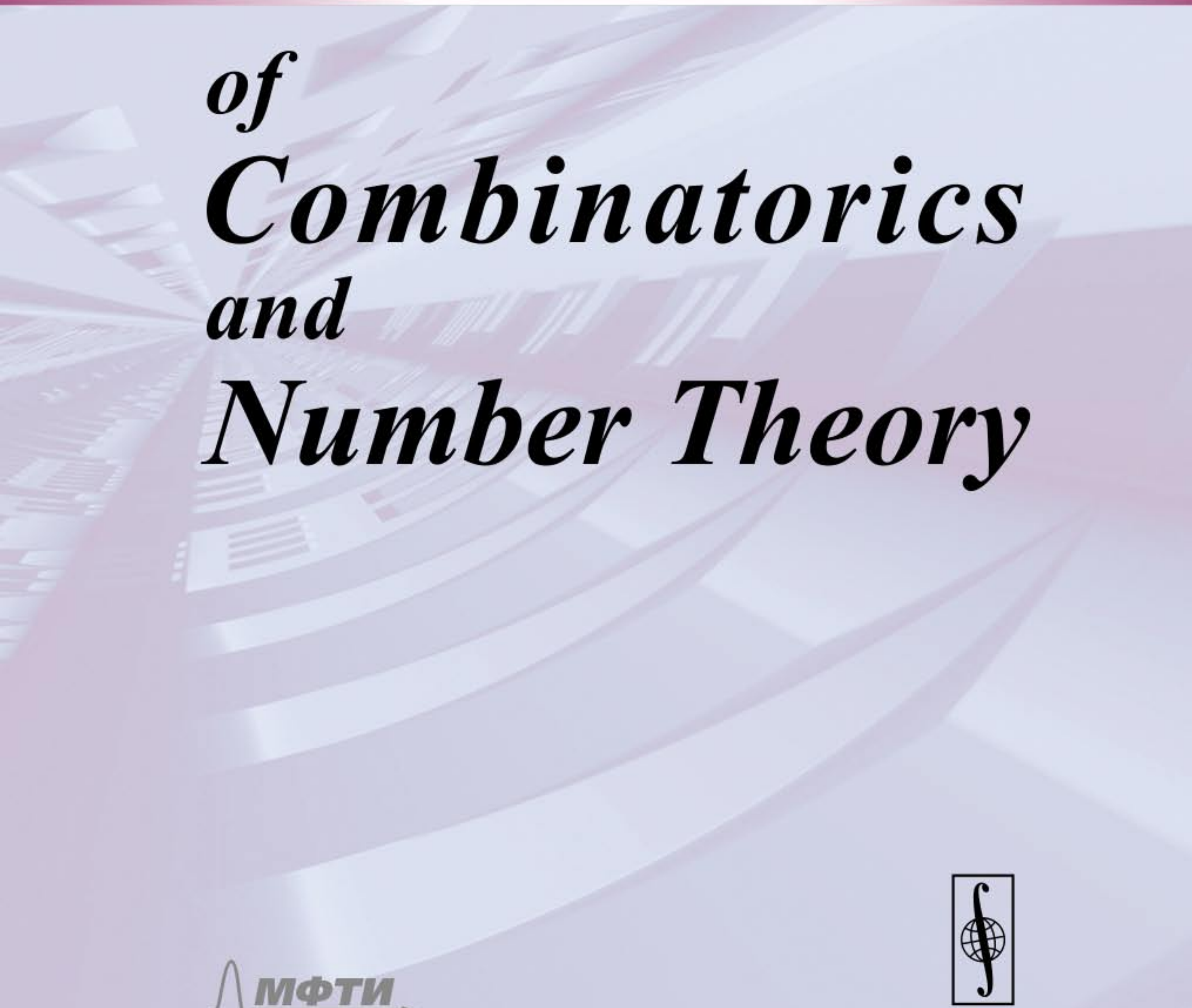
Reprint from

ISSN 2220-5438

Moscow Journal *of Combinatorics and Number Theory*



Volume 1 • Issue 4

2011

Moscow Journal of Combinatorics and Number Theory. 2011. Vol. 1. Iss. 4. 112 p.

The journal was founded in 2010.

Published by the Moscow Institute of Physics and Technology
with the support of Yandex and Microsoft.

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

Website of our journal
`http://mjcnt.phystech.edu`

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URSS Publishers
56, Nakhimovsky Prospekt,
Moscow,
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117335

Издательство «УРСС»
Нахимовский пр-т, 56
Москва,
Российская Федерация,
117335

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77–41900.

Формат 70×100/16. Печ. л. 7. Зак. № ЖТ-81.
Отпечатано в ООО «ЛЕНАНД».
117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.

ISSN 2220–5438

11762 ID 159732
9 785453 000289

Moscow Journal
of Combinatorics
and Number Theory
2011, vol. 1, iss. 4,
pp. 91–111, [pp. 379–399]

Sufficient and equivalent criteria for the Riemann Hypothesis

Davide Schipani (Zurich)

Abstract: The paper presents several new criteria, which are equivalent to or sufficient for proving the classical Riemann Hypothesis. Several other noteworthy statements and remarks on the zeta function are also included.

