Since $|M \cdot x|_\infty \leqslant \lambda \cdot |x|_\infty$ for all $x \in \mathbb{R}^s$, it follows that $M \cdot U(\partial(M^{-1} \cdot G), 1) \subset \subset U(\partial G, \lambda)$. Therefore,

$$\operatorname{mes} U\big(\partial(M^{-1} \cdot G), 1\big) \leqslant \frac{1}{D} \cdot \operatorname{mes} U(\partial G, \lambda). \tag{48}$$

It remains to substitute (47) and (48) into (46). □

Lemma 11. *Suppose that* $R \in (1, +\infty)$, *and*

$$H_{s-1}(R) = \left\{x \in [1, +\infty)^{s-1} : |x|_\infty \cdot \prod_{i=1}^{s-1} x_i \leqslant R\right\}.$$

Then we have

$$\int\limits_{H_{s-1}(R)} \frac{dx}{x_1 \cdot \ldots \cdot x_{s-1}} = \frac{1}{s!} \cdot \ln^{s-1} R. \tag{49}$$

Proof. Let

$$H_{t,s}(R) = \left\{ x \in [1; +\infty)^t : x_1 \leqslant x_2 \leqslant \ldots \leqslant x_t,\ x_t^{s-t+1} \cdot \prod_{i=1}^{t-1} x_i \leqslant R \right\},$$

$$I_{t,s}(R) = \int\limits_{H_{t,s}(R)} \frac{dx}{x_1 \cdot \ldots \cdot x_t}.$$

It is clear that

$$\int\limits_{H_{s-1}(R)} \frac{dx}{x_1 \cdot \ldots \cdot x_{s-1}} = (s-1)! \cdot I_{s-1,s}(R). \tag{50}$$

Let us show that

$$I_{t,s}(R) = \frac{(s-t)!}{t! \cdot s!} \cdot \ln^t R. \tag{51}$$

The proof is by induction on t.

In the case $t = 1$ there is nothing to prove.

Induction step from $(t-1)$ to t. Let us perform the following substitution:

$$y_i = x_i, \quad i = \overline{1, t-1}, \quad y_t = x_t^{s-t+1} \cdot \prod_{i=1}^{t-1} x_i.$$

Since we have

$$x_t = \left(\frac{y_t}{y_1 \cdot \ldots \cdot y_{t-1}} \right)^{1/(s-t+1)}, \qquad \frac{dx}{x_1 \cdot \ldots \cdot x_t} = \frac{1}{s-t+1} \cdot \frac{dy}{y_1 \cdot \ldots \cdot y_t},$$

it follows that

$$I_{t,s}(R) = \frac{1}{s-t+1} \int\limits_1^R \frac{I_{t-1,s}(y_t)}{y_t}\, dy_t.$$

Using the induction assumption, we obtain (51). Now (49) follows from (50) and (51). □

For any $X \in M_s(\mathbb{R})$ by $A_j(X)$ we denote the (s, j) cofactor of the matrix X.

Lemma 12. *Suppose that $N \in \mathbb{N}$, $N \geqslant 2$, the set $\Omega \subset \mathrm{GL}_s(\mathbb{R})$ satisfies the conditions (A) and (B) of Theorem 1, and there exists a constant $C_1 \in [1; +\infty)$ such that*

$$\mathcal{N}_i(X) \leqslant C_1 \cdot x_{ii}, \quad i = \overline{1, s}; \quad \prod_{i=1}^{s-1} \mathcal{N}_i(X) \leqslant C_1 \cdot A_s(X) \quad \textit{for any} \quad X \in \Omega. \tag{52}$$

Let

$$\Omega'(\mathbb{Z}, N, C_2) = \left\{ ((m_{ij})) \in \Omega \cap \mathrm{M}_s(\mathbb{Z}, N) : \max_{1 \leqslant i \leqslant s-1} m_{ii} \cdot \prod_{j=1}^{s-1} m_{jj} \leqslant C_2 \cdot N \right\}.$$

Then we have

$$\#\Omega'(\mathbb{Z}, N, C_2) = \frac{\mathcal{R}_s(N)}{\zeta(2) \cdot \ldots \cdot \zeta(s)} \cdot \left(\frac{\mu(\mathcal{P}(\Omega))}{s!} \cdot \ln^{s-1} N + O_{\Omega, C_2}(\chi(N) \cdot \ln^{s-2} N) \right)$$

for any positive C_2 such that

$$C_2 < \frac{1}{C_1^2 \cdot C_0 \cdot s! \cdot (s-1)! \cdot (s-1)}, \tag{53}$$

where C_0 is the constant in the estimate (45).

