ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory

МФТИ

URSS

Volume 1 • Issue 3 2011

Moscow Journal of Combinatorics and Number Theory. 2011. Vol. 1. Iss. 3. 80 p.

The journal was founded in 2010.

Published by the Moscow Institute of Physics and Technology with the support of Yandex and Microsoft.

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

Website of our journal
http://mjcnt.phystech.edu

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URSS Publishers

56, Nakhimovsky Prospekt,
Moscow,
Russia,
117335

Издательство «УРСС»

Нахимовский пр-т, 56
Москва,
Российская Федерация,
117335

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77-41900.

Формат 70×100/16. Печ. л. 5. Зак. № ПЖ-43.

Отпечатано в ООО «ЛЕНАНД».
117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.

ISSN 2220-5438

11210 ID 158900

9 785453 000272

Moscow Journal
of Combinatorics
and Number Theory
2011, vol. 1, iss. 3,
pp. 3–11, [pp. 211–219]

On hypergraph cliques with chromatic number 3

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Abstract: This paper is devoted to a problem in extremal hypergraph theory, which goes back to P. Erdős and L. Lovász (see [1]). Before giving an exact statement of the problem, we recall some definitions and introduce some notation.

Keywords: hypergraph, clique, chromatic number, covering number

AMS Subject classification: 05C15

Received: 29.07.2011; **revised:** 13.08.2011

Let $H = (V, E)$ be a hypergraph without multiple edges. We call it n-*uniform*, if any of its edges has cardinality n: for every $e \in E$, we have $|e| = n$. By the *chromatic number* of a hypergraph $H = (V, E)$ we mean the minimum number $\chi(H)$ of colors needed to paint all the vertices in V so that any edge $e \in E$ contains at least two vertices of some different colors. Finally, a hypergraph is said to form a *clique*, if its edges are pairwise intersecting.

In 1973 Erdős and Lovász noticed that if an n-uniform hypergraph $H = (V, E)$ forms a clique, then $\chi(H) \in \{2, 3\}$. They also observed that in the case of $\chi(H) = 3$, one certainly has $|E| \leqslant n^n$ (see [1]). Thus, the following definition has been motivated:

$$M(n) = \max\left\{|E| : \exists \text{ an } n - \text{uniform clique } H = (V, E) \text{ with } \chi(H) = 3\right\}.$$

Obviously such definition has no sense in the case of $\chi(H) = 2$.

THEOREM 1 (P. ERDŐS, L. LOVÁSZ, [1]). *The following inequalities hold*:

$$n!\left(\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right) \leqslant M(n) \leqslant n^n.$$

Almost nothing better than this has been done during the last 35 years. In the book [3] the estimate $M(n) \leqslant (1-1/e)n^n$ is mentioned as the one "to appear". However, we have not succeeded in finding the corresponding paper.

At the same time, another quantity $r(n)$ was introduced in [4]:

$$r(n) = \max\left\{|E| : \text{ there is an } n\text{-uniform clique } H=(V,E) \text{ s. t. } \tau(H)=n\right\},$$

where $\tau(H)$ is the *covering number* of H, i. e.,

$$\tau(H) = \min\left\{|f| : f \subset V, \quad \forall\, e \in E \quad f \cap e \neq \varnothing\right\}.$$

Clearly, for any n-uniform clique H, we have $\tau(H) \leqslant n$ (since every edge forms a cover), and if $\chi(H) = 3$, then $\tau(H) = n$. Thus, $M(n) \leqslant r(n)$. Lovász noticed that for $r(n)$ the same estimates as in Theorem 1 apply and conjectured that the lower estimate is best possible. In 1996 P. Frankl, K. Ota, and N. Tokushige (see [2]) disproved this conjecture and showed that $r(n) \geqslant (n/2)^{n-1}$.

We discovered a new upper bound for the initial quantity $M(n)$.

