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Website
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E-mail
`mjcnt@phystech.edu`

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Faculty of Innovations
and High Technology,
Laboratorny Korpus, k. 209,
9, Institutskii pereulok,
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The totally real algebraic integers with diameter less than 4

Valérie Flammang (Metz),
Georges Rhin (Metz),
Qiang Wu (Chongqing)

Abstract: The diameter of a totally real algebraic integer α of degree d with conjugates $\alpha_1 < \alpha_2 < \ldots < \alpha_d$ is $\mathrm{diam}(\alpha) = \alpha_d - \alpha_1$. For all positive integers k and n, $\mathrm{diam}(2\cos(2k\pi/n))$ is less than 4. R. M. Robinson has computed, modulo integer translations, all the other totally real algebraic integers α with $\mathrm{diam}(\alpha) < 4$ for $d \leqslant 8$. We have extended the computations to all $d \leqslant 15$. We use a large family of explicit auxiliary functions related to generalized integer transfinite diameter of real intervals. They give better bounds than the previous methods for the coefficients of the minimal polynomial of α, making such computations practicable. For $d = 15$ we prove a recent conjecture of Capparelli et al.

Keywords: totally real algebraic integers, diameter, auxiliary functions, integer transfinite diameter, Chebyshev polynomials

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1. Introduction

Let α be an algebraic integer of degree $d \geqslant 2$ with minimal polynomial

$$P = X^d + b_1 X^{d-1} + \ldots + b_d = \prod_{i=1}^{d} (X - \alpha_i).$$

The diameter of α (or of P) is $\mathrm{diam}(\alpha) = \max_{1 \leqslant i,j \leqslant d} |\alpha_i - \alpha_j|$. The diameter is clearly invariant by integer translation, i.e. $\mathrm{diam}(\alpha + n) = \mathrm{diam}(\alpha)$ for any rational integer n. It is also invariant by reflection, i.e. $\mathrm{diam}(-\alpha) = \mathrm{diam}(\alpha)$. In this paper, we shall examine the case of totally real algebraic integers α, i.e., when all the conjugates of α are real (in this case, their minimal polynomials P are called hyperbolic polynomials).

Let I be a real interval $[a, b]$ containing the numbers $\alpha_1 < \alpha_2 < \ldots < \alpha_d$. If I has length $b - a > 4$ then R. M. Robinson [7] proved that there are infinitely many numbers α with all α_i in I. If $b - a < 4$ Schur and Pólya [10] proved that there is only a finite number of algebraic integers having this property. If $\mathrm{diam}(\alpha) = 4$, then $\alpha_d - \alpha_1 = 4$. By Galois theory there exists some i with $1 < i < d$ such that

$\alpha_{d-1} - \alpha_i = 4$. That is clearly impossible. Thus the following question naturally arises:

$$\text{what happens when we only suppose that } \operatorname{diam}(\alpha) < 4?$$

If I is a real interval of length equal to 4 with integer endpoints, for instance $I = [-2, 2]$, then the existence of infinitely many numbers α with all their conjugates in I is known. By Kronecker's second theorem [4], the numbers $2\cos(2k\pi/n)$ with positive integers k and n are the only algebraic integers which possess this property.

We say that two numbers α, or their minimal polynomials, are *equivalent* if their diameters are equal.

R. M. Robinson [8] has computed all diameters less than 4 up to degree 8.

Recently Capparelli et al. [2] have computed diameters of all α up to degree 14. They also gave lists of diameters for the degrees 15 to 17 which have been computed with heuristic conditions on the coefficients of the polynomials P. They conjecture that these lists are complete. Here we prove the following theorem:

THEOREM 1. *There are exactly* 6 *totally real algebraic integers, up to equivalence, of degree* 15 *and diameter less than* 4. *Their minimal polynomials are*:
$x^{15} - 3x^{14} - 11x^{13} + 37x^{12} + 45x^{11} - 181x^{10} - 79x^9 + 445x^8 + 34x^7 - 572x^6 + 67x^5 + 357x^4 - 84x^3 - 81x^2 + 27x - 1$ of diameter $3.974724\ldots$,
$x^{15} - x^{14} - 15x^{13} + 12x^{12} + 92x^{11} - 55x^{10} - 294x^9 + 119x^8 + 515x^7 - 120x^6 - 474x^5 + 45x^4 + 196x^3 - x^2 - 22x + 1$ of diameter $3.985304\ldots$,
$x^{15} - 6x^{14} + x^{13} + 54x^{12} - 66x^{11} - 188x^{10} + 325x^9 + 313x^8 - 697x^7 - 238x^6 + 747x^5 + 38x^4 - 382x^3 + 44x^2 + 72x - 17$ of diameter $3.993078\ldots$,
$x^{15} - 7x^{14} + 7x^{13} + 50x^{12} - 103x^{11} - 127x^{10} + 393x^9 + 129x^8 - 692x^7 - 35x^6 + 597x^5 - x^4 - 217x^3 - 10x^2 + 15x + 1$ of diameter $3.996557\ldots$,
$x^{15} - 2x^{14} - 13x^{13} + 24x^{12} + 69x^{11} - 113x^{10} - 191x^9 + 263x^8 + 291x^7 - 312x^6 - 235x^5 + 173x^4 + 87x^3 - 34x^2 - 10x + 1$ of diameter $3.996983\ldots$,
$x^{15} - x^{14} - 15x^{13} + 12x^{12} + 91x^{11} - 54x^{10} - 283x^9 + 111x^8 + 470x^7 - 100x^6 - 393x^5 + 30x^4 + 140x^3 - 15x - 1$ of diameter $3.999952\ldots$.

Remark. In this paper we use a different method to compute all diameters for all degrees up to 15. Using the arithmetical properties of the numbers α, we construct a large family of "explicit auxiliary functions" which are related to generalizations of the integer transfinite diameter. This gives better bounds for the coefficients of the minimal polynomial of α. Despite the exponential growth of the computing time (by a factor close to 13 between degree 14 and degree 15), we succeed in computing all these numbers α up to degree 15 on a PC and prove that the conjecture made by Capparelli et al. for degree 15 is true.

We give our results in Table 1 (excluding the numbers of cosine type):

Table 1

Number of polynomials of diam < 4 for $2 \leqslant d \leqslant 15$.

d	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number	1	3	10	14	13	15	21	19	15	10	9	4	9	6

The minimal polynomials of these numbers can be found on the Web site [11]. In view of the data in Table 1 we make the following conjecture:

CONJECTURE. There are infinitely many diam(α) of the type given in Table 1, but probably not for every degree.

The auxiliary functions that we use give bounds for s_k which is the sum of the k-th powers of the conjugates of α. Then, by Newton's formula

$$s_k + s_{k-1}b_1 + \ldots + s_1 b_{k-1} + kb_k = 0,$$

we may deduce bounds for b_k for $k \leqslant d$.

As explained in Section 3 below, up to degree 12, we built the auxiliary functions with the algorithm given by the third author [12]. For degrees 13 to 15 we used the important refinement introduced by the first author [3].

The paper is organized as follows. In Section 2 we give the scheme of the computation, once we have given the bounds for the s_k. In Section 3 we explain the principle of the explicit auxiliary functions and their relation with the integer transfinite diameter. We also give all kinds of auxiliary functions that we use and how we use the Chebyshev polynomials of some real intervals. In Section 4 we recall Robinson's method of computation. Some numerical results and comments are given in Section 5. In this last section, we also study polynomials whose roots satisfy a symmetry.

2. Scheme of computations

If diam(α) < 4 then all conjugates of $\{\alpha_1\} - 2$ or of $3 - \{\alpha_1\}$, where $0 < \{\alpha_1\} < 1$ is the fractional part of α_1, lie in the interval $H = (-2, 2.5)$. Since we already know all the polynomials which have their roots in $(-2, 2)$ we only need to compute the numbers α such that α_d lies in $(2, 2.5)$. We divide this interval in 5 subintervals of length 0.1. Then all the conjugates of α_d belong to an interval $I_i = (a_i, b_i)$ of length 4.1 where $a_i = -2 + 0.1(i-1)$ and $b_i = 2 + 0.1\,i$ for $1 \leqslant i \leqslant 5$.

For any such interval I_i we compute all the possible values of $s_1 = \text{trace}(\alpha)$. We give in Table 2 the bounds obtained with the auxiliary functions for degree 15.

Table 2

Bounds of s_1 for $d = 15$

Interval	$(-2.0, 2.1)$	$(-1.9, 2.2)$	$(-1.8, 2.3)$	$(-1.7, 2.4)$	$(-1.6, 2.5)$
Lower Bound	-2	-2	0	2	3
Upper Bound	4	5	7	9	10

They noticeably greatly improve the naïve bounds, which are, for I_1 for instance, -29 and 31, respectively. Then we compute all possible values of s_2 for a given value of s_1. For degree 15 we get 212 triples (I_i, s_1, s_2). As explained in Section 3 below, we give bounds for s_k for $k \geqslant 3$ depending on the values of s_j for some $j < k$ by other methods.