Proof. Using the conditions of the Lemma, we can see that any set V_i can be represented in the form

$$\begin{aligned} V_i = \big\{ t \cdot (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_s) : t \in \mathbb{R}_+, \\ (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s) \in W_i, \big\}, \end{aligned} \tag{54}$$

where $W_i \subset \mathbb{R}^{s-1}$, and the boundary ∂W_i is piecewise differentiable. Put

$$\Theta = \big\{ ((x_{ij})) : (x_{i1}, \ldots, x_{is}) \in V_i, \quad i = \overline{1, s-1} \big\},$$

$$\Theta(N) = \left\{ X \in \Theta : \max_{1 \leqslant i \leqslant s-1} x_{ii} \cdot \prod_{j=1}^{s-1} x_{jj} \leqslant C_2 \cdot N \right\}.$$

For any integer matrix $M \in \Theta(N)$ we define

$$K(M) = \#\left\{ m \in \mathbb{Z}^s : \begin{pmatrix} M \\ m \end{pmatrix} \in \Omega \cap \mathrm{M}_s(\mathbb{Z}, N) \right\}.$$

Then

$$\#\Omega'(\mathbb{Z}, N, C_2) = \sum_{M \in \Theta(N)} K(M).$$

Let us calculate $K(M)$.

If $d(M)$ does not divide N, then $K(M) = 0$.

Suppose that $M \in \Theta(N)$, and $d = d(M)$ is a divisor of N. Then

$$K(M) = \#\left\{ m \in \mathbb{Z}^s \cap V_s : \sum_{i=1}^{s} m_i A_i = N \right\},$$

where

$$A_i = A_i\left(\begin{pmatrix} M \\ m \end{pmatrix}\right), \quad i = \overline{1, s}.$$

Therefore,

$$K(M) = \#\left\{ m \in \mathbb{Z}^{s-1} \cap V(M, N) : \sum_{i=1}^{s-1} m_i A_i \equiv N \pmod{A_s} \right\},$$

where

$$V(M,N) = \left\{ (x_1, \ldots, x_{s-1}) \in \mathbb{R}^{s-1} :\ (x_1, \ldots, x_{s-1}, x_s) \in V_s, \quad x_s = \frac{N - \sum\limits_{i=1}^{s-1} x_i A_i}{A_s} \right\}.$$

Since $d = \gcd(A_1, \ldots, A_s)$ divides N, it follows that there exists a solution $n \in \mathbb{Z}^{s-1}$ to the congruence

$$\sum_{i=1}^{s-1} n_i A_i \equiv N \pmod{A_s}.$$

Now we have

$$K(M) = \#(V(M, N) \cap (\Gamma + n)),$$

where

$$\Gamma = \left\{ (\gamma_1, \dots, \gamma_{s-1}) \in \mathbb{Z}^{s-1} : \sum_{i=1}^{s-1} \gamma_i A_i \equiv 0 \pmod{A_s} \right\}.$$

It is easy to prove that $\det \Gamma = |A_s|/d$. Using this fact together with Cor 8 yields

$$K(M) = \frac{d}{|A_s|} \cdot \operatorname{mes} V(M,N) + O_s\left(\frac{d}{|A_s|} \cdot \operatorname{mes}\big(U(\partial V(M,N), \lambda)\big)\right). \tag{55}$$

The rows of M are a linearly independent system in Γ. Combining this with (45), we obtain

$$\lambda \leqslant C_0 \cdot |M|_\infty. \tag{56}$$

To calculate the integral

$$\operatorname{mes} V(M,N) = \int\limits_{V(M,N)} dx,$$

we make the substitution

$$x_i' = \frac{x_i}{x_s}, \quad i = \overline{1, s-1}, \quad \text{where} \quad x_s = \frac{1}{A_s}\left(N - \sum_{i=1}^{s-1} x_i A_i \right). \tag{57}$$

This means that

$$x_i = N \cdot \frac{x_i'}{D(M,x')}, \quad \text{where} \quad D(M,x') = \det \begin{pmatrix} m_{11} & \dots & m_{1(s-1)} & m_{1s} \\ \vdots & & \vdots & \vdots \\ m_{(s-1)1} & \dots & m_{(s-1)(s-1)} & m_{(s-1)s} \\ x_1' & \dots & x_{s-1}' & 1 \end{pmatrix}.$$

Using (B), we have

$$D(M,x') \geqslant \frac{1}{C} \cdot \mathcal{N}(M) \cdot \max\{1, x_1', \dots, x_{s-1}'\} \gg \mathcal{N}(M) \quad \text{for} \quad (x_1', \dots, x_{s-1}') \in W_s. \tag{58}$$

In this proof, constants in asymptotic estimates $\ll$ and $O(\dots)$ depend only on Ω and C_2.

We have

$$dx_1 \dots dx_{s-1} = N^{s-1} \cdot |A_s| \cdot \frac{dx_1' \dots dx_{s-1}'}{(D(M',x'))^s}. \tag{59}$$

Since

$$V_s = \left\{ x \in \mathbb{R}^s : \left(\frac{x_1}{x_s}, \dots, \frac{x_{s-1}}{x_s} \right) \in W_s,\ x_s \in \mathbb{R}_+ \right\},$$

we can see that the mapping (57) transforms the set $V(M, N)$ onto W_s. Finally, we obtain

$$\operatorname{mes} V(M, N) = N^{s-1} \cdot |A_s| \cdot f(M), \quad f(M) = \int\limits_{W_s} \frac{dx'_1 \dots dx'_{s-1}}{(D(M, x'))^s}. \tag{60}$$

