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An estimate for the number of edges between vertices of given degrees in random graphs in the Bollobás—Riordan model

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Abstract: We study the Bollobás—Riordan model for a random graph. We investigate the total number of edges connecting a node of degree d_1 and a node of degree d_2. Here d_1 and d_2 are fixed positive integers. We prove that this number is close to its expectation with high probability. We calculate the expectation with an error term $O_{d_1,d_2}(1/t)$, where t is the number of nodes in a graph. One part of our proofs includes a significant improvement of the known estimate for the expectation of the number of nodes of degree d. We calculate this expectation with an error term $O(d/t)$ without any restrictions on d.

Keywords: random graphs, scale-free graphs, web graphs, Bollobás—Riordan model, degree distribution, power law

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1. Introduction

There are many interesting structures in the real world, which can be thought of as graphs. A typical example is the World Wide Web: one can consider web pages to be the nodes of the graph and hyperlinks to be the edges. One of efficient methods for studying these graphs involves investigating a suitable random graph model.

First models of random graphs were constructed and studied long ago. Classical models and results are systematized, for example, in [3] and [9]. However, they are not suitable for approximating dynamically changing and non-uniform networks. In particular, such characteristics as the degree sequence differ significantly from those observed in real networks.

Recently other models of random graphs were constructed to match more closely the growth of real networks. The first one is due to Barabási and Albert [2]. However, they did not construct a precise model, leaving some parameters unspecified. Variation of those parameters can change the properties of arising graphs significantly, as it was shown in [4], so one needs something concrete for theoretical investigation. Bollobás et al. proposed a concrete model in [6] based on the same ideas. In the same paper they gave a rigorous proof of a theorem concerning the degree sequence in this model.

In the current paper we study the Bollobás—Riordan model proposed in [6]. A survey of other models and their properties can be found, for example, in the paper [4], and also in the book [7].

We now define the model of a random graph with t nodes and kt edges, $k, t \in \mathbb{N}$. The model is constructed in two stages. The first stage defines a probabilistic space G_n. The second stage constructs a probabilistic space $\widehat{G}_{k,kt} = \widehat{G}_{kt}$ basing on G_{kt}. Hereafter we consider k as a fixed parameter of the model. An element of all probabilistic spaces is an undirected graph with numbered nodes.

- The probabilistic space G_1 contains only one graph with one node and one loop.
- Given G_n, let us construct G_{n+1}. Let G be a graph from G_n. We add a new node with the number $n+1$ to G. We also add a new edge incident to this node. The other end of this edge can be any vertex $a \in G$ or the new node (this creates a new loop). The probability of the first case is $\dfrac{d}{2n+1}$, where d is the degree of a. The probability of the loop is $\dfrac{1}{2n+1}$. Since G has exactly n nodes, the sum of the degrees of all the nodes in G is $2n$, so the definition is correct.
- Let $G \in G_{kt}$. We construct the graph $\widehat{G} \in \widehat{G}_{kt}$ as follows. We take as nodes of $\widehat{G}$ the following collections of nodes of G: $\{1, \dots, k\}$, $\{k+1, \dots, 2k\}$, $\dots$, $\{kt-k+1, \dots, kt\}$. We shall call such a collection a conglomerate. An edge ij of G defines an edge in $\widehat{G}$ between the two conglomerates containing i and j.

The probability of $\widehat{G} \in \widehat{G}_{kt}$ is the sum of the probabilities of all the graphs $G \in G_{kt}$ corresponding to $\widehat{G}$.

Many properties of the Bollobás—Riordan model were investigated. The paper [5] studies the diameter of a random graph. The paper [4] studies subgraphs of a random graph. One of the most important properties is the degree distribution for a random graph. The main paper [6] proves the power law for the degree distribution. This result matches well with the empirical observations of Barabási and Albert (see [2]). However, it was proved under very strong restrictions on parameters. In this paper we study a far more complicated property concerning the degrees. As a part of our proofs, we also deduce a significant improvement of the theorem by Bollobás et al.

We formulate the old and new results in Section 2. We prove new theorems in Section 3.

2. Main results

Let us introduce some notation. For a condition A, we define $[A]$ as the indicator of A: $[A] = 1$ if A holds, $[A] = 0$ otherwise. Let d_1, d_2 be arbitrary positive integers. Let G be a random graph from $\widehat{G}_{kt}$. We define a random quantity

$$X = X(G) = \Big|\big\{(i, j) : (i, j) \text{ is an edge of } G, \deg i = d_1, \deg j = d_2, i \neq j\big\}\Big|.$$

The following two theorems are the main results of this paper.

