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On the number of common values of arithmetic functions ϕ and σ below x

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Abstract: Let ϕ be Euler's totient function, σ be the sum-of-divisors function. We effectively show that for any $A > 0$ and large $x > x_0(A)$ there are at least $\exp((\log\log x)^A)$ integers $n \leqslant x$ which are common values of ϕ and σ. This improves a recent result of Ford, Luca and Pomerance, where the existence of such $A > 0$ was established.

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1. Introduction

Let ϕ be Euler's function, σ be the sum-of-divisors function. Ford, Luca and Pomerance [3] proved (effectively) the following statement, which solves an old conjecture of Erdős.

THEOREM. ([3]). *The equation $\phi(a) = \sigma(b)$ has infinitely many solutions. Moreover, for some positive a and large x, there are at least $\exp((\log\log x)^a)$ integers $n \leqslant x$ which are common values of ϕ and σ.*

They noted that the proof relies on a recent work of Ford, Konyagin and Luca on prime chains (see Lemma 5 below) and a result of Heath-Brown connecting the possible existence of Siegel zeros with the distribution of twin primes. In a remark (see [3, p. 487]) they also described an alternative approach of Konyagin, which avoids using Heath-Brown's result. It is mentioned there that Konyagin's approach would give a somewhat weaker conclusion about the number of common values below x. In the present paper we effectively prove the following improvement of a result from [3].

THEOREM 1. *For any* $A > 0$ *and large* $x > x_0(A)$ *there are at least* $\exp\big((\log\log x)^A\big)$ *integers* $n \leqslant x$ *which are common values of* ϕ *and* σ.

The proof of Theorem 1 is based on a refined version of Konyagin's approach and the argument used in [3]. We note that instead of Lemma 5 one can use a precise formulation given in [2, Theorem 1]. This would allow to substitute A by some function $A(x) \to \infty$ as $x \to \infty$. However the function $A(x)$ obtained in this way would grow extremely slowly. That is the reason why we decided not to pursue this issue.

We begin to collect common values of ϕ and σ in a way as it has been described in [3, p. 2]. These values are found among numbers of the form $n = \sigma\Big(\prod_p p\Big) = \prod_p (p+1)$, where p belongs to some set of primes $p \leqslant x$ for which all prime factors of $p+1$ are smaller than $x^{1/2-\delta}$, where $\delta > 0$ is a small numerical constant. Then to deduce that n is a value of ϕ one exploits the implication

$$\phi(\mathrm{rad}(m))|m \Rightarrow m = \phi\left(\frac{m\,\mathrm{rad}(m)}{\phi(\mathrm{rad}(m))}\right),$$

where $\mathrm{rad}(m)$ is the product of the distinct prime factors of m.

In what follows p, q denote prime numbers, $P(n)$ denotes the largest prime factor of n.

2. Lemmas

As usual, below $L(s,\chi)$, $s = \sigma + it$, is the Dirichlet L-function.

LEMMA 1. *For some constant* $c_0 > 0$, *if* χ_1, χ_2 *are two real primitive characters to the moduli* m_1, m_2 *and if* $L(s,\chi_1)$ *and* $L(s,\chi_2)$ *have real zeros* $\beta_1 \neq \beta_2$, *then*

$$\min\{\beta_1, \beta_2\} < 1 - \frac{c_0}{\log(m_1 m_2)}.$$

Lemma 1 follows from Landau's result and the general theory of L-functions. The following lemma is a refined version of Konyagin's approach described in [3].

Lemma 2. *There are absolute positive constants* c *and* C *such that the following holds: for any large* X *and any* $\alpha > C(\log\log X)^2(\log X)^{-1/2}$, *there exists* $x \in [(\log X)^{1/\alpha}, X]$ *such that for any* $3 \leqslant m \leqslant x^{\alpha}$ *and any primitive character* χ *modulo* m,

$$L(s,\chi) \neq 0 \quad \textit{in the region} \quad \operatorname{Re} s > 1 - \frac{c}{\log x^{\alpha}\big(|t|+1\big)}.$$

Proof. The statement is well-known if χ is complex or if χ is real and $t \neq 0$. Thus, it suffices to deal with real characters and real s.