Let us estimate $\operatorname{mes} U(\partial V(M, N), \lambda)$. Using (52), we have

$$\left| \frac{A_i}{A_s} \right| \leqslant C_1 \cdot (s-1)!. \tag{61}$$

Take any point $y = (y_1, \dots, y_{s-1}) \in U(\partial V(M, N), \lambda)$. Then there exists $x = (x_1, \dots, x_{s-1}) \in \overline{V(M, N)}$ such that

$$|x_i - y_i| \leqslant \lambda \leqslant C_0 \cdot |M|_\infty, \quad i = \overline{1, s-1}. \tag{62}$$

Let

$$x_s = \frac{1}{A_s}\left(N - \sum_{i=1}^{s-1} x_i A_i \right), \quad y_s = \frac{1}{A_s}\left(N - \sum_{i=1}^{s-1} y_i A_i \right).$$

Then

$$\begin{pmatrix} M \\ \widetilde{x} \end{pmatrix} \in \overline{\Omega}, \quad \det \begin{pmatrix} M \\ \widetilde{x} \end{pmatrix} = N,$$

where $\widetilde{x} = (x_1, \dots, x_{s-1}, x_s)$. Hence

$$N \leqslant s! \cdot \mathcal{N}(M) \cdot |\widetilde{x}|_\infty \leqslant s! \cdot \mathcal{N}(M) \cdot C_1 \cdot x_s \ \Rightarrow\ x_s \geqslant \frac{1}{C_1 \cdot s!} \cdot \frac{N}{\mathcal{N}(M)}. \tag{63}$$

Using (61), (62), and the condition $\mathcal{N}(M) \cdot |M|_\infty \leqslant C_2 \cdot N$, we obtain

$$|x_s - y_s| \leqslant \sum_{i=1}^{s-1} \left| \frac{A_i}{A_s} \right| \cdot |x_i - y_i| \leqslant C_1 \cdot (s-1)! \cdot (s-1) \cdot C_0 \cdot |M|_\infty \leqslant$$
$$\leqslant C_1 \cdot (s-1)! \cdot (s-1) \cdot C_0 \cdot C_2 \cdot \frac{N}{\mathcal{N}(M)}.$$

From this estimate and (63) it follows that

$$y_s \geqslant x_s - |x_s - y_s| \geqslant C_3 \cdot \frac{N}{\mathcal{N}(M)}, \tag{64}$$

where

$$C_3 = \frac{1}{C_1 \cdot s!} - C_1 \cdot (s-1)! \cdot (s-1) \cdot C_0 \cdot C_2 > 0 \quad \text{(see (53))}.$$

Suppose that

$$x_i' = \frac{x_i}{x_s}, \quad y_i' = \frac{y_i}{y_s}, \quad i = \overline{1, s-1}.$$

Using (62), (63), and (64), we have

$$|y_i' - x_i'| = \left| \frac{y_i}{y_s} - \frac{x_i}{y_s} + \frac{x_i}{y_s} - \frac{x_i}{x_s} \right| \leqslant \frac{|y_i - x_i|}{y_s} + |x_i| \cdot \frac{|x_s - y_s|}{x_s \cdot y_s} \ll \frac{|M|_\infty \cdot \mathcal{N}(M)}{N}.$$

Hence, the mapping (57) transforms $U(\partial V(M, N), \lambda)$ onto some set $\widetilde{U}$ such that $\widetilde{U} \subset U(\partial W_s, \mu)$, where $\mu = O\big(|M|_\infty \cdot \mathcal{N}(M) \cdot N^{-1}\big)$. From the definition of y_s it follows that $y_s \cdot D(M, y') = N$. Using this equality together with (64), we have

$$D(M, y') \geqslant \frac{N}{y_s} \gg \mathcal{N}(M).$$

Now applying the above to (59), we obtain

$$\begin{aligned}
\operatorname{mes} U_\lambda(\partial V(M, N)) &\ll N^{s-1} \cdot |A_s| \cdot \int\limits_{U(\partial W_s, \mu)} \frac{dy_1' \ldots dy_{s-1}'}{|D(M, y')|^s} \ll \\
&\ll N^{s-1} \cdot \frac{|A_s|}{|\mathcal{N}(M)|^s} \cdot \operatorname{mes} U(\partial W_s, \mu) \ll N^{s-1} \cdot \frac{|A_s|}{|\mathcal{N}(M)|^s} \cdot \mu \ll \\
&\ll N^{s-2} \cdot |A_s| \cdot \frac{|M|_\infty}{(\mathcal{N}(M))^{s-1}}.
\end{aligned} \tag{65}$$

Substituting (60) and (65) into (55) yields that

$$K(M) = N^{s-1} \cdot d \cdot f(M) + O\left(N^{s-2} \cdot d \cdot \frac{|M|_\infty}{(\mathcal{N}(M))^{s-1}} \right).$$

This, in turn, allows us to write

$$\#\Omega'(\mathbb{Z},N,C_2)=S(N)+O(\xi(N)),$$

$$S(N)=N^{s-1}\cdot\sum_{d|N}\sum_{\substack{M\in\Theta(N),\\ d(M)=d}} d\cdot f(M),\quad \xi(N)=N^{s-2}\cdot\sum_{d|N}\sum_{\substack{M\in\Theta(N),\\ d(M)=d}} d\cdot\frac{|M|_\infty}{(\mathcal{N}(M))^{s-1}}. \tag{66}$$