3. Auxiliary functions and integer transfinite diameter

3.1. Relation between auxiliary functions and integer transfinite diameter

Let I be a fixed interval I_i, i.e., $I = (a_i, b_i)$. We consider the auxiliary function

$$f(x) = x - a \log |Q(x)| \geqslant m \qquad \text{for } x \in I, \tag{1}$$

where a is a positive real number and $Q \in \mathbb{Z}[x]$.

Summing over the α_i, we get

$$\sum_{i=1}^{d} f(\alpha_i) \geqslant dm.$$

Then

$$s_1 = \sum_{i=1}^{d} \alpha_i \geqslant dm + a \log \left| \prod_{i=1}^{d} Q(\alpha_i) \right|.$$

The product $\prod_{i=1}^{d} Q(\alpha_i)$ is the resultant of P and Q. If P does not divide the polynomial Q, then this resultant is a nonzero integer. Therefore

$$s_1 \geqslant dm.$$

For a historical survey on the use of these auxiliary functions to get a lower bound for $s_1 = \mathrm{trace}(\alpha)$ when α is totally positive, see Aguirre and Peral [1]. If we take the auxiliary function as

$$f(x) = -x - a \log |Q(x)|,$$

we have an upper bound for s_1. And if we replace x by $\pm x^k$ we get bounds for s_k. If we replace a by t/h where $h = \deg Q$, we have

$$f(x) = x - \frac{t}{h} \log |Q(x)|. \tag{2}$$

We seek a polynomial Q in $\mathbb{Z}[x]$ such that

$$\max_{x \in I} |Q(x)|^{t/h} e^{-x} \leqslant e^{-m}.$$

If t is fixed (say $t = 1$), we relate this inequality to

$$t_{\mathbb{Z},\varphi}(I) = \liminf_{\substack{h \geqslant 1 \\ h \to \infty}} \min_{\substack{Q \in \mathbb{Z}[x] \\ \deg Q = h}} \max_{x \in I} |Q(x)|^{1/h} \varphi(x)$$

which is the integer transfinite diameter of I with the weight $\varphi(x) = e^{-x}$.

3.2. Construction of auxiliary functions

If $Q_1, Q_2, \dots, Q_J$ are the irreducible factors of Q in (1), then

$$f(x) = x - \sum_{j=1}^{J} e_j \log |Q_j(x)|. \tag{3}$$

When the polynomials Q_j are given, we use the semi-infinite linear programming to optimize f (i.e. to get the greatest possible m). This method was introduced in number theory by C. J. Smyth in 1984 [9]. So the main problem is to obtain a "good set" of polynomials Q_j.

Up to degree 12, we use the method given by the third author in [12], using LLL to get a good polynomial Q. For degree 13 to 15, a more elaborate algorithm was given by the first author [3]. The main idea is to get good polynomials Q_j by induction. Suppose that we have $Q_1, Q_2, \dots, Q_J$ and an optimal f for this set of polynomials in the form (1). Then, for several values of the integer k, we seek a polynomial $R \in \mathbb{Z}[x]$ of degree k such that

$$\sup_{x \in I} |Q(x)R(x)|^{\frac{t}{h+k}} e^{-x} \leqslant e^{-m},$$

i.e., such that

$$\sup_{x \in I} |Q(x)R(x)| \exp\left(\frac{-x(h+k)}{t}\right)$$

is as small as possible. We apply LLL to the set of linear forms

$$Q(x_n)R(x_n) \exp\left(\frac{-x_n(h+k)}{t}\right),$$

where the numbers x_n are suitable points in I, including the points where f has its least local minima. We get a polynomial R whose factors R_j are good candidates to enlarge the set of polynomials $(Q_1, Q_2, \dots, Q_J)$. We only keep the polynomials R_j which have a nonzero coefficient e_j in the new optimized auxiliary function f. After optimization, some previous polynomials Q_j may have a zero coefficient and are removed.

We give in Table 3 the polynomials Q_j and the coefficients e_j of the auxiliary function f in (3) which gives the inequality $s_1 \geqslant -0.179\dots d$ for α_d in I_1.

The polynomials with * are polynomials having some nonreal roots.