Theorem 1. *If $d_1 < k$, $d_2 < k$ or $d_1 = d_2 = k$, then $X = 0$. If $d_1 \geqslant k$, $d_2 \geqslant k$ and $d_1 + d_2 \geqslant 2k + 1$, then the expected value of X is*

$$EX = \frac{k(k+1)}{d_1(d_1+1)d_2(d_2+1)}\left(1 - \frac{C_{2k+2}^{k+1} C_{d_1+d_2-2k}^{d_1-k}}{C_{d_1+d_2+2}^{d_1+1}}\right)(2kt+1) -$$

$$- \sum_{n=1}^{k} \frac{C_{d_1+d_2-2n}^{d_1-n}}{d_1 d_2 C_{d_1+d_2}^{d_1}}\left(\frac{(2n)!}{n!(n+1)!}\frac{k+1}{2k} + [n = k]\frac{(2k)!}{2(k-1)!^2}\right) -$$

$$- [d_1 = k]\frac{(k-1)(k+1)}{2kd_2(d_2+1)} - [d_2 = k]\frac{(k-1)(k+1)}{2kd_1(d_1+1)} + O_{k,d_1,d_2}\left(\frac{1}{t}\right).$$

THEOREM 2. *Let $c > 0$. Then*

$$P\left(|X - EX| \geqslant c(d_1 + d_2)\sqrt{kt}\right) \leqslant 2\exp\left(-\frac{c^2}{8}\right).$$

In particular, if $c(t) \to \infty$ as $t \to \infty$, then $|X - EX| < c(t)(d_1 + d_2)\sqrt{kt}$ with the probability tending to 1 as $t \to \infty$.

While proving Theorem 1 we deduce the following simpler statement.

THEOREM 3. *The expected value of the number of nodes of degree d in a random graph from $\widehat{G}_{kt}$ is*

$$[d \geqslant k]\frac{(2kt+1)(k+1)}{d(d+1)(d+2)} - \frac{[d=k]}{k} + O_k\left(\frac{d}{t}\right).$$

Theorem 3 improves the result of Bollobás et al for the expected value. The main result of [6] asserts that $X \sim EX$ and $EX \sim \dfrac{2k(k+1)t}{d(d+1)(d+2)}$ when $d \leqslant t^{1/15}$ and $t \to \infty$. Our result is valid for all values of d and includes the explicit error term.

3. Proofs

This section consists of several subsections. We write recurrent equations for the quantity EX and related functions f, g, h, r in Subsection 3.1. The equation for EX involves a function f, the equation for f involves a function g, the equation for g involves a function h, the equation for h involves a function r. The function r is related to the degree distribution.

We estimate r in Subsection 3.2. Theorem 3 follows as a special case. Using this result, we estimate h in Subsection 3.3, then we estimate g in Subsection 3.4. Next, we estimate f in Subsection 3.5. Theorem 1 follows from an estimate for f.

Finally, we prove Theorem 2 in the last Subsection 3.6.

3.1. Recurrent equations

We use Greek letters to denote nodes of a graph in $\widehat{G}_{kt}$, which are conglomerates of k nodes of a graph in G_{kt}. We define the degree of a conglomerate $\{v+1, v+2, \dots, v+k\}$ in G_{kt} as the sum of the degrees of $v+1, \dots, v+k$. The quantity X can be calculated without references to $\widehat{G}_{kt}$, using only conglomerates in G_{kt}. The expectation over $\widehat{G}_{kt}$ is the same as the expectation over G_{kt}.

Let us introduce some notation. We denote the expectation over G_n by E_n. If $G \in G_n$ and $n' < n$, there is exactly one graph $G^{(n')} \in G_{n'}$ preceding G in the construction of G_n. The degree of a conglomerate α in $G^{(n')}$ is denoted by $\deg_{n'} \alpha$. We denote the number of edges connecting α and β by $N(\alpha, \beta)$. The same quantity for $G^{(n')}$ is denoted by $N_{n'}(\alpha, \beta)$.

From now on let m be an integer such that $0 \leqslant m \leqslant k$.

Let $f(d_1, d_2, t, m)$ be the expected value of the number of edges in a random graph $G \in G_{kt+m}$ connecting a conglomerate of degree d_1 and a conglomerate of degree d_2. Only conglomerates with the number from 1 to t are taken into account. Obviously, $EX = f(d_1, d_2, t, 0)$. Moreover,

$$f(d_1, d_2, t, m+1) = \sum_{\alpha=1}^{t} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{t} E_{kt+m+1}\big([\deg \alpha = d_1, \deg \beta = d_2] N(\alpha, \beta)\big).$$

Let $G \in G_{kt+m+1}$ be fixed. Let v be the node connected to the last node of G. Then

$$P(v \in \alpha) = \frac{\deg_{kt+m} \alpha}{2kt + 2m + 1}, \quad P(v \in \beta) = \frac{\deg_{kt+m} \beta}{2kt + 2m + 1},$$
$$P(v \notin \alpha, v \notin \beta) = 1 - \frac{\deg_{kt+m} \alpha + \deg_{kt+m} \beta}{2kt + 2m + 1}.$$

Note that $N(\alpha, \beta) = N_{kt+m}(\alpha, \beta)$. The following relations hold:

$$\begin{aligned}
&[\deg \alpha = d_1,\ \deg \beta = d_2] = [\deg \alpha = d_1,\ \deg \beta = d_2,\ v \in \alpha] + \\
&+ [\deg \alpha = d_1,\ \deg \beta = d_2,\ v \in \beta] + [\deg \alpha = d_1,\ \deg \beta = d_2,\ v \notin \alpha,\ v \notin \beta] = \\
&= [\deg_{kt+m} \alpha = d_1 - 1,\ \deg_{kt+m} \beta = d_2,\ v \in \alpha] + \\
&+ [\deg_{kt+m} \alpha = d_1,\ \deg_{kt+m} \beta = d_2 - 1,\ v \in \beta] + \\
&+ [\deg_{kt+m} \alpha = d_1,\ \deg_{kt+m} \beta = d_2,\ v \notin \alpha,\ v \notin \beta].
\end{aligned}$$