By the condition (B), we have

$$\mathcal{N}(M)\leqslant s\cdot C\cdot D(M)\quad\text{for}\quad M\in\Theta(N)\cap \mathrm{M}_{s-1,s}(\mathbb{Z}).$$

Clearly, we can write

$$D(M)\leqslant (s-1)!\cdot\mathcal{N}(M)\leqslant (s-1)!\cdot N.$$

Therefore, we can apply (20) to estimate $\xi(N)$:

$$\xi(N)\ll \mathcal{R}_s(N)\cdot\ln^{s-2}N. \tag{67}$$

Using (58), we can see that the function

$$f(X)=\int\limits_{W_s}\frac{dx'}{(D(X,x'))^s}$$

satisfies the conditions

$$f(X)\leqslant\frac{C^s}{|\mathcal{N}(X)|^s}\cdot\operatorname{mes}W_s,$$

$$\left|\frac{\partial f(X)}{\partial x_{kl}}\right|\leqslant s\cdot\int\limits_{W_s}\frac{|A_{kl}(X,x')|}{|D(X,x')|^{s+1}}\,dx'\ll\prod_{1\leqslant i\leqslant s-1,\ i\neq k}\mathcal{N}_i(X)\cdot\frac{1}{(\mathcal{N}(X))^{s+1}}=$$

$$=\frac{1}{\mathcal{N}_k(X)\cdot(\mathcal{N}(X))^s},$$

where $A_{kl}(X,x')$ is the (k,l) cofactor of the matrix

$$\begin{pmatrix} x_{11} & \dots & x_{1(s-1)} & x_{1s}\\ \vdots & & \vdots & \vdots\\ x_{(s-1)1} & \dots & x_{(s-1)(s-1)} & x_{(s-1)s}\\ x'_1 & \dots & x'_{s-1} & 1\end{pmatrix}.$$

Therefore the condition c) of Lemma 9 is satisfied ($t = s-1$, the constant R depends only on Ω). Hence, we can apply Cor 7 to calculate the sum $S(N)$, obtaining

$$S(N) = \frac{\mathcal{R}_s(N)}{\zeta(2)\cdot\ldots\cdot\zeta(s)}\Bigg(\int\limits_{\Theta(N)} f(X)\,dX + O(\chi(N)\cdot\ln^{s-2}N)\Bigg). \tag{68}$$

Substituting (67) and (68) into (66), we have

$$\#\Omega'(\mathbb{Z}, N, C_2) = \frac{\mathcal{R}_s(N)}{\zeta(2)\cdot\ldots\cdot\zeta(s)}\Bigg(\int\limits_{\Theta(N)} f(X)\,dX + O(\chi(N)\cdot\ln^{s-2}N)\Bigg). \tag{69}$$

To compute the integral (69), let us make the following substitution:

$$x_{ii} = t_i, \quad x_{ij} = t_i\cdot x'_{ij}, \quad i = \overline{1, s-1}, \quad j = \overline{1, s}, \quad j \neq i. \tag{70}$$

Then we can write

$$dX = (t_1\cdot\ldots\cdot t_{s-1})^{s-1}\,dt_1\ldots dt_{s-1}\prod_{\substack{1\leqslant i\leqslant s-1, 1\leqslant j\leqslant s,\\ i\neq j}} dx'_{ij},$$

$$f(X) = \frac{1}{(t_1\cdot\ldots\cdot t_{s-1})^s}\cdot\int\limits_{W_s}\frac{dx'_{s1}\ldots dx'_{s(s-1)}}{|\det X'|^s},$$

$$f(X)\,dX = \frac{dt_1\ldots dt_{s-1}}{t_1\cdot\ldots\cdot t_{s-1}}\cdot\Bigg(\int\limits_{W_s}\frac{dx'_{s1}\ldots dx'_{s(s-1)}}{|\det X'|^s}\Bigg)\cdot\prod_{\substack{1\leqslant i\leqslant s-1, 1\leqslant j\leqslant s,\\ i\neq j}} dx'_{ij}, \tag{71}$$

where

$$X' = \begin{pmatrix} 1 & x'_{12} & \ldots & x'_{1s}\\ x'_{21} & 1 & \ldots & x'_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ x'_{s1} & x'_{s2} & \ldots & 1\end{pmatrix}.$$

The set $\Theta(N)$ consists of matrices X such that

$$X = \begin{pmatrix} t_1 & t_1\cdot x'_{12} & \ldots & t_1\cdot x'_{1(s-1)} & t_1\cdot x'_{1s}\\ t_2\cdot x'_{21} & t_2 & \ldots & t_2\cdot x'_{2(s-1)} & t_2\cdot x'_{2s}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ t_{s-1}\cdot x'_{(s-1)1} & t_{s-1}\cdot x'_{(s-1)2} & \ldots & t_{s-1} & t_{s-1}\cdot x'_{(s-1)s}\end{pmatrix},$$

where

$$(t_1, \ldots, t_{s-1}) \in H_{s-1}(C_2 \cdot N) = \left\{ t \in [1; +\infty)^{s-1} : |t|_\infty \cdot \prod_{i=1}^{s-1} t_i \leqslant C_2 \cdot N \right\},$$

$$\left(x'_{i1}, \ldots, x'_{i(i-1)}, x'_{i(i+1)}, \ldots, x'_{is}\right) \in W_i, \;\; i = \overline{1, s-1}.$$