Keywords: zeta function, Riemann hypothesis, sufficient conditions, equivalent criteria

AMS Subject classification: 11M06, 11M26

Received: 29.10.2010; **revised:** 10.02.2011

1. Introduction

It is well known that the Riemann zeta function $\zeta(s)$ is zero at the points $s = -2, -4, -6, \ldots$, which are called the trivial zeros. The Riemann Hypothesis conjectures that all non-trivial zeros have their real parts equal to $1/2$. This conjecture inspired many mathematicians to contribute to a vast amount of literature on the subject. The Riemann Hypothesis has also been computationally verified for the first 10^{13} nontrivial zeros ordered by increasing positive imaginary part (see e. g. [14]). Several authors also came up with sufficient and/or necessary conditions; many of these, as well as a more general overview of the problem, can be found in such works as [5], [6], [10] and the references therein. This work is mainly devoted to presenting new sufficient conditions and equivalent criteria.

The paper is organized as follows. After recalling some properties of the zeta function below, in Section 2 we are going to state and prove a sufficient condition for the Riemann Hypothesis. In section 3 we are going to discuss some equivalent criteria.

In a companion paper we state analogues of some of these results for several generalized versions of the Riemann Hypothesis ([26]).

Throughout this paper we make use of basic properties of the Riemann zeta function; we summarize them below to make the argument accessible to a non-specialist. We will refer to these properties by using numbers in round brackets. Some facts related to the Γ function are also included in the list.

1. It is known that there are no nontrivial zeros of ζ in the half-planes $x \leqslant 0$ and $x \geqslant 1$ (nor in some regions in between, for example on the real axis between 0 and 1, see e. g. [1, chapter 13] or [27]). Thus, essentially the Riemann Hypothesis states that the only zeros in the critical strip, i. e. in the strip $0 < \mathrm{Re}(s) < 1$, lie on the line $x = 1/2$. Therefore, the critical strip $0 < \mathrm{Re}(s) < 1$ becomes our region of interest, with particular focus on analytical properties or convergence of sequences of functions.
2. The function $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is analytic in the whole complex plane apart from poles at 0 and 1. Furthermore, $\Lambda(s) = \Lambda(1-s)$ (see e. g. [17]).
3. Zeros of ζ in the critical strip are the same as zeros of Λ (Γ has no zeros). It follows from (2) that if s is a zero of ζ, then also $1-s$ is a zero.
4. Since ζ is analytic in the critical strip, its zeros are isolated.
5. In the critical strip the function ζ can be defined as follows:

   $$\zeta(s) = \left(1 - 2^{1-s}\right)^{-1} \cdot \phi(s),$$

   where

   $$\phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

   (see e. g. [19, section 10.2]). Again, zeros of ζ in the strip are the same as zeros of ϕ.
6. $\overline{\zeta(s)} = \zeta(\bar{s})$. (This relation can be easily derived from the representation (5). One can also apply the Schwarz reflection principle in complex analysis [9] after observing that ζ is real on the real axis.)

7. From (3) and (6) it follows that the zeros of ζ are symmetric with respect to both the critical line $x = 1/2$ and the real axis.
8. By Euler—Maclaurin summation formula (see e. g. [11, section 6.4]), if $s = \sigma + it$ belongs to $\mathbb{C}\backslash\{1\}$, the following holds:

$$\zeta(s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} + O(N^{-\sigma-1}).$$

   Moreover, by the upper bound on the remainder terms obtained in [11], the implicit constant in the big O notation can be chosen to be the same for all s in any compact subset of $\mathbb{C}\backslash\{1\}$.
   Actually, if we consider a region $\sigma \geqslant \sigma_0 > 0$, where σ_0 is fixed, we could even say that

$$\zeta(s) = \sum_{n=1}^{N} n^{-s} + \frac{N^{1-s}}{s-1} + O(N^{-\sigma})$$

   holds uniformly, provided that $N > C|t|/2\pi$, where C is a given constant greater than 1 (see [30, Theorem 4.11]).
9. $\overline{\Gamma(s)} = \Gamma(\bar{s})$. (It follows, for example, from the definition of Γ as a limit, or again from the Schwarz reflection principle.)
10. If $-1/2 \leqslant \sigma \leqslant 1/2$, then

$$\left|\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right| \leqslant \left|\frac{1+s}{2}\right|^{1/2-\sigma}$$

   ([23, Lemma 1] specialized with $q = 0$).
11. There is an infinite number of zeros of ζ lying on the critical line (Hardy's theorem: see e. g. [11, chapter 11]).

2. Sufficient conditions for the Riemann Hypothesis

Before stating our first condition, we recall a result of Selberg ([28], [16, section 2.1.11]) saying that

$$\lim_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in (0,T] : \forall\, k \in \left(\tau, \tau + \frac{\Phi(\tau)}{\log \tau}\right)\ \zeta\left(\frac{1}{2} + ik\right) \neq 0\right\} = 0,$$

where $\Phi(\tau)$ is any positive function which tends to ∞ as $\tau \to \infty$ and μ is the Lebesgue measure.