Taking the expectation over $G \in G_{kt+m+1}$ is equivalent to taking the expectation over v and then over $G^{(kt+m)}$. Therefore,

$$\begin{aligned}
f(d_1, d_2, t, m+1) = f(d_1 - 1, d_2, t, m) \frac{d_1 - 1}{2kt + 2m + 1} + \\
+ f(d_1, d_2 - 1, t, m) \frac{d_2 - 1}{2kt + 2m + 1} + f(d_1, d_2, t, m) \left(1 - \frac{d_1 + d_2}{2kt + 2m + 1}\right). \quad (1)
\end{aligned}$$

The values $f(d_1, d_2, t, k)$ and $f(d_1, d_2, t+1, 0)$ are different from the point of view of the conglomerate with the number $t+1$ which is taken into account only in the second case. Let $\gamma = t+1$ and

$$g(d_1, d_2, t, m) = E_{kt+m}\left([\deg\gamma = d_1] \sum_{\beta=1}^{t} [\deg\beta = d_2] N(\gamma, \beta) \right).$$

Then

$$f(d_1, d_2, t+1, 0) = f(d_1, d_2, t, k) + g(d_1, d_2, t, k) + g(d_2, d_1, t, k). \qquad (2)$$

The equation (1) defines the values of f at all natural m by its value at $m = 0$. The equation (2) defines the values of f at all natural t by its value at $t = 1$. Any graph from G_k contains only one conglomerate. All the edges in any graph from G_k connect the conglomerate with itself. Thus the only graph in $\widehat{G}_k$ has one node and k loops. Therefore,

$$f(d_1, d_2, 1, 0) = 0. \qquad (3)$$

Let us study the quantity

$$g(d_1, d_2, t, m+1) = \sum_{\beta=1}^{t} E_{kt+m+1}\left([\deg\gamma = d_1,\ \deg\beta = d_2] N(\gamma, \beta) \right).$$

Note that the last node of the graph G is in the conglomerate γ and

$$P(v \in \gamma) = \frac{\deg_{kt+m}\gamma + 1}{2kt + 2m + 1}.$$

The following recurrent relation holds:

$$\begin{aligned}
&[\deg\gamma = d_1,\ \deg\beta = d_2] N(\gamma, \beta) = [\deg\gamma = d_1,\ \deg\beta = d_2,\ v \in \gamma] N(\gamma, \beta) + \\
&+ [\deg\gamma = d_1,\ \deg\beta = d_2,\ v \in \beta] N(\gamma, \beta) + \\
&+ [\deg\gamma = d_1,\ \deg\beta = d_2,\ v \notin \beta,\ v \notin \gamma] N(\gamma, \beta) = \\
&= [\deg_{kt+m}\gamma = d_1 - 2,\ \deg_{kt+m}\beta = d_2,\ v \in \gamma] N_{kt+m}(\gamma, \beta) + \\
&+ [\deg_{kt+m}\gamma = d_1 - 1,\ \deg_{kt+m}\beta = d_2 - 1,\ v \in \beta](N_{kt+m}(\gamma, \beta) + 1) + \\
&+ [\deg_{kt+m}\gamma = d_1 - 1,\ \deg_{kt+m}\beta = d_2,\ v \notin \gamma,\ v \notin \beta] N_{kt+m}(\gamma, \beta).
\end{aligned}$$

Let

$$h(d_1, d_2, t, m) = \sum_{\beta=1}^{t} P_{kt+m}(\deg\gamma = d_1,\ \deg\beta = d_2).$$

Same as for f, we have

$$\begin{aligned} g(d_1, d_2, t, m+1) &= h(d_1-1, d_2-1, t, m)\frac{d_2-1}{2kt+2m+1} + \\ &+ g(d_1-2, d_2, t, m)\frac{d_1-1}{2kt+2m+1} + g(d_1-1, d_2-1, t, m)\frac{d_2-1}{2kt+2m+1} + \\ &+ g(d_1-1, d_2, t, m)\left(1 - \frac{d_1+d_2}{2kt+2m+1}\right). \end{aligned} \tag{4}$$

If $m = 0$, the conglomerate γ has no nodes and $N(\gamma, \beta) = 0$. Therefore,

$$g(d_1, d_2, t, 0) = 0. \tag{5}$$

The definition of h is similar to that of g, except for the absent factor $N(\gamma, \beta)$. The calculation for h is the same as for g, except for the corresponding absent term:

$$\begin{aligned} h(d_1, d_2, t, m+1) &= h(d_1-2, d_2, t, m)\frac{d_1-1}{2kt+2m+1} + \\ &+ h(d_1-1, d_2-1, t, m)\frac{d_2-1}{2kt+2m+1} + \\ &+ h(d_1-1, d_2, t, m)\left(1 - \frac{d_1+d_2}{2kt+2m+1}\right). \end{aligned} \tag{6}$$