From this fact and (71) it follows that

$$\int\limits_{\Theta(N)} f(X)\, dX = \int\limits_{H_{s-1}(C_2 \cdot N)} \frac{dt_1 \ldots dt_{s-1}}{t_1 \cdot \ldots \cdot t_{s-1}} \cdot \int\limits_{W_1 \times \ldots \times W_s} \frac{1}{|\det X'|^s} \prod_{\substack{1 \leqslant i,j \leqslant s,\\ i \neq j}} dx'_{ij}.$$

Using (49) and the definition of the measure μ, we obtain

$$\int\limits_{H_{s-1}(C_2 N)} \frac{dt_1 \ldots dt_{s-1}}{t_1 \cdot \ldots \cdot t_{s-1}} = \frac{\ln^{s-1} N}{s!} + O(\ln^{s-2} N),$$

$$\int\limits_{W_1 \times \ldots \times W_s} \frac{1}{|\det X'|^s} \prod_{\substack{1 \leqslant i,j \leqslant s,\\ i \neq j}} dx'_{ij} = \mu(\mathcal{P}(\Omega)).$$

Thus, we have

$$\int\limits_{\Theta(N)} f(X)\, dX = \frac{\ln^{s-1} N}{s!} \cdot \mu(\mathcal{P}(\Omega)) + O(\ln^{s-2} N).$$

It remains to substitute the last formula into (69). □

Corollary 9. *Under the conditions of Lemma* 12, *we have*

$$\#\Omega^{(s)}(\mathbb{Z}, N) = \frac{\mathcal{R}_s(N)}{\zeta(2) \cdot \ldots \cdot \zeta(s)} \cdot \left(\frac{\mu(\mathcal{P}(\Omega))}{s!} \cdot \ln^{s-1} N + O_\Omega(\chi(N) \cdot \ln^{s-2} N) \right),$$

where

$$\Omega^{(s)}(\mathbb{Z}, N) = \{M \in \Omega \cap \mathrm{M}_s(\mathbb{Z}, N) : \mathcal{N}_s(M) = \max_{1 \leqslant i \leqslant s} \mathcal{N}_i(M)\}.$$

Proof. Take any positive constant C_2 such that C_2 satisfies the condition (53) and $C_2 \leqslant 1/s!$. Then for any $M \in \Omega'(\mathbb{Z}, N, C_2)$ we have

$$\prod_{j=1}^{s-1} \mathcal{N}_j(M) \cdot \max_{1 \leqslant i \leqslant s-1} \mathcal{N}_i(M) \leqslant C_2 \cdot N \leqslant C_2 \cdot s! \cdot \mathcal{N}(M) \leqslant \mathcal{N}(M).$$

Therefore,

$$\max_{1 \leqslant i \leqslant s-1} \mathcal{N}_i(M) \leqslant \mathcal{N}_s(M),$$

and hence

$$\Omega'(\mathbb{Z}, N, C_2) \subset \Omega^{(s)}(\mathbb{Z}, N). \tag{72}$$

Take a matrix $M \in \Omega^{(s)}(\mathbb{Z}, N) \setminus \Omega'(\mathbb{Z}, N, C_2)$. Then

$$\max_{1 \leqslant i \leqslant s-1} \mathcal{N}_i(M) \leqslant \mathcal{N}_s(M), \quad \prod_{j=1}^{s-1} \mathcal{N}_j(M) \cdot \max_{1 \leqslant i \leqslant s-1} \mathcal{N}_i(M) > C_2 \cdot N. \tag{73}$$

Using the condition (B), we have

$$\prod_{j=1}^{s-1} \mathcal{N}_j(M) \cdot \max_{1 \leqslant i \leqslant s-1} \mathcal{N}_i(M) > C_2 \cdot N \geqslant \frac{C_2}{C} \cdot \mathcal{N}(M).$$

Therefore, it follows that

$$\max_{1 \leqslant i \leqslant s-1} \mathcal{N}_i(M) > \frac{C_2}{C} \cdot \mathcal{N}_s(M).$$

From this inequality and (73) it follows that

$$\max_{1 \leqslant i \leqslant s-1} \mathcal{N}_i(M) \underset{C, C_2}{\asymp} \mathcal{N}_s(M).$$

Hence, by Cor 3 b), we obtain that

$$\#\left(\Omega^{(s)}(\mathbb{Z}, N) \setminus \Omega'(\mathbb{Z}, N, C_2)\right) \underset{s, C, C_2}{\ll} \mathcal{R}_s(N) \cdot \ln^{s-2} N.$$

Using this relation together with (72), we have

$$\#\Omega^{(s)}(\mathbb{Z}, N) = \#\Omega'(\mathbb{Z}, N, C_2) + O_{s, C, C_2}(\mathcal{R}_s(N) \cdot \ln^{s-2} N).$$

It remains to apply Lemma 12. □

Lemma 13. *Suppose that* $X \in \mathrm{M}_s(\mathbb{R})$, $C \in [1; +\infty)$, *and* $\mathcal{N}(X) \leqslant C \cdot |\det X|$. *Then there exists a permutation* $(k_1, \dots, k_s)$ *of* $\{1, \dots, s\}$ *such that*

$$\mathcal{N}_i(X) \leqslant C \cdot s! \cdot |x_{ik_i}|, \quad i = \overline{1, s}, \quad \prod_{i=1}^{s-1} \mathcal{N}_i(X) \leqslant C \cdot s \cdot |A_{k_s}|,$$

where $A_{k_s} = A_{k_s}(X)$ *is the* (s, k_s) *cofactor of the matrix* X.