Taking, for example, $\Phi(\tau) = \log\log\tau$, this implies that for any fixed η we have

$$\lim_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in (0,T] : \forall\, k \in (\tau, \tau+\eta)\ \zeta\left(\frac{1}{2}+ik\right) \neq 0\right\} = 0,$$

or

$$\lim_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in (0,T] : \exists\, k \in (\tau, \tau+\eta) \mid \zeta\left(\frac{1}{2}+ik\right) = 0\right\} = 1.$$

We will also need another important result, namely Voronin's universality theorem, which states the following (see e. g. [29]): suppose that K is a compact subset of the strip $1/2 < \sigma < 1$ with a connected complement, and let $g(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K; then for all $\epsilon > 0$ we have

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{s\in K} \left|\zeta(s+i\tau) - g(s)\right| < \epsilon\right\} > 0.$$

Note (see [29, section 8.1]) that if Voronin's theorem were true even if $g(s)$ were allowed to vanish, then the Riemann Hypothesis would be false. As shown in the preceding reference, this cannot happen, i. e. a function having zeros cannot be approximated uniformly by ζ, which in fact hints at the relation between universality and the distribution of zeros. In this regard, we can observe that given another notion of universality, which would be slightly different from one expressed by Voronin's theorem, it would be possible to find functions which would possess the principal properties of ζ (namely the functional equation in (2), the property of being analytic except for a pole in 1, and that of being real on the real axis), but wouldn't satisfy the Riemann Hypothesis (see [20]).

Furthermore, not only is the zeta function universal (in the sense of the formula above), we can even say that its derivative is strongly universal in Voronin's sense ("strongly" meaning that the function to be approximated needs not to be non-vanishing [29, section 1.3 and 1.6], [18], [3]).

We are now ready to formulate our first statement:

THEOREM 1. *Suppose that a positive real number η and a zero s_* of ζ lying on the critical line are such that ζ' is strongly universal in the region*

$$U \doteq \left\{\frac{1}{2} < \sigma < 1\right\} \cup \big\{[s_*, s_* + i\eta]\big\},$$

i. e.

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{s\in K} \left|\zeta'(s+i\tau) - g(s)\right| < \epsilon\right\} > 0$$

for any $\epsilon > 0$, any compact subset K of U with a connected complement, and any function $g(s)$ continuous on K and analytic in its interior.

Then the Riemann Hypothesis is true.

PROOF. We are going to give a proof by contradiction: assume that the Riemann Hypothesis is false, which, by (7), implies that there is a zero s_0 in the strip $1/2 < \sigma < 1$. Let K be a compact subset in this strip containing s_0 and having a connected complement.

For any given $\epsilon > 0$, by continuity, there exists $\eta > 0$ such that $|s_* - s| < \eta$ implies $|\zeta(s)| < \epsilon/2$.

The hypothesis implies that for any $\delta > 0$ we have

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{s\in K_2} \left|\zeta'(s+i\tau) - \zeta'(s)\right| \leqslant \delta\right\} > 0$$

for any compact K_2 with a connected complement in

$$\left\{\frac{1}{2} < \sigma < 1\right\} \cup \big\{[s_*, s_* + i\eta]\big\}.$$

Denote as K_2 the compact set containing all lines drawn between points of K and points of the segment $[s_*, s_* + i\eta]$.

Also let δ be equal to

$$\frac{\epsilon}{2\big(\max_{s\in K} |s - s_*| + \eta\big)},$$

and consider $|\zeta(s+i\tau) - \zeta(s)|$ for each of those τ such that

$$\max_{s\in K_2} \left|\zeta'(s+i\tau) - \zeta'(s)\right| \leqslant \delta.$$

By the triangular inequality, we have for any $0 \leqslant \alpha \leqslant 1$ and $s \in K$ that

$$\left|\zeta(s+i\tau)-\zeta(s)\right| \leqslant \left| \int\limits_{s_*+i\alpha\eta}^{s} \big(\zeta'(s+i\tau)-\zeta'(s)\big)\, ds \right| + \left|\zeta(s_*+i\alpha\eta+i\tau)\right| + \left|\zeta(s_*+i\alpha\eta)\right|,$$

where the path of integration is a straight line connecting the two points. The first term in the right-hand side of this inequality is then bounded by $|s-(s_*+i\alpha\eta)|\cdot\delta$, which is less than or equal to $\epsilon/2$.

By Selberg's theorem, we know that for almost all τ there exists $0 \leqslant \alpha \leqslant 1$ such that $\zeta(s_*+i\alpha\eta+i\tau)=0$; moreover $\left|\zeta(s_*+i\alpha\eta)\right| \leqslant \epsilon/2$ by the choice of η, and thus the second part is also less than or equal to $\epsilon/2$.

Thus, in general, we can conclude that

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{s\in K}\left|\zeta(s+i\tau)-\zeta(s)\right| \leqslant \epsilon\right\} > 0.$$

As noted before stating the theorem, this cannot be true since ζ is supposed to vanish on K, which leads to a contradiction. ☐

Remark 1. A direct proof can be obtained by applying a theorem by Bagchi (see e. g. [29, section 8.2]), which says that the Riemann Hypothesis is true if and only if for any compact subset K of the strip $\frac{1}{2} < \sigma < 1$ with a connected complement and for any $\epsilon > 0$ we have

$$\liminf_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0,T] : \max_{s\in K}\left|\zeta(s+i\tau)-\zeta(s)\right| < \epsilon\right\} > 0.$$

Remark 2. Looking at the above proof, one may think of modifying it by considering the possibility that the zeta function takes a particular value on the line $\sigma = \sigma_0$, where $1/2 < \sigma_0 < 1$, as often as zeros occur on the critical line. However, it has been shown that for any complex number $c \neq 0$ the number of solutions to $\zeta(s) = c$ lying below a height T in the strip $1/2 < \sigma_1 < \sigma_2 < 1$ is only asymptotically equal to KT with a finite positive constant K (see [4], [29, Theorem 1.5]).