Let $c = 0.1c_0$, where c_0 is the constant in the statement of Lemma 1. Assuming contrary, we get a real primitive character χ_1 modulo $m_1 \leqslant \log X$ such that there is a real zero β_1 of $L(s,\chi_1)$ with

$$\beta_1 > 1 - \frac{0.1c_0}{\log\log X}. \tag{1}$$

Recall that (see, [1, p. 95]) for some absolute constant $c_1 > 0$,

$$\beta_1 < 1 - \frac{c_1}{m_1^{1/2}\log^2 m_1} \leqslant 1 - \frac{c_1}{(\log X)^{1/2}(\log\log X)^2}.$$

Therefore, together with (1), we have, for $C = c_0/c_1$,

$$(\log X)^{1/\alpha} < \exp\left(\frac{0.1c_0}{\alpha(1-\beta_1)}\right) < X.$$

Recall that we conjectured Lemma 2 not to be true. Thus, there is an integer m_2 from the interval $3 \leqslant m_2 \leqslant \exp(0.1c_0/(1-\beta_1))$ such that for some real primitive character χ_2 modulo m_2 there exists a real zero β_2 of $L(s,\chi_2)$ with

$$\beta_2 > 1 - \frac{0.1c_0}{\log\Big(\exp\big(\frac{0.1c_0}{(1-\beta_1)}\big)\Big)} = \beta_1.$$

According to Lemma 1,

$$\frac{c_0}{1-\beta_1} < \log m_1 + \log m_2 \leqslant \log\log X + \frac{0.1c_0}{1-\beta_1},$$

whence

$$\log\log X > \frac{0.9c_0}{1-\beta_1}.$$

This contradicts (1) and proves Lemma 2. □

LEMMA 3. *There exist absolute positive constants* $\varepsilon_0 < 0.01$ *and* C *such that the following holds: for any large* X *and any* $\alpha \in \left[C(\log\log X)^2(\log X)^{-1/2}, \varepsilon_0\right]$ *there exists* $x \in \left[(\log X)^{1/\alpha}, X\right]$ *such that*

$$\sum_{3\leqslant m\leqslant x^\alpha}\ \sum_{\chi\in\mathcal{C}(m)}\left|\sum_{n\leqslant x}\Lambda(n)\chi(n)\right| \leqslant 0.1x,$$

where $\mathcal{C}(m)$ *is the set of primitive characters modulo* m.

PROOF. Define $x \in [(\log X)^{1/\alpha}, X]$ as in Lemma 2. Note that $x^\alpha \geqslant \log X \geqslant \log x$.

We follow Gallagher's paper [4]. For a character $\chi \in \mathcal{C}(m)$ and $2 \leqslant T \leqslant x^{0.9}$ we use the representation (see, [1, p. 117])

$$\sum_{x^{0.9}<n\leqslant x}\Lambda(n)\chi(n) = -\sum_\rho \frac{x^\rho - x^{0.9\rho}}{\rho} + O\left(\frac{x\log^2 x}{T}\right).$$

Here ρ are zeros of $L(s,\chi)$ in the region $0 < \operatorname{Re} s < 1$, $|\operatorname{Im} s| \leqslant T$. Choose $T = x^{7\alpha}$. Then, if $\alpha \in \left[C(\log\log X)^2(\log X)^{-1/2}, 0.01\right]$ and X is large,

$$\left|\sum_{n\leqslant x}\Lambda(n)\chi(n)\right| < x^{1-4\alpha} + \sum_{|\operatorname{Im}\rho|\leqslant T}\left|\int_{x^{0.9}}^{x} y^{\rho-1}\,dy\right| \leqslant x^{1-4\alpha} + \int_{x^{0.9}}^{x}\sum_{|\operatorname{Im}\rho|\leqslant T} y^{\beta(\rho)-1}\,dy.$$

Here $\beta(\rho) = \operatorname{Re}\rho$. Hence

$$\sum_{3\leqslant m\leqslant x^\alpha}\ \sum_{\chi\in\mathcal{C}(m)}\left|\sum_{n\leqslant x}\Lambda(n)\chi(n)\right| < x^{1-2\alpha} + \int_{x^{0.9}}^{x}\sum_{3\leqslant m\leqslant x^\alpha}\ \sum_{\chi\in\mathcal{C}(m)}\ \sum_{|\operatorname{Im}\rho|\leqslant T} y^{\beta(\rho)-1}\,dy.$$

Since $|\operatorname{Im}\rho| \leqslant x^{10\alpha}$ and $m \leqslant x^\alpha$, by the definition of x (see Lemma 2) we see that, for some absolute constant c,

$$\beta(\rho) < 1 - \frac{c}{\log x^\alpha}.$$

Therefore, since $x^{0.9} \leqslant y \leqslant x$ and the number of zeros ρ with $|\operatorname{Im}\rho| \leqslant T$ is of order $T\log(mT) = o(x^{8\alpha})$, we obtain

$$\sum_{|\operatorname{Im}\rho|\leqslant T} y^{\beta(\rho)-1} \leqslant \log y \sum_{|\operatorname{Im}\rho|\leqslant T} \int_0^{\beta(\rho)} y^{u-1}\,du + \frac{x^{8\alpha}}{y} \leqslant$$