Let

$$r(d, t, m) = \sum_{\beta=1}^{t} P_{kt+m}(\deg\beta = d).$$

Note that $r(d, t, 0)$ is the expected value of the number of nodes of degree d in a graph from $\widehat{G}_{kt}$. Obviously,

$$h(d_1, d_2, t, 0) = [d_1 = 0]\, r(d_2, t, 0). \tag{7}$$

Now,

$$
\begin{aligned}
&P_{kt+m+1}(\deg\beta=d)=P_{kt+m+1}(\deg\beta=d,\ v\in\beta)+P_{kt+m+1}(\deg\beta=d,\ v\notin\beta)=\\
&=P_{kt+m+1}(\deg_{kt+m}\beta=d-1,\ v\in\beta)+P_{kt+m+1}(\deg_{kt+m}\beta=d,\ v\notin\beta)=\\
&=P_{kt+m}(\deg_{kt+m}\beta=d-1)\frac{d-1}{2kt+2m+1}+\\
&+P_{kt+m}(\deg_{kt+m}\beta=d)\left(1-\frac{d}{2kt+2m+1}\right),\\
&r(d,t,m+1)=r(d-1,t,m)\frac{d-1}{2kt+2m+1}+r(d,t,m)\left(1-\frac{d}{2kt+2m+1}\right). \qquad (8)
\end{aligned}
$$

Additionally,

$$r(d, t+1, 0) = r(d, t, k) + P_{kt+k}(\deg\gamma = d). \qquad (9)$$

The only graph from $\widehat{G}_k$ has exactly one node. The degree of this node is $2k$. Therefore,

$$r(d, 1, 0) = [d = 2k]. \qquad (10)$$

Let us calculate $P_{kt+k}(\deg\gamma = d)$ with an error term $O\left(\frac{1}{t^2}\right)$.

If $x \geqslant 0$ and $y \geqslant 0$, then $(1-x)(1-y) \geqslant 1-x-y$. One can easily prove by induction that

$$(1-x_1)\cdot\ \ldots\ \cdot(1-x_l) \geqslant 1-x_1-\ldots-x_l$$

for $x_1, \ldots, x_l \in [0, 1]$.

We suppose first that $d = k$. In this case the equality $\deg_{kt+k}\gamma = d$ holds if and only if the edge from the node $kt+1$ does not go to the node $kt+1$, the edge from the node $kt+2$ does not go neither to the node $kt+1$, nor to the node $kt+2, \ldots,$ the edge from the node $kt+k$ does not go to any of the nodes $kt+1, \ldots, kt+k$. Therefore,

$$
\begin{aligned}
&P_{kt+k}(\deg\gamma=k)=\left(1-\frac{1}{2kt+1}\right)\left(1-\frac{2}{2kt+3}\right)\cdot\ldots\cdot\left(1-\frac{k}{2kt+2k-1}\right)\geqslant\\
&\geqslant 1-\frac{1}{2kt+1}-\frac{2}{2kt+3}-\ldots-\frac{k}{2kt+2k-1}=1-\sum_{j=1}^{k}\frac{j}{2kt+2j-1}. \qquad (11)
\end{aligned}
$$

We suppose now that $d = k+1$. In this case the equality $\deg_{kt+k}\gamma = d$ holds if and only if the following is true: there is exactly one value of j in the range $1 \leqslant j \leqslant k$, for which the edge incident to the node $kt+j$ goes to one of the nodes $kt+1, \dots, kt+j$. Therefore,

$$\begin{aligned}
&P_{kt+k}(\deg\gamma = k+1) = \\
&= \sum_{j=1}^{k}\prod_{i=1}^{j-1}\left(1-\frac{i}{2kt+2i-1}\right)\frac{j}{2kt+2j-1}\times\prod_{i=j+1}^{k}\left(1-\frac{i+1}{2kt+2i-1}\right) \geqslant \\
&\geqslant \sum_{j=1}^{k}\frac{j}{2kt+2j-1}\times\left(1-\sum_{i=1}^{j-1}\frac{i}{2kt+2i-1}-\sum_{i=j+1}^{k}\frac{i+1}{2kt+2i-1}\right) = \\
&= \sum_{j=1}^{k}\frac{j}{2kt+2j-1}+O_k\left(\frac{1}{t^2}\right).
\end{aligned} \tag{12}$$

Let $\widetilde{s}(d,t)$ be the function defined by

$$\begin{aligned}
P_{kt+k}(\deg\gamma = d) = [d=k]\left(1-\sum_{j=1}^{k}\frac{j}{2kt+2j-1}\right) + \\
+ [d=k+1]\sum_{j=1}^{k}\frac{j}{2kt+2j-1}+\widetilde{s}(d,t).
\end{aligned}$$

Since $\sum_d P_{kt+k}(\deg\gamma = d) = 1$ and probability is always non-negative, we have $\sum_d \widetilde{s}(d,t) = 0$, $\widetilde{s}(d,t) \geqslant 0$ for $d \neq k+1$ and $\widetilde{s}(k+1,t) \geqslant O_k\left(\frac{1}{t^2}\right)$. Therefore,

$$\widetilde{s}(d,t) = O_k\left(\frac{1}{t^2}\right).$$

So, we have expressed the function f as a function of g, the function g as a function of h, and the function h as a function of r. We have proved a recurrent equation for r. We have also estimated the quantity $P_{kt+k}(\deg\gamma = d)$ which is included in the recurrent equation for r. We are now ready to prove an estimate for r.