COROLLARY. *Suppose that* ζ' *is strongly universal in* $1/2 \leqslant \sigma < 1$. *Then the Riemann Hypothesis is true.*

3. Equivalent criteria for the Riemann Hypothesis

In the first part of this section we focus on results that involve zeros of partial sums. One of the most famous theorems of that type is the classical result of Turán, who has proved the following (see e. g. [2, section 8.14] or [6, section 8.3]): let

$$\zeta_N(s) \doteq \sum_{n=1}^{N} \frac{1}{n^s};$$

if there exists an integer N_0 such that $\zeta_N(s) \neq 0$ for all $N \geqslant N_0$ and all $\sigma > 1$, then the Riemann Hypothesis is true.

Note that the converse of this statement doesn't hold, and that this result isn't suitable for proving the Riemann Hypothesis (it has been proved by Montgomery that some roots of these partial sums can be found in certain half planes to the right of the critical strip, see the above references for the details).

For similar results and general results on zeros of partial sums refer to [7, 12, 13].

Our first result in this direction is the following:

THEOREM 2. *Let $H_N(s)$ be defined as*

$$\sum_{n=1}^{N} n^{-s} + \frac{N^{1-s}}{s-1}.$$

The Riemann Hypothesis is true if and only if for any compact disc K in the right (or left) open half of the critical strip there exists an N_0 such that for infinitely many $N > N_0$ the function $H_N(s)$ is non-vanishing for all s in K.

PROOF. If the Riemann Hypothesis is true, then there are no zeros in the right open half. Given any compact K, there exists a minimum $|\zeta|$, which will be denoted as $m > 0$. Now by (8) we have that

$$|H_N(s) - \zeta(s)| = \left| \frac{1}{2} N^{-s} + O(N^{-\sigma-1}) \right|$$

(keeping in mind the remark about big O notation in (8)). By taking a sufficiently large N we can make the above expression smaller than $m/2$ for all s in K, and then by the triangular inequality we can write

$$|H_N(s)| = \left| H_N(s) - \zeta(s) + \zeta(s) \right| \geqslant |\zeta(s)| - \left| H_N(s) - \zeta(s) \right| \geqslant \frac{m}{2} > 0.$$

To prove the statement in the other direction, i. e. to show that the Riemann Hypothesis is true, we can use Hurwitz's theorem (e. g. [29, Theorem 5.13]) which states the following: let G be a region and let $\{f_n\}$ be a sequence of functions analytic on G which converges uniformly on G to some function f; also suppose that $f(s)$ is not identically zero, then an interior point s_0 of G is a zero of $f(s)$ if and only if there exists a sequence $\{s_n\}$ in G which tends to s_0 as $n \to \infty$, and $f_n(s_n) = 0$ for all sufficiently large n.

Since our hypothesis implies that for any compact disc K there cannot exist a sequence s_N in K such than s_N is a zero of the function H_N for all sufficiently large N, Hurwitz's theorem tells us that there are no zeros of ζ in K. However, any s lying in the right open half is the center of some compact disc contained in the interior of this strip. □

We can derive a similar statement using the approximants

$$\phi_N(s) \doteq \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s}.$$

Theorem 3. *The Riemann Hypothesis is true if and only if for any compact disc K in the right open half of the critical strip there exists a number N_0 such that for infinitely many $N > N_0$ the function $\phi_N(s)$ is non-vanishing for all s in K.*

Proof. We can follow the same argument as above, provided that we consider $(1-2^{1-s})^{-1} \cdot \phi_N(s)$. Note that in this case we know that $\phi_N(s)$ converges uniformly in any compact subset of the strip since any Dirichlet series converges uniformly on any compact subset of its convergence half-plane (see e. g. [1, chapter 11]). □

We can slightly generalize these results by considering, for example, the case of $H_N(s)$:

Theorem 4. *The Riemann Hypothesis is true if and only if for any compact disc K in the right open half of the critical strip there exists an analytic function $f(N,s)$ which tends to zero uniformly in K as $N \to \infty$, as well as numbers $\alpha_1, \dots, \alpha_d \in \mathbb{C}$, $L_1, \dots, L_d, N_0 \in \mathbb{N}$ such that for infinitely many $N > N_0$ the expression*

$$f(N,s) + \alpha_1 H_N(s) + \alpha_2 H_{N+L_1} + \cdots + \alpha_d H_{N+L_{d-1}} + \left(1 - \sum_{i=1}^{d} \alpha_i\right) H_{N+L_d}$$

is non-vanishing for all s in K.

Proof. If the Riemann Hypothesis is true, then, as shown in the proof of the previous equivalence, for any K there exists a number N_0 such that for infinitely many $N > N_0$ the function $H_N(s)$ is non-vanishing for all s in K. This corresponds to taking $f(N, s) = 0$ for all s in K with $\alpha_1 = 1$ and all other α_i equal to 0. The proof going in the other direction can be constructed similarly to the previous proof for the less general case since

$$f(N, s) + \alpha_1 H_N(s) + \alpha_2 H_{N+L_1} + \cdots + (1 - \sum_{i=1}^{d} \alpha_i) H_{N+L_d}$$

is also converging to $\zeta(s)$ uniformly in K. □

For example, it would be sufficient to prove that the arithmetic mean of H_N and H_{N-1} is always nonzero for infinitely many $N > N_0$ and for all s in any given K. This might be easier, though it would be essentially equivalent; it might be useful to note that H_N and H_{N-1} cannot be both equal to zero, at least for large values of N that we are interested in (in fact, in the critical strip this also holds for small values, as stated in [21]); the difference between the two is equal to 0 if

$$1 - s = n - (n-1)^{1-s} n^s = n - n\left(1 - \frac{1}{n}\right)^{1-s}.$$

We do have

$$1 - s = n - n\left(1 - \frac{1-s}{n}\right),$$

but this is only an approximation of the binomial expansion of

$$n - n\left(1 - \frac{1}{n}\right)^{1-s}.$$

We continue by presenting several equivalent forms of the Riemann Hypothesis of a different nature.