$$\leqslant \log x \int_0^{1-c/\log x^{\alpha}} y^{u-1}\Big\{\sum_{\substack{|\operatorname{Im}\rho|\leqslant T\\ \beta(\rho)\geqslant u}} 1\Big\}\,du + x^{-0.8} =$$

$$= \log x \int_0^{1-c/\log x^{\alpha}} y^{u-1} N_{\chi}(u,T)\,du + x^{-0.8},$$

where $N_{\chi}(u,T)$ is the number of zeros of $L(s,\chi)$ in the rectangle $u \leqslant \sigma \leqslant 1, |t| \leqslant T$. From Gallagher's density estimate [4], for some absolute constant $C_0 > 0$ we get

$$\sum_{m\leqslant T}\sum_{\chi\in\mathcal{C}(m)} N_{\chi}(u,T) \leqslant C_0 T^{C_0(1-u)}.$$

Thus, if ε_0 is small enough, then for $\alpha \in [C(\log\log X)^2(\log X)^{-1/2},\ \varepsilon_0]$ and large X we have

$$\sum_{3\leqslant m\leqslant x^{\alpha}}\sum_{\chi\in\mathcal{C}(m)}\Big|\sum_{n\leqslant x}\Lambda(n)\chi(n)\Big| < 2x^{1-2\alpha} + C_0\log x\int_{x^{0.9}}^{x}\int_0^{1-c/\log x^{\alpha}} y^{u-1}T^{C_0(1-u)}\,du\,dy <$$

$$< 2x^{1-2\alpha} + C_0 x\log x\int_0^{1-c/\log x^{\alpha}} x^{-0.5(1-u)}\,du \leqslant$$

$$\leqslant 3x^{1-2\alpha} + 2C_0 x e^{-c/2\alpha} < 0.1x. \qquad \square$$

For positive reals δ, γ, y, x with $1 \leqslant y \leqslant x^{1/2-\delta}$ and a nonzero integer a, the following notation has been introduced in [3]:

$$S_q(x;\delta,a) = \#\{p \leqslant x : P(p+a) \leqslant x^{1/2-\delta},\ q|p+a\},$$

$$\mathcal{E}(x,y;\delta,\gamma) = \left\{q \leqslant y : S_q(x;\delta,1) \leqslant \frac{\gamma x}{q\log x} \quad \text{or} \quad S_q(x;\delta,-1) \leqslant \frac{\gamma x}{q\log x}\right\}.$$

LEMMA 4. *There are absolute constants* $0<\delta<0.1$, $\gamma>0$, $\varepsilon_1>0$ *and* $C>0$ *such that the following holds: for any large* X *and any* $\alpha \in [C(\log\log X)^2(\log X)^{-1/2}, \varepsilon_1]$ *there exists* $x \in [(\log X)^{1/\alpha}, X]$ *such that for all* $y \leqslant x^{1/2-\delta}$,

$$\#\mathcal{E}(x,y;\delta,\gamma) \leqslant yx^{-0.5\alpha}.$$

PROOF. For small positive ε_0 and $C(\log\log X)^2(\log X)^{-1/2} < \alpha < \varepsilon_0$ we choose x to satisfy the conclusion of Lemma 3. Thus,

$$\Psi(x,m) := \sum_{\chi\in\mathcal{C}(m)} \left|\sum_{n\leqslant x}\Lambda(n)\chi(n)\right| \leqslant 0.1x \quad \text{for any} \quad m \leqslant x^{\alpha}. \tag{2}$$

The result follows from the proof of [3, Lemma 2.6]. □

A prime chain is defined as a sequence of primes $q=t_0,t_1,t_2,\ldots$, where for every j one has $t_{j+1}\equiv 1 \pmod{t_j}$. Let $\mathcal{T}(y,q)$ be the set of primes $t\leqslant y$ which are in a prime chain starting with q. The following statement is due to Ford, Konyagin and Luca [2].

LEMMA 5. *For every* $\varepsilon>0$ *there is a constant* $C(\varepsilon)$ *such that if* $y>q$ *then*

$$\#\mathcal{T}(y,q) \leqslant C(\varepsilon)\left(\frac{y}{q}\right)^{1+\varepsilon}.$$

3. Proof of Theorem 1

Let $\delta,\gamma,\varepsilon_1$ be positive absolute constants defined in Lemma 4. For small $0<\alpha<\varepsilon_1$ and large $X>X_0(\alpha)>0$ we choose $x\in[(\log X)^{1/\alpha},X]$ according to the conclusion of Lemma 4. Let $\mathcal{E}=\mathcal{E}(x,x^{1/2-\delta};\delta,\gamma)$ and let

$$\mathcal{T} = \bigcup_{q\in\mathcal{E}} \mathcal{T}(x^{1/2-\delta},q).$$

Denote

$$S = \{p\leqslant x : P(p+1)\leqslant x^{1/2-\delta} \quad \text{and} \quad t\nmid p+1 \quad \text{for all} \quad t\in\mathcal{T}\}.$$

By partial summation, Lemmas 4, 5 we see that if $X_0(\alpha)$ is sufficiently large then

$$\sum_{t\in\mathcal{T}} \frac{1}{t} < \frac{\gamma}{20\log x} \tag{3}$$

and $\#S > \gamma x/(3\log x)$ (see [3] for the details).