3.2. Asymptotic behaviour of r and proof of Theorem 3

STATEMENT 1. If d, t are positive integers and m is an integer such that $0 \leqslant m \leqslant k$, then

$$r(d,t,m) = [d \geqslant k]\frac{(2kt+2m+1)(k+1)}{d(d+1)(d+2)} - \frac{[d=k]}{k}(m+1) + O_k\left(\frac{d}{t}\right). \quad (13)$$

Theorem 3 is the special case of this Statement with $m = 0$.

PROOF. Let

$$a(d,t,m) = r(d,t,m) - [d \geqslant k]\frac{(2kt+2m+1)(k+1)}{d(d+1)(d+2)} + \frac{[d=k]}{k}(m+1) - \\ - \sum_{i=0}^{m-1} \frac{([d=k]-[d=k+1])(i+1)}{2kt+2i+1}.$$

The error term in (13) is equal to $a(d,t,m) + O_k\left(\frac{1}{t}\right)$. Thus, it is sufficient to prove that $a(d,t,m) = O_k\left(\frac{d}{t}\right)$.

Let us rewrite the equations (8)–(10) in terms of the function a. Start by rewriting (8):

$$a(d,t,m+1) - a(d-1,t,m)\frac{d-1}{2kt+2m+1} - a(d,t,m)\left(1-\frac{d}{2kt+2m+1}\right) =$$

$$= -[d\geqslant k]\frac{(2kt+2m+3)(k+1)}{d(d+1)(d+2)} + \frac{[d=k]}{k}(m+2) - \sum_{i=0}^{m}\frac{([d=k]-[d=k+1])(i+1)}{2kt+2i+1} +$$

$$+ [d-1\geqslant k]\frac{k+1}{d(d+1)} - \frac{[d-1=k]}{k}(m+1)\frac{d-1}{2kt+2m+1} +$$

$$+ \frac{d-1}{2kt+2m+1}\sum_{i=0}^{m-1}\frac{([d-1=k]-[d-1=k+1])(i+1)}{2kt+2i+1} +$$

$$+ \left(1-\frac{d}{2kt+2m+1}\right)\left([d\geqslant k]\frac{(2kt+2m+1)(k+1)}{d(d+1)(d+2)} - \frac{[d=k]}{k}(m+1) + \right.$$

$$\left. + \sum_{i=0}^{m-1}\frac{([d=k]-[d=k+1])(i+1)}{2kt+2i+1}\right) = -[d\geqslant k]\frac{2(k+1)}{d(d+1)(d+2)} + \frac{[d=k]}{k} -$$

$$-\frac{([d=k]-[d=k+1])(m+1)}{2kt+2m+1}+[d\geqslant k+1]\frac{k+1}{d(d+1)}-[d=k+1]\frac{m+1}{2kt+2m+1}-$$
$$-[d\geqslant k]\frac{k+1}{(d+1)(d+2)}+[d=k]\frac{m+1}{2kt+2m+1}+[m>0,\ k\leqslant d\leqslant k+2]O_k\left(\frac{1}{t^2}\right)=$$
$$=[m>0,\ k\leqslant d\leqslant k+2]O_k\left(\frac{1}{t^2}\right).$$

If $k=1$, we always have $m=0$ in this equation. If $k>1$, then $k+2\leqslant 2k$. Thus, we can replace the latter indicator function with $[k\leqslant d\leqslant 2k]$.

Further, we have

$$a(d,t+1,0)-a(d,t,k)=\left(r(d,t+1,0)-[d\geqslant k]\frac{(2kt+2k+1)(k+1)}{d(d+1)(d+2)}+\frac{[d=k]}{k}\right)-$$
$$-\left(r(d,t,k)-[d\geqslant k]\frac{(2kt+2k+1)(k+1)}{d(d+1)(d+2)}+[d=k]\frac{k+1}{k}-\right.$$
$$\left.-([d=k]-[d=k+1])\sum_{i=0}^{k-1}\frac{i+1}{2kt+2i+1}\right)=$$
$$=P_{kt+k}(\deg\gamma=d)-[d=k]+([d=k]-[d=k+1])\sum_{i=0}^{k-1}\frac{i+1}{2kt+2i+1}=$$
$$=\widetilde{s}(d,t)=[k\leqslant d\leqslant 2k]O_k\left(\frac{1}{t^2}\right).$$

Finally,

$$a(d,1,0)=[d=2k]-[d\geqslant k]\frac{(2k+1)(k+1)}{d(d+1)(d+2)}+\frac{[d=k]}{k}=O_k(1).$$

Therefore, there is a constant C_1 such that

$$\begin{cases}\left|a(d,t,m+1)-a(d-1,t,m)\dfrac{d-1}{2kt+2m+1}-a(d,t,m)\left(1-\dfrac{d}{2kt+2m+1}\right)\right|\leqslant\\ \leqslant\dfrac{C_1}{(2kt+2m+1)^2}[d\leqslant k\leqslant 2k],\\ |a(d,t+1,0)-a(d,t,k)|\leqslant\dfrac{C_1}{(2k(t+1)+1)^2}[d\leqslant k\leqslant 2k].\end{cases}\tag{14}$$

We now consider six cases.