THEOREM 2. *There exists a constant $c > 0$ such that*

$$M(n) \leqslant cn^{n-1/2} \ln n.$$

We shall prove Theorem 2 in the next section. Note that our proof will also work for $r(n)$. So in fact we give the bound $r(n) \leqslant cn^{n-1/2} \ln n$, too.

1. Proof of Theorem 2

We shall proceed by citing or proving successive propositions that will eventually lead us to the proof of the theorem.

PROPOSITION 1 (P. ERDŐS, L. LOVÁSZ, [1]). *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Let k be an arbitrary integer such that $1 \leqslant k \leqslant n$. Take any set $W \subseteq V$ of cardinality k. Let $E(W)$ denote the set of all edges $e \in E$ such that $W \subseteq e$. Then $|E(W)| \leqslant n^{n-k}$.*

Note that, in particular, the degree $\deg v$ of any vertex $v \in V$ does not exceed n^{n-1} (here $k = 1$). This fact entails immediately the estimate $M(n) \leqslant n^n$. Although we intend to prove a much better bound, we shall frequently use Proposition 1 during the proof.

To any n-uniform hypergraph $H = (V, E)$ we assign the set

$$B(H) = \left\{ v \in V : \deg v > \frac{|E|}{n^2} \right\}.$$

PROPOSITION 2. *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Then the following two assertions hold*:

1. $|B(H)| < n^3$;
2. *any edge $e \in E$ intersects the set $B(H)$.*

PROOF. We start by proving the first assertion. Fix an $H = (V, E)$. Let $B = B(H)$. We know that $\sum_{v \in V} \deg v = n|E|$. Furthermore,

$$\sum_{v \in V} \deg v \geqslant \sum_{v \in B} \deg v > \frac{|B||E|}{n^2}.$$

Thus, $\frac{|B||E|}{n^2} < n|E|$, which means that actually $|B| < n^3$.

To prove the second assertion let us fix an arbitrary edge e. Since H is a clique, any $f \in E$ intersects e. Therefore, $\sum_{v \in e} \deg v \geqslant |E|$. By the pigeon-hole principle, there is a vertex $v \in e$ with $\deg v \geqslant |E|/n \geqslant |E|/n^2$. So, $v \in B$, and the proof is complete. □

PROPOSITION 3. *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Let $t \in \{1, \dots, n\}$ and suppose there is an edge $e \in E$ intersecting the set $B = B(H)$ by at most t vertices. Then there is a vertex $v \in V$ with* $\deg v \geqslant \frac{|E|}{t+1}$.

PROOF. Fix a hypergraph $H = (V, E)$ and an $e \in E$ with $|e \cap B| \leqslant t$. Put $f = e \cap B$ and $a = |f| \leqslant t$. We know that for any $v \in f$, one has $\deg v > |E|/n^2$. We also know that for any $v \in (e \setminus f)$, one has $\deg v \leqslant |E|/n^2$. Finally, we know that H is a clique. Consequently,

$$\sum_{v\in f}\deg v = \sum_{v\in e}\deg v - \sum_{v\in(e\setminus f)}\deg v \geqslant |E| - (n-a)\frac{|E|}{n^2}.$$

By the pigeon-hole principle, there is a vertex $v \in f$ with

$$\deg v \geqslant \frac{|E| - (n-a)\frac{|E|}{n^2}}{a}.$$

The right-hand side of the above inequality decreases in $a \leqslant t$, so that anyway

$$\deg v \geqslant \frac{|E| - (n-t)\frac{|E|}{n^2}}{t} = |E|\cdot\frac{n^2-n+t}{n^2t} \geqslant \frac{|E|}{t+1},$$

where the latter inequality is true, since $t \in \{1,\dots,n\}$. Proposition 3 is proved. □

Proposition 4. *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Let $t \in \{1,\dots,n\}$. Then, either $|E| \leqslant tn^{n-1}$, or for any $e \in E$, we have $|e \cap B(H)| \geqslant t$.*