Proof. Assume the converse. This means that any permutation $(k_1, \dots, k_s)$ of $\{1, \dots, s\}$ satisfies the following condition:

- either $|A_{k_s}| < \dfrac{1}{C \cdot s} \displaystyle\prod_{i=1}^{s-1} \mathcal{N}_i(X)$,
- or there exists a number $j \in \{1, \dots, s\}$ such that $|x_{jk_j}| < \dfrac{1}{C \cdot s!} \cdot \mathcal{N}_j(X)$.

  Then we must have

$$|\det X| \leqslant \sum_{l=1}^{s} |A_l| \cdot |x_{sl}| < \frac{1}{C} \prod_{i=1}^{s} \mathcal{N}_i(X),$$

which contradicts the conditions of the lemma. □

Let $\Xi(s)$ be the set of all sets $\Omega \subset \mathrm{GL}_s(\mathbb{R})$ satisfying the conditions (A) and (B).

Corollary 10. *Any set* $\Omega \in \Xi(s)$ *can be represented in the following form*:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_L,$$

where $L \in \mathbb{N}$, $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, *and for any* $l \in \{1, \dots, L\}$ *the following conditions hold*:

a) $\Omega_l \in \Xi(s)$;

b) *there exists a permutation* $(k_1, \dots, k_s)$ *of* $\{1, \dots, s\}$ *such that*

$$\prod_{i=1}^{s-1} \mathcal{N}_i(X) \underset{\Omega}{\ll} |A_{k_s}(X)|, \quad \mathcal{N}_i(X) \underset{\Omega}{\ll} |x_{ik_i}|, \quad i = \overline{1, s} \quad \textit{for} \quad X \in \Omega_l.$$

Proof. We put $h = (2 \cdot C \cdot s \cdot s!)^{-1}$ and divide the set Ω into the following parts:

$$\Omega_n = \left\{ X \in \Omega : n_{ij} h \leqslant \frac{x_{ij}}{\mathcal{N}_i(X)} < (n_{ij} + 1)h, \quad i, j = \overline{1, s} \right\},$$

where $n_{ij} \in \mathbb{Z}$, $|n_{ij}| \leqslant h^{-1}$. It is obvious that $\Omega_n \in \Xi(s)$. It remains to prove that any Ω_n satisfies the condition b). Take a matrix $Y \in \Omega_n$. By Lemma 13, there exists a permutation $(k_1, \dots, k_s)$ of $\{1, \dots, s\}$ such that

$$\mathcal{N}_i(Y) \leqslant C \cdot s! \cdot |y_{ik_i}|, \quad i = \overline{1, s}, \tag{74}$$

$$\prod_{i=1}^{s-1} \mathcal{N}_i(Y) \leqslant C \cdot s \cdot |A_{k_s}(Y)|. \tag{75}$$

From this and the definition of Ω_n, we obtain that $|(n_{ik_i} + 1)h| \geqslant 1/(C \cdot s!)$. Hence, $|n_{ik_i}| \geqslant (2s-1)$, $i = \overline{1, s}$, and

$$|x_{in_i}| \geqslant h \cdot (2s-1) \cdot \mathcal{N}_i(X), \quad i = \overline{1, s} \tag{76}$$

for any matrix $X \in \Omega_n$.

Let us show that

$$\left| \frac{A_{k_s}(X)}{\prod_{i=1}^{s-1} \mathcal{N}_i(X)} - \frac{A_{k_s}(Y)}{\prod_{i=1}^{s-1} \mathcal{N}_i(Y)} \right| \leqslant (s-1)! \cdot (s-1) \cdot h \quad \text{for} \quad X \in \Omega_n. \tag{77}$$

Without loss of generality, it can be assumed that $k_s = s$. We define matrices

$$A = ((a_{ij})), \quad B = ((b_{ij})), \quad a_{ij} = \frac{x_{ij}}{\mathcal{N}_i(X)}, \quad b_{ij} = \frac{y_{ij}}{\mathcal{N}_i(Y)}, \quad i, j = \overline{1, s-1}.$$

Then we can write

$$|a_{ij}| \leqslant 1, \quad |b_{ij}| \leqslant 1, \quad |a_{ij} - b_{ij}| \leqslant h, \quad i, j = \overline{1, s-1},$$

and hence

$$|\det A - \det B| \leqslant (s-1)! \cdot (s-1) \cdot h.$$

Taking into account the relations

$$\det A = \frac{A_s(X)}{\prod_{i=1}^{s-1} \mathcal{N}_i(X)}, \quad \det B = \frac{A_s(Y)}{\prod_{i=1}^{s-1} \mathcal{N}_i(Y)},$$

we obtain (77).