First, let us define

$$F(s) \doteq \frac{\zeta(s)}{\zeta(1-s)}.$$

Note that $F(s)$ can be analytically continued to the zeros of ζ, where its values can be defined as

$$F(s) = \frac{\pi^{s-1/2}\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$

(see (2)).

In fact, this shows not only that the zeros of the denominator are simply removable singularities, but also that if s is a zero, then s and $1-s$ have the same multiplicity[1].

Note that if $s_0 = \sigma_0 + it_0$ is a zero of ζ in the critical strip, then $F(s_0)$ is nonzero, and it can be written as

$$F(s_0) = \frac{\pi^{s_0-1/2}\Gamma\left(\frac{1-s_0}{2}\right)}{\Gamma\left(\frac{s_0}{2}\right)}.$$

THEOREM 5. *The Riemann Hypothesis is true if and only if* $|F(s_0)| = 1$ *for any zero* s_0 *of* ζ.

PROOF.

If $\sigma = 1/2$, then by (6) we have $|F(s)| = 1$.

On the other hand, if $\sigma \neq 1/2$ and $t \geqslant 2\pi + 1$, then by [25, Theorem 1] we have $|F(s)| \neq 1$. Note that due to (6) and the fact that $F(s) = 1/F(1-s)$, we can say that if $|F(s)| \neq 1$, then the same inequality can be written for the points which are symmetric to s with respect to the line 1/2 or the critical line.

Finally, the region $\{t<2\pi+1\}$ isn't relevant in this context since it is well known that the imaginary part of the first zero in the upper half-plane is bigger than 14. □

Remark 3. In fact, we can almost complement the result from [25] that is mentioned in the proof, somewhat improving the known asymptotic estimates and almost equivalences (in addition to [25] and the known asymptotic estimate resulting from Stirling's approximation, see also [12, Lemma 6.1] and [8, Theorem 4.3]).

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[1] Assuming that this is not the case, s must have the higher multiplicity so that $F(s)$ remains bounded in a neighboorhood of s (and thus extendable: see [9, section III.4.4]). However, the same argument shows that $\zeta(1-s)/\zeta(s)$ would then not be extendable. Or we can observe that if s is a zero with a higher multiplicity compared to $1-s$, then $F(s) = 0$, but from the explicit expression above we can see that $F(s)$ is never equal to zero in the critical strip.

Observe that the proof in [25] is ultimately derived from certain estimates, and in fact requires $t > t_0$, where t_0 is such that

$$\left|\frac{1}{2} + it_0\right| = 2\pi e^{0.0212411} = 6.418\ldots.$$

Instead, we consider the region below, proving the following proposition.

Proposition 1. *We have* $|F(s)| \neq 1$ *in the regions defined as*

$$\left\{0 < \sigma < \frac{1}{2}\right\} \cap \left\{\sqrt{(1+\sigma)^2 + t^2} < 2\pi\right\}$$

and

$$\left\{\frac{1}{2} < \sigma < 1\right\} \cap \left\{\sqrt{(2-\sigma)^2 + t^2} < 2\pi\right\}.$$

Proof. First, note that $|F(s)| = 1$ if

$$\pi^{\sigma-1/2} = |\pi^{s-1/2}| = \left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right|.$$

From the property (10) it follows that for $\sigma < 1/2$ we have

$$\left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right| \geqslant \left|\frac{1+s}{2}\right|^{\sigma-1/2}.$$

This implies that for $\sigma < 1/2$ and $\sqrt{(1+\sigma)^2 + t^2} < 2\pi$ we can write

$$\left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right| > \pi^{\sigma-1/2},$$

and therefore we can never have $|F(s)| = 1$.

We have noted above that for $\sigma > 1/2$ it suffices to consider symmetry with respect to the critical line. Or, equivalently, let $z = 1 - s$ so that $\operatorname{Re} z < 1/2$; then

$$\left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right| = \left|\frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)}\right| \leqslant \left|\frac{1+z}{2}\right|^{1/2-\mathrm{Re}(z)} = \left|\frac{2-s}{2}\right|^{\sigma-1/2}.$$

This means that for $\sigma > 1/2$ and $\sqrt{(2-\sigma)^2+t^2} < 2\pi$ we have

$$\left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right| < \pi^{\sigma-1/2},$$

and once again the equality $|F(s)| = 1$ is impossible. □

In fact, the above proves that $|F(s)|$ is less than 1 for

$$s \in \{0 < \sigma < 1/2\} \cap \left\{\sqrt{(1+\sigma)^2+t^2} < 2\pi\right\}$$

(since $|F(s)|$ is continuous, it is also sufficient to consider the segment of the real axis lying inside the strip, where ζ is known to be negative and decreasing, see [15, notes to chapter 1]). For $s \in \{0 < \sigma < 1/2\} \cap \{t > t_0\}$ we have the contrary, i. e. $|F(s)|$ is bigger than 1, see [25, Theorem 1].