Proof. Fix an $H = (V, E)$ with $B(H) = B$. Assume that $|E| > tn^{n-1}$ and that there exists an $e \in E$ such that $|e\cap B| \leqslant t-1$. By Proposition 3 we can find a vertex v with $\deg v \geqslant |E|/t > n^{n-1}$, which is in conflict with Proposition 1. Thus, our assumption is false, and the proof is complete. □

Proposition 5. *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$ and $|E| > n^{n-1/2}$. Suppose that $n \geqslant 100$. Then there exist edges $e, f \in E$ such that $[\sqrt{n}] \leqslant |e \cap f| \leqslant n - [\sqrt{n}]$.*

Proof. Fix an $H = (V, E)$ with $B(H) = B$. Put $t = [\sqrt{n}]$. Since $tn^{n-1} \leqslant n^{n-1/2} <$ $< |E|$, Proposition 4 tells us that for any $e \in E$, we have $|e \cap B| \geqslant t$.

Consider the family $\mathcal{B}_t = C_B^t$ consisting of all the t-element subsets of the set B. By the first assertion of Proposition 2 we have

$$|\mathcal{B}_t| = C_{|B|}^t \leqslant |B|^t < n^{3t}.$$

We also know that any $e \in E$ must contain a set $T \in \mathcal{B}_t$, since $|e \cap B| \geqslant t$.

At the same time, $|E| > n^{n-1/2} > n^{5t}$, as $n \geqslant 100$. Thus, using the notation adopted in the statement of Proposition 1, we get that there exists a set $T \in \mathcal{B}_t$ such that $|E(T)| > n^{5t}/n^{3t} = n^{2t}$.

Clearly, for any $e, f \in E(T)$, we have $T \subseteq (e \cap f)$, so that $|e \cap f| \geqslant t = \left[\sqrt{n}\right]$. If there exist $e, f \in E(T)$ with $|e \cap f| \leqslant n - \left[\sqrt{n}\right] = n - t$, then the proposition is proved. Otherwise, every two edges from $E(T)$ intersect by at least $n-t+1$ vertices.

Take an arbitrary $A \in E(T)$. Put $s = n-t+1$ and consider the family $\mathcal{A}_s = C_A^s$ consisting of all the s-element subsets of the set A. We know that the following facts hold simultaneously:

a) $|E(T)| > n^{2t}$;

b) any $e \in E(T)$ contains a set $S \in \mathcal{A}_s$ (since $|e \cap A| \geqslant s$);

c) $|\mathcal{A}_s| = C_{|A|}^s = C_n^s = C_n^{t-1} < n^t$.

Therefore, there is a set $S \in \mathcal{A}_s$ such that $|E(S)| > n^t$. Since $|S| = s$, by Proposition 1, we have $|E(S)| \leqslant n^{n-s} = n^{t-1}$, which leads to a contradiction. Proposition 5 is proved. □

Remark 1. Note that the proof of Proposition 5 can be easily extended to support the following assertion: *Let $H = (V, E)$ be an n-uniform clique with $\chi(H) = 3$. Let $F \subseteq E$ satisfy the inequality $|F| > n^{n-1/2}$. Suppose that $n \geqslant 100$. Then there exist edges $f_1, f_2 \in F$ such that $\left[\sqrt{n}\right] \leqslant |f_1 \cap f_2| \leqslant n - \left[\sqrt{n}\right]$.*

Note also that a hypergraph $H' = (V, F)$ does not necessarily have chromatic number 3. It can be bipartite as well.

PROPOSITION 6. *Let $n \in \mathbb{N}$, $t \in \{1, \dots, n\}$,*

$$t' = \min\left\{t, 4\sqrt{n}\ln n\right\}, \quad N(t) = (t+1)\left(n - \frac{\sqrt{n}}{4}\right)^{t'-1} n^{n-t'}.$$

Then $N(t) = O(n^{n-1/2}\ln n)$.