From (77) and (75) we have

$$\frac{A_{k_s}(X)}{\prod_{i=1}^{s-1}\mathcal{N}_i(X)} \geqslant \left|\frac{A_{k_s}(Y)}{\prod_{i=1}^{s-1}\mathcal{N}_i(Y)}\right| - \left|\frac{A_{k_s}(X)}{\prod_{i=1}^{s-1}\mathcal{N}_i(X)} - \frac{A_{k_s}(Y)}{\prod_{i=1}^{s-1}\mathcal{N}_i(Y)}\right| \geqslant$$
$$\geqslant \frac{1}{C\cdot s} - (s-1)!\cdot(s-1)\cdot h \geqslant \frac{1}{2\cdot C\cdot s}. \qquad (78)$$

By applying (76) and (78), it follows that Ω_n satisfies the condition b). □

Proof of Theorem 1. We divide the set Ω into the following parts:

$$\Omega = \bigcup_{k=1}^{s}\Omega^{(k)}, \quad \Omega^{(k)} = \{X\in\Omega : \mathcal{N}_k(X) = \max_{1\leqslant i\leqslant s}\mathcal{N}_i(X)\}.$$

From Cor 3 b) we obtain

$$\#\big(\Omega^{(k)}\cap\Omega^{(l)}\cap \mathrm{M}_s(\mathbb{Z},N)\big) = O_{C,s}(\mathcal{R}_s(N)\cdot \ln^{s-2}N)$$

for $k\neq l$. Thus, we can write

$$\#\big(\Omega\cap \mathrm{M}_s(\mathbb{Z},N)\big) = \sum_{k=1}^{s}\#\big(\Omega^{(k)}\cap \mathrm{M}_s(\mathbb{Z},N)\big) + O_{C,s}(\mathcal{R}_s(N)\cdot\ln^{s-2}N). \qquad (79)$$

Let us show that

$$\#(\Omega^{(k)}\cap \mathrm{M}_s(\mathbb{Z},N)) =$$
$$= \frac{\mathcal{R}_s(N)}{\zeta(2)\cdot\ldots\cdot\zeta(s)}\cdot\left(\frac{\mu(\mathcal{P}(\Omega))}{s!}\cdot\ln^{s-1}N + O_\Omega(\chi(N)\cdot\ln^{s-2}N)\right) \qquad (80)$$

for any $k\in\{1,\ldots,s\}$. Obviously, it suffices to consider the case $k=s$. By Cor 10, without loss of generality it can be assumed that

$$\mathcal{N}_i(X) \underset{\Omega}{\ll} x_{ii},\ i=\overline{1,s}, \quad \prod_{i=1}^{s-1}\mathcal{N}_i(X) \ll A_s(X) \text{ for } X\in\Omega^{(s)}.$$

Then, using Cor 9, we have (80). To conclude the proof, it remains to substitute (80) into (79). □

7. The average number of local minima of integer lattices

For a rational number $P/Q \in (0,1)$ let $s = s(P/Q)$ be the length of the continued fraction expansion

$$\frac{P}{Q} = [q_1, q_2, \ldots, q_s] = \cfrac{1}{q_1 + \cfrac{1}{q_2 + \ldots + \cfrac{1}{q_s}}}, \quad q_i \in \mathbb{N}.$$

In [3] H. Heilbronn proved the following asymptotic formula:

$$\frac{1}{\phi(Q)} \sum_{\substack{1\leqslant P<Q,\\ \gcd(P,Q)=1}} s\left(\frac{P}{Q}\right) = \frac{2\ln 2}{\zeta(2)} \cdot \ln Q + R(Q), \tag{81}$$

where $\phi(Q) = \#\{n \in N \cap [1;Q] : \gcd(Q,n) = 1\}$ is the Euler function, $R(Q) = O\big(\sigma_{-1}^4(Q)\big)$. In [10] J. Porter established that $R(Q) = C + O_\epsilon(Q^{-5/6+\epsilon})$ for any $\epsilon > 0$, where C is a positive constant.

One of the most interesting generalizations of continued fractions was proposed by G. Voronoi [12] and H. Minkowski [9]. Their approach is based on local minima of lattices. Let us recall some definitions.

By saying that Γ is a complete *lattice* of dimension s we mean that it has the form

$$\Gamma = \left\{k_1 m^{(1)} + \ldots + k_s m^{(s)} : k_i \in \mathbb{Z},\ i = \overline{1,s}\right\},$$

where $m^{(i)}$ are linearly independent vectors from $\mathbb{R}^s$ (a basis of Γ). The value $\det\Gamma = |\det((m_j^{(i)}))|$ is called the determinant of Γ.

A nonzero point $\gamma \in \Gamma$ is called a *local minimum* of $\Gamma \in \mathcal{L}_s(\mathbb{R})$ if there is no other nonzero point $\gamma' \in \Gamma$ such that

$$|\gamma_i'| \leqslant |\gamma_i|, \quad i = \overline{1,s}; \quad |\gamma'| < |\gamma|.$$

Denote as $\mathfrak{M}(\Gamma)$ the set of all local minima of Γ.

The Voronoi—Minkowski construction is motivated by Lagrange's classical theorem on best approximations by continued fractions. For example, if $\alpha \in (0, 1/2)$, then for a lattice Γ_α defined by the basis $(1, \alpha)$, $(0, 1)$ we have

$$\mathfrak{M}(\Gamma_\alpha) = \{\pm(Q_i, \alpha Q_i - P_i) : i = 0, 1, \ldots\}, \tag{82}$$

where $Q_0 = 0$, $P_0 = 1$, and $P_i/Q_i = [q_1, \ldots, q_{i-1}]$ is the ith convergent to α for $i \geqslant 1$.