Case 1. Let $d < k$. The degree of any conglomerate in any graph from G_{kt+m} is not less than k, since we do not take into account the last «unfinished» conglomerate. Thus, $r(d, t, m) = 0$ and $a(d, t, m) = 0$.

Case 2. Let $d = 1$. Suppose that the case 1 does not hold, i. e. that $k = 1$. The equation (8) has sense only for $m = 0$. Since $a(0, t, m) = 0$, we have

$$a(1, t, 1) - a(1, t, 0)\left(1 - \frac{1}{2t+1}\right) =$$
$$= -\frac{2t+3}{3} + 2 - \frac{1}{2t+1} + \left(\frac{2t+1}{3} - 1\right)\left(1 - \frac{1}{2t+1}\right) = 0.$$

It follows from (11) that $P_{t+1}(\deg\gamma = 1) = 1 - \dfrac{1}{2t+1}$. Consequently, the equation (9) is equivalent to

$$a(1, t+1, 0) - a(1, t, 1) = P_{t+1}(\deg\gamma = 1) - 1 + \frac{1}{2t+1} = 0.$$

Finally, (10) is equivalent to

$$a(1, 1, 0) = 0.$$

One can easily show by induction that $a(1, t, m) = 0$.

Case 3. Let $d = 2$, $k = 1$. Similar to the previous case, the equation (8) has sense only for $m = 0$. We have

$$a(2, t, 1) - a(2, t, 0)\left(1 - \frac{2}{2t+1}\right) =$$
$$= -\frac{2t+3}{12} + \frac{1}{2t+1} + \frac{1}{3} - \frac{1}{2t+1} + \left(\frac{2t+1}{12}\right)\left(1 - \frac{2}{2t+1}\right) = 0.$$

Using (12), we obtain from (9) the following:

$$\widetilde{s}(2, t) = P_{t+1}(\deg\gamma = 2) - \frac{1}{2t+1} = 0,$$
$$a(2, t+1, 0) - a(2, t, 1) = 0.$$

Finally, (10) is equivalent to

$$a(2,1,0) = 1 - \frac{3\cdot 2}{2\cdot 3\cdot 4} = \frac{3}{4}.$$

Therefore, $a(2,t,1) = a(2,t+1,0)$ and

$$a(2,t,0) = a(2,t-1,0)\frac{2t-3}{2t-1} = \ldots = a(2,1,0)\cdot\frac{1}{3}\cdot\frac{3}{5}\cdot\ldots\cdot\frac{2t-3}{2t-1} = \frac{3}{4(2t-1)}.$$

Case 4. Let $d = 2$, $k = 2$. When $m = 0$, the equation (8) is equivalent to

$$a(2,t,1) - a(2,t,0)\left(1 - \frac{2}{4t+1}\right) =$$
$$= -\frac{4t+3}{8} + 1 - \frac{1}{4t+1} + \left(\frac{4t+1}{8} - \frac{1}{2}\right)\left(1 - \frac{2}{4t+1}\right) = 0.$$

When $m = 1$, the same equation is equivalent to

$$a(2,t,2) - a(2,t,1)\left(1 - \frac{2}{4t+3}\right) = -\frac{4t+5}{8} + \frac{3}{2} - \frac{1}{4t+1} - \frac{2}{4t+3} +$$
$$+ \left(\frac{4t+3}{8} - 1 + \frac{1}{4t+1}\right)\left(1 - \frac{2}{4t+3}\right) = -\frac{2}{(4t+1)(4t+3)}.$$

Using (11), we obtain

$$\widetilde{s}(2,t) = \left(1 - \frac{1}{4t+1}\right)\left(1 - \frac{2}{4t+3}\right) - 1 + \frac{1}{4t+1} + \frac{2}{4t+3} = \frac{2}{(4t+1)(4t+3)}.$$

Therefore, (9) is equivalent to

$$a(2,t+1,0) - a(2,t,2) = \frac{2}{(4t+1)(4t+3)}.$$

Thus,

$$a(2,t+1,0) = a(2,t,0)\left(1 - \frac{2}{4t+1}\right)\left(1 - \frac{2}{4t+3}\right) = a(2,t,0)\frac{4t-1}{4t+3}.$$

The equation (10) gives

$$a(2,1,0) = -\frac{5\cdot 3}{2\cdot 3\cdot 4} + \frac{1}{2} = -\frac{1}{8}.$$

Finally,

$$a(2,t,0)=a(2,t-1,0)\frac{4t-5}{4t-1}=\ldots=a(2,1,0)\cdot\frac{3}{7}\cdot\frac{7}{11}\cdot\ldots\cdot\frac{4t-5}{4t-1}=-\frac{3}{8(4t-1)}.$$

We now prove the following bound for $d\leqslant 2k$ by induction on d:

$$|a(d,t,m)|\leqslant\frac{C(d,k)}{2kt+2m+1}. \tag{15}$$

Here $C(d,k)$ depends on the degree d and on k, but does not depend on t.