PROOF. First, assume that $t \leqslant 4\sqrt{n}\ln n$. Then

$$N(t) \leqslant (t+1)\cdot n^{t-1}\cdot n^{n-t} = (t+1)n^{n-1} = O(n^{n-1/2}\ln n),$$

and we are done. Now, assume that $t > 4\sqrt{n}\ln n$. In this case,

$$N(t) \leqslant (n+1)\left(n - \frac{\sqrt{n}}{4}\right)^{t'-1} n^{n-t'} = (n+1)\cdot n^{n-1}\left(1 - \frac{1}{4\sqrt{n}}\right)^{4\sqrt{n}\ln n - 1} = O(n^{n-1}),$$

and we are done again. Proposition 6 is proved. □

Remark 2. Note that we may write, say, $N(t) \leqslant 10n^{n-1/2} \ln n$ for $n \geqslant n_0$ and all t.

COMPLETION OF THE PROOF OF THEOREM 2. Fix an n-uniform clique $H = (V, E)$ with $\chi(H) = 3$ and $n \geqslant \max\{n_0, 10\,000\}$. We shall prove that $|E| \leqslant 10n^{n-1/2} \ln n$. This will be enough to complete the proof of Theorem 2.

Let

$$T = \max\left\{t : \ \forall \ e \in E \ \ |e \cap B| \geqslant t\right\}.$$

By the second assertion of Proposition 2, $T \in \{1, \dots, n\}$.

Define T' in the same way as t' was defined by t in Proposition 6. Since $n \geqslant 10\,000$, we have $T' < n$, and thus $T' \in \{1, \dots, n-1\}$. Also, since $n \geqslant n_0$, we have $N(T) \leqslant 10n^{n-1/2} \ln n$ (see Remark 2).

Assume that $|E| > 10n^{n-1/2} \ln n$. So we automatically assume that $|E| > N(T)$. By the definition of T, there exists an edge $e \in E$ intersecting B by at most T vertices. Hence we get by Proposition 3 that there is a vertex $v \in V$ with

$$\deg v \geqslant \frac{|E|}{T+1} > \frac{N(T)}{T+1} = \left(n - \frac{\sqrt{n}}{4}\right)^{T'-1} n^{n-T'}.$$

Put $I = \{v\}$, $i = 1$. Then

$$|E(I)| = \deg v > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-1} n^{n-T'} = \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i} n^{n-T'}. \tag{1}$$

If $T' = 1$, then (1) contradicts Proposition 1. Therefore, we shall assume that $T' \in \{2, \dots, n-1\}$.

Let us consider (1) as the base for an inductive procedure with $\leqslant T'$ steps. So, let us assume that we have already found a set $I \subset V$ with $|I| = i \in \{1, \dots, T'-1\}$ (do not forget that $T' \geqslant 2$) and

$$|E(I)| > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i} n^{n-T'}.$$

We shall prove that either we can take an $a \in (V \setminus I)$ such that

$$|E(I \cup \{a\})| > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-1} n^{n-T'}, \tag{2}$$

or we can take $a, b \in (V \setminus I)$ such that

$$|E(I \cup \{a, b\})| > \left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-2} n^{n-T'}. \tag{3}$$

(Here, for all i, $T' - i - 2 \geqslant -1$ and $i + 2 \leqslant T' + 1 \leqslant n$, so that the choice of the parameters is correct.)

In order to prove (2) or (3), let us set

$$E_I = \{e \in E : e \cap I \neq \varnothing\}.$$

By Proposition 1, $|E_I| \leqslant in^{n-1}$. Hence $|E_I| < T'n^{n-1} \leqslant 4n^{n-1/2} \ln n$. Putting

$$E^I = \{e \in E : e \cap I = \varnothing\}$$

we immediately get the estimate

$$|E^I| = |E| - |E_I| > 10n^{n-1/2} \ln n - 4n^{n-1/2} \ln n = 6n^{n-1/2} \ln n > n^{n-1/2}.$$

Since $n > 100$, Remark 1 tells us that there exist $f_1, f_2 \in E^I$ with $\left[\sqrt{n}\right] \leqslant |f_1 \cap f_2| \leqslant \leqslant n - \left[\sqrt{n}\right]$.