From known results (see [5]) it follows that there is an absolute positive constant $\theta < 0.1$ such that

$$\{p \leqslant x : P(p+1) < x^{\theta}\} < \frac{\gamma x}{10\log x}.$$

Hence

$$\#\{p \in S : P(p+1) > x^{\theta}\} > \frac{\gamma x}{6\log x}.$$

We divide the prime numbers $p \in S$ with $P(p+1) > x^{\theta}$ into classes such that each class consists of primes p with equal values of $P(p+1)$. Since each class contains at most $x^{1-\theta}$ numbers, there are at least $x^{0.5\theta}$ disjoint classes. Thus, picking up from each class at most one prime, we obtain a subset $S_0 \subset S$ consisting of $L = [x^{0.5\theta}]$ primes p with pairwise distinct $P(p+1)$. In particular, we can form 2^L pairwise distinct numbers

$$\{k_1, k_2, \ldots, k_{2^L}\} = \Big\{\prod_{p\in S_0'}(p+1) : \; S_0' \subset S_0\Big\}.$$

Let $n_j = \Big(\prod_{p\in S}(p+1)\Big)k_j^{-1}$. Since $\sigma(p) = p+1$, all n_j are values of σ. We will show that all n_j are also values of ϕ. We recall that for this purpose it is sufficient to show that $\phi(\mathrm{rad}(n_j))|n_j$.

The prime factors of n_j are $\leqslant x^{1/2-\delta}$. Repeating the argument from [3, page 484], we see that for $q \leqslant x^{1/2-\delta}$, if $q \in \mathcal{T}$ then $q \nmid \phi(\mathrm{rad}(n_j))$, and if $q \notin \mathcal{T}$ then

$$v_q\big(\phi(\mathrm{rad}(n_j))\big) \leqslant \frac{x^{1/2-\delta}}{q-1},$$

where $v_q(m)$ denotes the exponent of q in the factorization of m. On the other hand for such q Lemma 4 and the inequality (3) imply

$$\begin{aligned}v_q(n_j) &\geqslant \#\{p \in S \setminus S_0 : q|p+1\} \geqslant \#\{p \in S : q|p+1\} - x^{0.5\theta} \geqslant \\ &\geqslant \#\{p \leqslant x : P(p+1) \leqslant x^{1/2-\delta}, q|p+1\} - 2\sum_{t\in\mathcal{T}} \frac{x}{qt} - x^{0.5\theta} \geqslant \\ &\geqslant \frac{\gamma x}{q\log x} - \frac{\gamma x}{9q\log x} \geqslant \frac{\gamma x}{2q\log x} > v_q\big(\phi(\mathrm{rad}(n_j))\big).\end{aligned}$$

Thus, all $2^{[x^{0.5\theta}]}$ numbers n_j are common values of σ and ϕ, and moreover $n_j \leqslant e^{2x}$. Hence, since $x \in \big[(\log X)^{1/\alpha}, X\big]$, there are at least $\exp\big\{(\log X)^{0.1\theta/\alpha}\big\}$ common values of σ and ϕ below e^{2X}. Since $\theta > 0$ is an absolute constant the result follows by taking α to be sufficiently small.

Bibliography

1. ***H. Davenport,*** *Multiplicative number theory*, Third edition, revised by H. L. Montgomery. Graduate Texts in Mathematics, 74. Springer-Verlag, New York, 2000.
2. ***K. Ford, S. Konyagin, F. Luca,*** *Prime chains and Pratt trees*, Geom. Funct. Anal. **20**, № 5 (2010), 1231–1258.
3. ***K. Ford, F. Luca, C. Pomerance,*** *Common values of the arithmetic functions ϕ and σ*, Bull. London Math. Soc. **42**, № 3 (2010), 478–488.
4. ***P. X. Gallagher,*** *A large sieve density estimate near $\sigma = 1$*, Invent. Math. **11** (1970), 329–339.
5. ***C. Pomerance, I. E. Shparlinski,*** *Smooth orders and cryptographic applications*, Lecture Notes in Computer Science **2369** (2002), 359–373.

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