Note that per se Theorem 5 and Proposition 1 do not hold specifically for ζ: for instance, it isn't hard to use ζ to construct other analytic functions (possibly having zeros not lying on the critical line) which satisfy the respective functional equation and the property of having real values for real arguments. Such functions may be constructed by multiplying ζ with expressions of the form

$$f(z)f(1-z)\overline{f(\bar{z})f(1-\bar{z})}.$$

As mentioned in [25] and [24], the Riemann Hypothesis would follow if we could improve these results and obtain a strict inequality ($|\zeta(\sigma+it)| > |\zeta(1-\sigma+it)|$ for $0 < \sigma < 1/2$ and $t > 2\pi+1$), since the roots of ζ have to be symmetric with respect to the critical line.

Now let us show how Theorem 5 can lead to other formulations of the Riemann Hypothesis.

Suppose that the multiplicity of s_0 is equal to $m = m(s_0)$ (as mentioned above, in this case the multiplicity of $1-s_0$ also equals m). Since $\zeta(s)$ and $\zeta(1-s)$ are analytic at s_0, the following power series expansions hold for s in a neighborhood of s_0:

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{\zeta^{(n)}(s_0)}{n!}(s-s_0)^n, \quad \zeta(1-s) = \sum_{n=0}^{\infty} \frac{(-1)^n\zeta^{(n)}(1-s_0)}{n!}(s-s_0)^n.$$

Therefore,

$$|F(s_0)| = \left| \lim_{s \to s_0} \frac{\zeta(s)}{\zeta(1-s)} \right| = \left| \frac{\zeta^{(m)}(s_0)}{\zeta^{(m)}(1-s_0)} \right|.$$

By (6), $\overline{\zeta(s)} = \zeta(\bar{s})$ for any s, and thus $|\zeta^{(m)}(s)| = |\zeta^{(m)}(\bar{s})|$.

Then we also have

$$|F(s_0)| = \left| \frac{\zeta^{(m)}(s_0)}{\zeta^{(m)}(1-\overline{s_0})} \right|.$$

Thus, we can state the following propositions.

PROPOSITION 2. *The Riemann Hypothesis is true if and only if*

$$\left| \frac{\zeta^{(m)}(s_0)}{\zeta^{(m)}(1-\overline{s_0})} \right| = 1$$

for any zero s_0 *of* ζ.

PROPOSITION 3. *The Riemann Hypothesis is true if and only if*

$$\left| \frac{\zeta^{(m)}(s_0)}{\zeta^{(m)}(1-s_0)} \right| = 1$$

for any zero s_0 *of* ζ.

We can write explicit expansions for these derivatives. For example, we can look at $F(s)$ with ζ represented as in (5). Then

$$F(s) = \frac{\phi(s)}{\phi(1-s)} \cdot \frac{(1-2^s)}{(1-2^{1-s})},$$

where $F(s_0)$, as above, is the analytic continuation of $F(s_0)$ to s_0.

Since the analytic continuation is unique, and since the second fraction is a nonzero constant at the point s_0, then

$$\frac{\phi(s_0)}{\phi(1-s_0)} \doteq \lim_{s \to s_0} \frac{\phi(s)}{\phi(1-s)}$$

must also be a nonzero constant.

Similarly to the above, we can show that

$$\left|\lim_{s\to s_0}\frac{\phi(s)}{\phi(1-s)}\right|=\left|\frac{\phi^{(m)}(s_0)}{\phi^{(m)}(1-s_0)}\right|.$$

Applying some analytic properties of Dirichlet series (see [1, section 11.7]), in the case $\mathrm{Re}(s)>0$ we can write (by differentiating term by term) that

$$\phi^{(m)}(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+m-1}(\log n)^m}{n^s}.$$

Thus, we would like to evaluate the following expression:

$$\lim_{N\to\infty}\left|\frac{\sum_{n=1}^{N}(-1)^{n+m-1}(\log n)^m n^{-s_0}}{\sum_{n=1}^{N}(-1)^{n+m-1}(\log n)^m n^{-1+s_0}}\right|.$$

Note that we can consider the limit of the whole fraction since the denominator doesn't tend to zero as $N\to\infty$.

We can now reformulate the previous proposition in the following way.

Proposition 4. *Assume that*

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s_0}=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1+s_0}=0,$$

where s_0 is a zero of order m. This condition implies that

$$\lim_{N\to\infty}\left|\frac{\sum_{n=1}^{N}(-1)^{n+m-1}(\log n)^m n^{-s_0}}{\sum_{n=1}^{N}(-1)^{n+m-1}(\log n)^m n^{-1+s_0}}\right|=\left|\frac{(1-2^{1-s_0})}{(1-2^{s_0})}\right|$$

if and only if the Riemann Hypothesis is true.

Clearly, a similar argument would hold if we considered $1-\overline{s_0}$ instead.

On the other hand, in order to prove the Riemann Hypothesis it would be sufficient to show that $|F(s_0)|$ or

$$\lim_{N\to\infty}\left|\frac{\sum\limits_{n=1}^{N}(-1)^{n+m-1}(\log n)^m n^{-s_0}}{\sum\limits_{n=1}^{N}(-1)^{n+m-1}(\log n)^m n^{-1+s_0}}\right|$$

equals 0 or ∞ (and doesn't equal a nonzero constant) if s_0 is not on the critical line.