Since H is a clique, any edge e from $E(I)$ intersects both f_1 and f_2. So, either e goes through a vertex $v \in (f_1 \cap f_2)$ or it contains a vertex $v_1 \in (f_1 \setminus f_2)$ and a vertex $v_2 \in (f_2 \setminus f_1)$. Formally, we may write down the equality

$$E(I) = \Bigg(\bigcup_{v \in (f_1 \cap f_2)} E_v(I)\Bigg) \bigcup \Bigg(\bigcup_{v_1 \in (f_1 \setminus f_2)} \bigcup_{v_2 \in (f_2 \setminus f_1)} E_{v_1,v_2}(I)\Bigg),$$

where

$$E_v(I) = \big\{e \in E(I) : \ v \in e\big\}, \quad E_{v_1,v_2}(I) = \big\{e \in E(I) : \ v_1, v_2 \in e\big\}.$$

Of course,

$$|E(I)| \leqslant \sum_{v \in (f_1 \cap f_2)} |E_v(I)| + \sum_{v_1 \in (f_1 \setminus f_2)} \sum_{v_2 \in (f_2 \setminus f_1)} |E_{v_1,v_2}(I)|.$$

If there is a summand in the first sum greater than $\left(n - \frac{\sqrt{n}}{4}\right)^{T'-i-1} n^{n-T'}$, then (2) is shown: indeed, the corresponding v is contained in this many edges

$e \in E(I)$ already containing I. Similarly, if there is a summand in the second sum greater than $\left(n-\frac{\sqrt{n}}{4}\right)^{T'-i-2} n^{n-T'}$, then (3) is shown. So, let us suppose that there are no such summands. In this case, putting $k = |f_1 \cap f_2|$, we get

$$|E(I)| \leqslant k\left(n-\frac{\sqrt{n}}{4}\right)^{T'-i-1} n^{n-T'} + (n-k)^2\left(n-\frac{\sqrt{n}}{4}\right)^{T'-i-2} n^{n-T'}.$$

On the other hand, $|E(I)| > \left(n-\frac{\sqrt{n}}{4}\right)^{T'-i} n^{n-T'}$. Having proved that

$$\left(n-\frac{\sqrt{n}}{4}\right)^{T'-i} n^{n-T'} > k\left(n-\frac{\sqrt{n}}{4}\right)^{T'-i-1} n^{n-T'} + (n-k)^2\left(n-\frac{\sqrt{n}}{4}\right)^{T'-i-2} n^{n-T'}$$

for any $k \in \left[\left[\sqrt{n}\right], n-\left[\sqrt{n}\right]\right]$, we would get a contradiction which would complete the proof of (2) or (3).

The needed inequality is equivalent to

$$\left(n-\frac{\sqrt{n}}{4}\right)^2 > k\left(n-\frac{\sqrt{n}}{4}\right) + (n-k)^2,$$

which can be proved by standard analytic calculations.

Thus, indeed, either (2), or (3) takes place. So, after $\leqslant T'$ steps of the inductive procedure, we get either a set I of cardinality T' such that $|E(I)| > n^{n-T'}$, or a set I of cardinality $T'+1$ such that $|E(I)| > \left(n-\frac{\sqrt{n}}{4}\right)^{-1} n^{n-T'} > n^{n-T'-1}$. Both estimates are in conflict with Proposition 1. Consequently, our initial assumption $|E| > 10n^{n-1/2}\ln n$ is false, and Theorem 2 is proved. □

Acknowledgements. This work was supported by the grant of RFBR № 09–01–00294.

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