Let us recall that $\mathcal{L}_s(\mathbb{Z}, N)$ is the set of all s-dimensional integer lattices Γ with $\det \Gamma = N$. Using Lemma 1, we can see that $\#\mathcal{L}_s(\mathbb{Z}, N) = \mathcal{R}_s(N)$. Let

$$E_s(N) = \frac{1}{\mathcal{R}_s(N)} \cdot \sum_{\Gamma \in \mathcal{L}_s(\mathbb{Z};N)} \#\mathfrak{M}(\Gamma)$$

be the average number of local minima of lattices Γ from $\mathcal{L}_s(\mathbb{Z}, N)$.

Using (81), (82), we have (see [7])

$$E_2(N) = \frac{4 \ln 2}{\zeta(2)} \cdot \ln N + O(\chi(N)).$$

It was shown in [6] that

$$\frac{1}{\sum_{N=1}^{D} \mathcal{R}_s(N)} \cdot \sum_{N=1}^{D} \sum_{\Gamma \in \mathcal{L}_s(\mathbb{Z};N)} \#\mathfrak{M}(\Gamma) = \mathcal{C}(s) \cdot \ln^{s-1} D + O_s(\ln^{s-2} D)$$

for any $D > 1$. The proof of this formula is based on the following result.

Theorem 2 [6]. *For any integer $s \geqslant 2$ there exists a set $\Omega_{\mathfrak{M},s} \subset \mathrm{GL}_s(\mathbb{R})$ such that*

a) *the interior of $\Omega_{\mathfrak{M},s}$ is not empty*;

b) $\Omega_{\mathfrak{M},s} = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_L$, *where $L \in \mathbb{N}$, $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$, and the sets Ω_i satisfy the conditions* (*A*) *and* (*B*);

c) *for any natural N, the following formula holds*:

$$\sum_{\Gamma \in \mathcal{L}_s(\mathbb{Z},N)} \#\mathfrak{M}_+(\Gamma) = \#(\Omega_{\mathfrak{M},s} \cap \mathrm{M}_s(\mathbb{Z}, N)) + O_s\big(\#(\partial\Omega_{\mathfrak{M},s} \cap \mathrm{M}_s(\mathbb{Z}, N))\big),$$

where $\mathfrak{M}_+(\Gamma) = \mathfrak{M}(\Gamma) \cap \mathbb{R}^s_+$.

From Theorems 1 and 2, we obtain the following result.

Corollary 11. *For any integers* $s \geqslant 2$, $N \geqslant 2$, *the following asymptotic formula holds*:

$$E_s(N) = \mathcal{C}(s) \cdot \ln^{s-1} N + O_s(\chi(N) \cdot \ln^{s-2} N), \tag{83}$$

where

$$\mathcal{C}(s) = \frac{2^s}{(s-1)!} \cdot \frac{\mu(\mathcal{P}(\Omega_{\mathfrak{M},s}))}{\zeta(2) \cdot \ldots \cdot \zeta(s)}.$$

Proof. The following estimate is well-known (see, e. g., [1, 4]):

$$\#\{\gamma \in \mathfrak{M}(\Gamma) : \gamma_1 \cdot \ldots \cdot \gamma_s = 0\} \ll_s \ln^{s-2} N$$

for

$$\Gamma \in \mathcal{L}_s(\mathbb{Z}, N).$$

Therefore, we have

$$\sum_{\Gamma \in \mathcal{L}_s(\mathbb{Z},N)} \#\mathfrak{M}(\Gamma) = 2^s \cdot \sum_{\Gamma \in \mathcal{L}_s(\mathbb{Z},N)} \#\mathfrak{M}_+(\Gamma) + O_s(\mathcal{R}_s(N) \cdot \ln^{s-2} N).$$

From this equation and Theorem 2 it follows that

$$E_s(N) = \frac{2^s}{\mathcal{R}_s(N)} \cdot \Big(\#\big(\Omega_{\mathfrak{M},s} \cap \mathrm{M}_s(\mathbb{Z}, N)\big) + \\ + O_s\big(\#(\partial\Omega_{\mathfrak{M},s} \cap \mathrm{M}_s(\mathbb{Z}, N))\big)\Big) + O_s(\ln^{s-2} N).$$

We clearly have

$$\mu(\mathcal{P}(\partial\Omega_{\mathfrak{M},s})) = 0, \quad \mu(\mathcal{P}(\Omega_{\mathfrak{M},s})) > 0.$$

It remains to apply Theorem 1. □

Remark. The formula (83) was proved in [8] in the case $s = 3$. The paper [8] also contains the results of an approximate computation of the constant $\mathcal{C}(3)$. If $s \geqslant 4$, it is impossible to calculate the constant $\mathcal{C}(s)$ even approximately. However, the following estimates hold [11]:

$$\frac{2^{-1}}{(s-1)!} \leqslant \mathcal{C}(s) \leqslant \frac{2^s}{(s-1)!}.$$

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