Next, let us investigate the values of $|F(s_0)|$ further by constructing a new function from the approximants H_N.

We will start by observing that

$$H_N(s)=\sum_{n=1}^{N}n^{-s}+\frac{N^{1-s}}{s-1}$$

can be considered as a truncated sum: if we define (see also [22])

$$h_n(s)=\begin{cases}\dfrac{1}{n^s}-\dfrac{n^{1-s}-(n-1)^{1-s}}{1-s} & \text{if}\quad n\geqslant 2,\\[2ex] 1+\dfrac{1}{s-1} & \text{if}\quad n=1,\end{cases}$$

then $H_N(s)$ is equal to $\sum\limits_{n=1}^{N}h_n(s)$, or the N-th partial sum of the series $\sum\limits_{n=1}^{\infty}h_n(s)$. Note that the summation formula given in (8) implies that

$$\zeta(s)=\sum_{n=1}^{N}h_n(s)-\frac{1}{2}N^{-s}+O(N^{-\sigma-1}),$$

and therefore

$$\zeta(s)=\sum_{n=1}^{\infty}h_n(s).$$

If we have $\zeta(s)=0$, then

$$\sum_{n=1}^{N}h_n(s)=\frac{1}{2}N^{-s}+O(N^{-\sigma-1}).$$

Let us suppose that $s_* = \sigma_* + it_*$ lies in the critical strip.

Proposition 5. *Given that* $\zeta(s_*) \neq 0$ (*and thus also* $\zeta(1-s_*) \neq 0$), *we have*

$$\lim_{s\to s_*} \frac{\zeta(s)}{\zeta(1-s)} = \lim_{N\to\infty} \frac{H_N(s_*)}{H_N(1-s_*)}.$$

Proof. Since $\lim\limits_{N\to\infty} H_N(1-s_*) = \zeta(1-s_*) \neq 0$, we have

$$\lim_{N\to\infty} \frac{H_N(s_*)}{H_N(1-s_*)} = \frac{\lim\limits_{N\to\infty} H_N(s_*)}{\lim\limits_{N\to\infty} H_N(1-s_*)} = \frac{\zeta(s_*)}{\zeta(1-s_*)} = F(s_*). \qquad \square$$

For functions $a(s)$ and $b(s)$ that assume positive real values, let us define $f_{a(s)}(N, s)$ as $\lceil a(s)^{-1/(2\sigma)} N^{1-\sigma} \rceil$, where, as usual, $s = \sigma + it$. Defining $f_{b(s)}(N, s)$ similarly, let

$$H^{a,b}(s) \doteq \lim_{N\to\infty} \left| \frac{H_{f_{a(s)}(N,s)}(s)}{H_{f_{b(1-s)}(N,1-s)}(1-s)} \right|.$$

Observe that the proof of the previous proposition didn't make use of the fact that both the numerator and the denominator were N-th partial sums. Using a similar argument, we can prove the following proposition.

Proposition 6. *If* $\zeta(s_*) \neq 0$, *then*

$$|F(s_*)| = \lim_{s\to s_*} \left| \frac{\zeta(s)}{\zeta(1-s)} \right| = \lim_{N\to\infty} \left| \frac{H_{f_{a(s_*)}(N,s_*)}(s_*)}{H_{f_{b(1-s_*)}(N,1-s_*)}(1-s_*)} \right| = H^{a,b}(s_*).$$

Now let us consider $H^{a,b}(s_*)$, where s_* is a zero of ζ. We claim that the following lemma holds.

Lemma 1. *For any zero* s_* *of* ζ, *assuming that* $a(s)$ *and* $b(s)$ *are equal to constants* A *and* B *respectively, we have that* $H^{a,b}(s_*) = |\sqrt{A}/\sqrt{B}|$.

Proof. We can rewrite $\lceil A^{-1/(2\sigma)} N^{1-\sigma} \rceil$ as $A^{-1/(2\sigma)} N^{1-\sigma} + \epsilon(N)$ with $0 \leqslant \epsilon(N) < 1$, and then again as

$$(A^{-1/(2\sigma)} N^{1-\sigma}) \cdot \left(1 + \frac{\epsilon(N)}{A^{-1/(2\sigma)} N^{1-\sigma}}\right).$$

After performing a similar decomposition of $f_B(N, 1-s)$ and recalling the assumption $\zeta(s_*) = 0$ (which implies

$$\sum_{n=1}^{N} h_n(s_*) = \frac{1}{2} N^{-s_*} + O(N^{-\sigma_*-1}),$$

as shown above by using the summation formula), we can write

$$H^{a,b}(s_*) = \\ = \lim_{N\to\infty} \left| \frac{\frac{1}{2}\left(A^{-1/(2\sigma_*)} N^{1-\sigma_*}\right)^{-s_*} \left(1 + \frac{\epsilon_1(N)}{A^{-1/(2\sigma_*)} N^{1-\sigma_*}}\right)^{-s_*} + O\left(f_A(N, s_*)^{-\sigma_*-1}\right)}{\frac{1}{2}\left(B^{-1/(2-2\sigma_*)} N^{\sigma_*}\right)^{-1+s_*} \left(1 + \frac{\epsilon_2(N)}{B^{-1/(2-2\sigma_*)} N^{\sigma_*}}\right)^{-1+s_*} + O\left(f_B(N, 1-s_*)^{-2+\sigma_*}\right)} \right|.$$