This bound for $d\leqslant 2$ follows from the results of the cases 1–4.

Case 5. Let $3\leqslant d\leqslant 2k$. Suppose that the bound (15) holds for $d-1$.

It follows from (14) that

$$\begin{cases}|a(d,t,m+1)|\leqslant|a(d-1,t,m)|\dfrac{d-1}{2kt+2m+1}+|a(d,t,m)|\left(1-\dfrac{d}{2kt+2m+1}\right)+\\ +\dfrac{C_1}{(2kt+2m+1)^2},\\ |a(d,t+1,0)|\leqslant|a(d,t,k)|+\dfrac{C_1}{(2k(t+1)+1)^2}.\end{cases}$$

By the induction hypothesis

$$|a(d,t,m+1)|\leqslant|a(d,t,m)|\left(1-\frac{d}{2kt+2m+1}\right)+\frac{C(d-1,k)(d-1)}{(2kt+2m+1)^2}+$$
$$+\frac{C_1}{(2kt+2m+1)^2}\leqslant|a(d,t,m)|\left(1-\frac{d}{2kt+2m+1}\right)+\frac{C(d-1,k)(d-1)+C_1}{(2kt+2m+1)^2}.$$

Applying this bound several times we get

$$|a(d,t,m)|\leqslant|a(d,1,0)|\prod_{i=k}^{kt+m-1}\left(1-\frac{d}{2i+1}\right)+$$
$$+\sum_{j=k}^{kt+m-1}\frac{C(d-1,k)(d-1)+C_1}{(2j+1)^2}\prod_{i=j+1}^{kt+m-1}\left(1-\frac{d}{2i+1}\right)+$$
$$+\sum_{j=2}^{t}\frac{C_1}{(2kj+1)^2}\prod_{i=kj}^{kt+m-1}\left(1-\frac{d}{2i+1}\right). \tag{16}$$

We now estimate the products involved in (16):

$$\prod_{i=i_0}^{kt+m-1}\left(1-\frac{d}{2i+1}\right)=\exp\left(\sum_{i=i_0}^{kt+m-1}\ln\left(1-\frac{d}{2i+1}\right)\right)\leqslant\exp\left(-d\sum_{i=i_0}^{kt+m-1}\frac{1}{2i+1}\right)=$$

$$=\exp\left(-d\sum_{i=i_0}^{kt+m-1}\int\limits_{i}^{i+1}\frac{dz}{2i+1}\right)\leqslant\exp\left(-d\sum_{i=i_0}^{kt+m-1}\int\limits_{i}^{i+1}\frac{dz}{2z+1}\right)=$$

$$=\exp\left(-d\int\limits_{i_0}^{kt+m}\frac{dz}{2z+1}\right)=\exp\left(-\frac{d}{2}\ln\left(\frac{2kt+2m+1}{2i_0+1}\right)\right)=\left(\frac{2i_0+1}{2kt+2m+1}\right)^{\frac{d}{2}}. \quad (17)$$

We now estimate each of the three terms on the right-hand side of (16).

The first term is obviously not greater than

$$|a(d,1,0)|\left(\frac{2k+1}{2kt+2m+1}\right)^{\frac{d}{2}}\leqslant|a(d,1,0)|\frac{2k+1}{2kt+2m+1}=O_{d,k}\left(\frac{1}{2kt+2m+1}\right).$$

Let us consider the second term:

$$\sum_{j=k}^{kt+m-1}\frac{C(d-1,k)(d-1)+C_1}{(2j+1)^2}\prod_{i=j+1}^{kt+m-1}\left(1-\frac{d}{2i+1}\right)\leqslant$$

$$\leqslant\frac{C(d-1,k)(d-1)+C_1}{(2kt+2m+1)^{\frac{d}{2}}}\sum_{j=k}^{kt+m-1}\frac{(2j+3)^{\frac{d}{2}}}{(2j+1)^2},$$

$$\sum_{j=k}^{kt+m-1}\frac{(2j+3)^{\frac{d}{2}}}{(2j+1)^2}=\sum_{j=k}^{kt+m-1}(2j+1)^{\frac{d}{2}-2}\left(1+\frac{2}{2j+1}\right)^{\frac{d}{2}}\leqslant$$

$$\leqslant\left(1+\frac{2}{2k+1}\right)^{\frac{d}{2}}\sum_{j=k}^{kt+m-1}(2j+1)^{\frac{d}{2}-2}.$$

When $d\geqslant 4$, the function $(2z+1)^{\frac{d}{2}-2}$ is monotonically increasing. Thus,

$$\sum_{j=k}^{kt+m-1}(2j+1)^{\frac{d}{2}-2}\leqslant\int\limits_{k}^{kt+m}(2z+1)^{\frac{d}{2}-2}dz\leqslant\frac{(2kt+2m+1)^{\frac{d}{2}-1}}{d-2}.$$

When $d = 3$, the same function is monotonically decreasing. Thus,

$$\sum_{j=k}^{kt+m-1} (2j+1)^{\frac{d}{2}-2} \leqslant \int\limits_{k-1}^{kt+m-1} (2z+1)^{\frac{d}{2}-2} dz \leqslant \frac{(2kt+2m-1)^{\frac{d}{2}-1}}{d-2} \leqslant \frac{(2kt+2m+1)^{\frac{d}{2}-1}}{d-2}.$$