3.3. Computation of the minimum of $f(x)$

Let f be the auxiliary function defined in (3). We give an algorithm to prove that f is a convex function in its domain of definition $\mathbf{D}$. It suffices to prove that the second derivative of f is positive in $\mathbf{D}$. By factorization of the polynomials Q_j in irreducible real factors, we see that f'' is a sum of terms of type 1:

$$\frac{e_j}{(x-\alpha)^2},$$

Table 3

The explicit auxiliary function for the lower bound of s_1 in I_1

e_j	d_j	Coefficients of Q_j from degree 0 to d_j													
0.011924266819	1	−1	1												
0.096866488945	1	0	1												
0.451013485389	1	1	1												
0.527960449230	1	2	1												
0.019566079389	2	−3	0	1											
0.023273574120	2	−2	0	1											
0.172825687626	2	−1	1	1											
0.005291433336	3	−1	−4	0	1										
0.012502796101	3	−1	−3	0	1										
0.094662108579	3	−1	−2	1	1										
0.003914026079	3	1	−3	0	1										
0.071375556773	4	1	−4	−4	1	1									
0.013261762401	4	1	−1	−4	0	1									
0.021700560081	5	1	3	−3	−4	1	1								
0.008611832838	5	1	5	−1	−5	0	1								
0.028107777149	5	3	4	−5	−5	1	1								
0.002577198250	5	3	8	−1	−6	0	1								
0.002519477706	6	−5	−1	13	0	−7	0	1							
0.005370851864	6	−4	3	13	−1	−7	0	1							
0.010760212439	6	−1	3	6	−4	−5	1	1							
0.020912491011	6	1	8	2	−9	−4	2	1*							
0.010610276065	6	1	8	8	−6	−6	1	1							
0.000063033687	6	1	14	11	−8	−7	1	1							
0.002043090659	8	−1	−7	−5	13	7	−10	−5	2	1*					
0.033279104801	8	−1	−1	10	23	5	−14	−6	2	1*					
0.002949394220	8	1	−4	−10	10	15	−6	−7	1	1					
0.002368600679	9	1	5	−10	−20	15	21	−7	−8	1	1				
0.000672281393	10	1	−1	−27	−39	25	57	3	−24	−6	3	1			
0.003803608820	10	1	8	−6	−45	−12	46	19	−17	−8	2	1*			
0.015037380017	11	−1	−1	22	26	−47	−51	34	35	−10	−10	1	1*		
0.000861196960	12	−5	−14	31	84	−21	−137	−29	85	32	−22	−10	2	1*	
0.000592076853	12	−1	−1	34	83	−3	−130	−42	79	34	−21	−10	2	1*	
0.000631943824	12	−1	3	60	121	−26	−186	−45	103	40	−24	−11	2	1*	
0.005649700707	13	−1	−6	9	64	51	−109	−146	39	109	12	−31	−8	3	1*

where α is a real root of a polynomial Q_j, and of type 2:

$$\frac{2e_k((x-\gamma)^2-\delta^2)}{((x-\gamma)^2+\delta^2)^2},$$

where $\gamma + i\delta$ $(\delta > 0)$ is a complex root of a polynomial Q_k (k may be equal to j).

We now assume that all the real roots α are taken in increasing order, and that the complex roots $\gamma + i\delta$ are taken in increasing order of their real parts.

Algorithm

Step 1:

Let S be a sequence of complex roots $\gamma + i\delta$ $(\delta > 0)$ with consecutive increasing real parts, i.e., there is no real root α between two values of γ. We sum up all terms of type 2 related to the sequence S. Then we add to the rational function obtained all the terms of type 1 associated with a real root α, from the greatest α less than all the γ in the sequence S, to the least α greater than all the γ in the sequence. Let F_S be the rational function thus obtained. Using Sturm's algorithm, we compute the number of real zeros of the numerator of the function F_S which are in the domain $\mathbf{D}$ of definition of f. If, for all sequences S, there is no zero in $\mathbf{D}$, then we conclude that f is convex. If this is not the case, we use the following

Step 2:

We add to the exceptional functions F_S, having zeros in $\mathbf{D}$, some terms of type 1 containing real roots α close to the real roots already used in F_S, so that the modified F_S has no zeros in $\mathbf{D}$.

Remark. For the function given in Table 3, it suffices to use Step 1 of the algorithm. Then, the local minima of f between two consecutive real roots are easily computable by the downhill simplex algorithm [6].