From the binomial expansion of $(1+x)^z$, we have

$$\left(1 + \frac{\epsilon_1(N)}{A^{-1/(2\sigma_*)} N^{1-\sigma_*}}\right)^{-s_*} = 1 + O\left(\frac{1}{N^{1-\sigma_*}}\right)$$

and

$$\left(1 + \frac{\epsilon_2(N)}{B^{-1/(2-2\sigma_*)} N^{\sigma_*}}\right)^{-1+s_*} = 1 + O\left(\frac{1}{N^{\sigma_*}}\right),$$

which finally leads to

$$\lim_{N\to\infty} \left| \frac{\frac{1}{2} A^{s_*/(2\sigma_*)} N^{-s_*(1-\sigma_*)} + O\left(\left(N^{1-\sigma_*}\right)^{-\sigma_*-1}\right)}{\frac{1}{2} B^{(1-s_*)/(2-2\sigma_*)} N^{(-1+s_*)\sigma_*} + O\left(\left(N^{\sigma_*}\right)^{-2+\sigma_*}\right)} \right| =$$

$$= \lim_{N\to\infty} \left| \frac{\frac{1}{2} A^{s_*/(2\sigma_*)} N^{-s_*(1-\sigma_*)}}{\frac{1}{2} B^{(1-s_*)/(2-2\sigma_*)} N^{(-1+s_*)\sigma_*}} \right| =$$

$$= \lim_{N\to\infty} \left| \frac{A^{\sigma_*/(2\sigma_*)} N^{-\sigma_*(1-\sigma_*)}}{B^{(1-\sigma_*)/(2-2\sigma_*)} N^{(-1+\sigma_*)\sigma_*}} \right| = \left| \frac{\sqrt{A}}{\sqrt{B}} \right|.$$

□

Consider the case $a(s) = b(s) = |F(s)|$. Recalling the identity

$$|F(s)| = \frac{1}{|F(1-s)|},$$

one can show in a similar manner that

$$H^{|F(s)|,\,|F(s)|}(s) = \left|\frac{\sqrt{|F(s)|}}{\sqrt{|F(1-s)|}}\right| = |F(s)|$$

if s is a zero. If it is not a zero, then the statement is obviously true, meaning that we have constructed a continuous function.

Now showing, for example, that $H^{|F(s_0)|,\,|F(1-s_0)|}(s)$, which is continuous at s_0, is also continuous at $1 - s_0$ would prove that $|F(s_0)| = 1$. Indeed, the continuity and Lemma 1 would imply $|F(1 - s_0)| = |F(s_0)|$, but on the other hand we have

$$|F(s_0)| = \frac{1}{|F(1-s_0)|}.$$

Showing that there exists a constant k such that $H^{|F(s)|,\,|F(s)|}(s) = H^{k,k}(s)$ for all s in the strip would also imply $|F(s_0)| = 1$. It is clear that this condition holds locally: a point $s \neq s_*$ lying in a sufficiently small neighborhood of s_* will not be a zero either by continuity (if s_* is not a zero) or by the property (4) (otherwise). Thus, we find ourselves in the same situation as in the above Proposition 5 and Proposition 6, and, essentially, we are concerned whether H-subscript N or $f(N, s)$ tends to infinity as $N \to \infty$.

In other words, in this case Theorem 5 can be reformulated as the following proposition.

Proposition 7. *The Riemann Hypothesis is true if and only if $H^{1,1}(s)$ is continuous throughout the strip.*

Note that we can also state the following proposition, which is, though, a weaker result since the continuity appears to be much harder to prove.

Proposition 8. *The Riemann Hypothesis is true if and only if*

$$\lim_{N\to\infty}\left|\frac{H_N(s)}{H_N(1-s)}\right|$$

is continuous throughout the strip.

Proof. If the Riemann Hypothesis is true, then every zero lies on the critical line, where we have

$$\lim_{N\to\infty}\left|\frac{H_N(s)}{H_N(\bar{s})}\right| = 1,$$

since for each N the value $H_N(s)$ is the conjugate of $H_N(\bar{s})$, and therefore has the same modulus.

Going in the other direction, let us show that if the Riemann Hypothesis were not true, then the limit would not be continuous. Assuming that there is a zero lying off the critical line, we can use an argument similar to the proof of Lemma 1 to show that the limit is now equal 0 or ∞, contradicting the fact that $F(s)$ is always a nonzero number. ☐

Acknowledgements. I would like to thank everybody who has helped me in some way, even if I do not mention them personally, from the colleagues who engaged me in fruitful discussions to the people who read the manuscript in part or in its entirety, the people who have introduced me to the subject, and everyone who has ever encouraged or supported me. Among others, I would like to thank Joachim Rosenthal, Alessandro Cobbe, Jens Zumbrägel, Felix Fontein, Gérard Maze, Alessio Martini, Emanuele Spadaro, Ivan Contreras, Elisa Gorla, Patrick Solé, Ashkan Nikeghbali, Christopher Hughes, Michele Elia, Pietro Peterlongo, and Kim Dang.

Research supported by Swiss National Science Foundation Grant no. 107887.

Support has also been given over time by Scuola Normale Superiore of Pisa, University of Zürich and IBM Zürich Research Lab.

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