3.4. Other kinds of auxiliary functions

Using

$$f(x) = x^2 - e_0 x - \sum_{j=1}^{J} e_j \log |Q_j(x)| \geqslant m,$$

where e_0 is real, we get $s_2 - e_0 s_1 \geqslant dm$. Thus if $s_1 = \sigma_1$, then $s_2 \geqslant e_0\sigma_1 + dm$. We optimize the linear form $e_0\sigma_1 + dm$ to get a lower bound for s_2 when $s_1 = \sigma_1$. If we replace x^2 by $-x^2$, we get an upper bound for s_2. As we explained in the introduction, we obtain inequalities involving the numbers s_k. Let us give an example for s_4 and s_2. We use the auxiliary function

$$f(x) = x^4 - e_0 x^2 - \sum_{j=1}^{J} e_j \log |Q_j(x)| \geqslant m$$

where e_0 is real and $m \geqslant 0$. Then we get $s_4 \geqslant e_0 s_2$, and this relation is independent of the degree d. We maximize e_0 and stop the optimization process when $m \geqslant 0.01$. For the upper bound, $x^4 - e_0 x^2$ is replaced by $-x^4 + e_0 x^2$ and we minimize e_0.

3.5. Chebyshev polynomials

Let T_k be the Chebyshev polynomial of degree k of the interval (a, b). This is the monic polynomial whose sup norm is the least on (a, b). The polynomials

T_k can be defined by the relations $T_1 = x - A$, $T_2 = x^2 - 2Ax + A^2 - 2B^2$ and $T_k = (x - A)T_{k-1} - B^2 T_{k-2}$ for $k > 2$, where $A = (a + b)/2$ and $B = (b - a)/4$. Then $\max_{a \leqslant x \leqslant b} |T_k(x)| = 2B^k$ for $k \geqslant 1$.

Then we have the inequality $\left| \sum_{1 \leqslant i \leqslant d} T_k(\alpha_i) \right| \leqslant 2dB^k$. This gives a lower and an upper bound for s_k depending on the known values of s_j for $0 \leqslant j \leqslant k - 1$.

4. Robinson's method

The coefficients of the minimal polynomial P of α are computed by induction. In our case I_i, b_1 and b_2 are fixed. Suppose we already know $b_1, b_2, \ldots, b_k$, and want to compute the bounds for b_{k+1}. Let Q_{k+1} be the following polynomial

$$Q_{k+1} = \frac{1}{(d-k+1)!} \left(\frac{d}{dx} \right)^{d-k+1} P = c_0 x^{k+1} + c_1 x^k + \ldots + c_k x + a,$$

where c_j is an integer multiple of b_j, and a is an unknown integer (which will be b_{k+1}). Then the polynomial $\widetilde{Q}_{k+1} = Q_{k+1} - a$ is known. In I_i, $\widetilde{Q}_{k+1}$ has k local extrema at the roots of its derivative. These roots $r_1, r_2, \ldots, r_k$ have been computed before. We take $r_0 = a_i$ and $r_{k+1} = b_i$. We compute the values of $\widetilde{Q}_{k+1}(r_i)$ for $0 \leqslant i \leqslant k + 1$. The admissible integers a are those for which all the minima of Q_{k+1} are negative, and all its maxima are positive. For each a we compute the roots of Q_{k+1} with the Newton-Raphson method, starting at the points $(r_i + r_{i+1})/2$ for $0 \leqslant i \leqslant k$. When the diameter of Q_{k+1} is less than 4, we replace k by $k + 1$. When $k = d$, we use Pari [5] to eliminate the reducible polynomials.

5. Numerical results

5.1. Symmetry

Some polynomials P of even degree $2d$ have their roots pairwise symmetric with respect to 1/2. Then we say that P is *symmetric*. If, in P, we put $y = -x^2 + x + 2$, we get a polynomial Q of degree d whose roots lie in the interval $(-1.75, 2.25)$. The polynomial Q may be of cosine type. The reverse transformation will not always give an irreducible polynomial. We give on the Web site [11] all pairs P, Q. There are 19 such pairs. For the degree 16, Capparelli et al. give 3 polynomials, 2 of which are symmetric.

Thus we may reformulate the classical question:

Are there infinitely many α with $\operatorname{diam}(\alpha) < 4$ and, if so, how many of them are symmetric?

5.2. The computations

All the computations have been made with Pascal on a PC. For the degree 14 the computing time was 34 days, and for the degree 15 it took 440 days.

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Valérie Flammang,
Georges Rhin

UMR CNRS 7122,
Département de Mathématiques
UFR MIM
Université de Metz
Ile du Saulcy
57045 Metz Cedex 01, France
flammang@univ-metz.fr;
rhin@univ-metz.fr

Qiang Wu

Department of Mathematics
Southwest University of China
2 Tiansheng Road Beibei
400715 Chongqing, China
qiangwu@swu.edu.cn