Therefore, the second term on the right-hand side of (16) is not greater than

$$\frac{C(d-1,k)(d-1)+C_1}{(d-2)(2kt+2m+1)} \left(1+\frac{2}{2k+1}\right)^{\frac{d}{2}} = O_{d,k}\left(\frac{1}{2kt+2m+1}\right).$$

Let us consider the third term:

$$\sum_{j=2}^{t} \frac{C_1}{(2kj+1)^2} \prod_{i=kj}^{kt+m-1} \left(1-\frac{d}{2i+1}\right) \leqslant \sum_{j=2}^{t} \frac{C_1}{(2kj+1)^2} \left(\frac{2kj+1}{2kt+2m+1}\right)^{\frac{d}{2}} =$$

$$= \frac{C_1}{(2kt+2m+1)^{\frac{d}{2}}} \sum_{j=2}^{t} (2kj+1)^{\frac{d}{2}-2}.$$

The cases $d \geqslant 4$ and $d < 4$ are again different. When $d \geqslant 4$, the function $(2kz+1)^{\frac{d}{2}}$ is monotonically increasing. If $t > 1$, then

$$\sum_{j=2}^{t} (2kj+1)^{\frac{d}{2}-2} \leqslant (2kt+1)^{\frac{d}{2}-2} + \int\limits_{2}^{t} (2kz+1)^{\frac{d}{2}-2} dz \leqslant (2kt+1)^{\frac{d}{2}-2} + \frac{(2kt+1)^{\frac{d}{2}-1}}{k(d-2)}.$$

When $t = 1$, the left-hand side is zero, so this bound holds, too. When $d = 3$, the function $(2kz+1)^{\frac{d}{2}}$ is monotonically decreasing. Thus,

$$\sum_{j=2}^{t} (2kj+1)^{\frac{d}{2}-2} \leqslant \int\limits_{1}^{t} (2kz+1)^{\frac{d}{2}-2} dz \leqslant \frac{(2kt+1)^{\frac{d}{2}-1}}{k(d-2)}.$$

Therefore, the third term on the right-hand side of (16) is not greater than

$$\frac{C_1}{(2kt+2m+1)^2} + \frac{C_1}{k(d-2)(2kt+2m+1)} = O_{d,k}\left(\frac{1}{2kt+2m+1}\right).$$

The three bounds prove the induction step of the case 5.

Let

$$C = \max\left\{1,\ \max_{1<d\leqslant 2k} \frac{C(d,k)}{d-1}\right\}.$$

Then $C \geqslant 1$ is a constant (depending only on k) such that the bound

$$|a(d,t,m)| \leqslant \frac{C(d-1)}{2kt+2m-1} \tag{18}$$

holds for $1 \leqslant d \leqslant 2k$. We now prove by induction on d that this bound also holds for $d > 2k$.

Case 6. Let $d > 2k$. Suppose that (18) holds for $d-1$. We now prove the bound (18) for d by induction on t and m.

If $d > 2kt+2m$, any graph from $\widehat{G}_{kt+m}$, like any graph from G_{kt+m}, has $kt+m$ edges and cannot have a node of degree d. Let $d = 2kt+2m$. Since $d > 2k$, any graph from G_{kt+m} has at least two nodes. The degree of any node is greater than 1. Thus, the degree of any node is strictly less than $2(kt+m) = d$. Therefore, if $d \geqslant 2kt+2m$, then $r(d,t,m) = 0$ and

$$|a(d,t,m)| = \left| r(d,t,m) - \frac{(2kt+2m+1)(k+1)}{d(d+1)(d+2)} \right| = \frac{(2kt+2m+1)(k+1)}{d(d+1)(d+2)} \leqslant$$
$$\leqslant \frac{k+1}{d(d+2)} < 1 \leqslant \frac{d-1}{2kt+2m-1}.$$

Suppose now that $2kt+2m > d$. By the induction hypothesis

$$|a(d,t,m)| \leqslant |a(d-1,t,m-1)| \frac{d-1}{2kt+2(m-1)+1} +$$
$$+ |a(d,t,m-1)| \left(1 - \frac{d}{2kt+2(m-1)+1}\right) \leqslant$$
$$\leqslant \frac{C(d-2)}{2kt+2m-3}\,\frac{d-1}{2kt+2m-1} + \frac{C(d-1)}{2kt+2m-3}\left(1 - \frac{d}{2kt+2m-1}\right) =$$
$$= \frac{C(d-1)}{(2kt+2m-3)(2kt+2m-1)}((d-2)+(2kt+2m-1-d)) = \frac{C(d-1)}{2kt+2m-1}.$$

This proves the step of the induction on m. Since $a(d,t+1,0) = a(d,t,k)$, the step of the induction on t is trivial. This proves the bound (18) in the general case.

Statement 1 is proved. □

3.3. Asymptotic behaviour of h

STATEMENT 2. If d_1, d_2, t are positive integers and m is an integer such that $0 \leqslant m \leqslant k$, then

$$h(d_1, d_2, t, m) =$$

$$= [d_2 \geqslant k]\left([d_1 = m]\frac{(k+1)\left(2kt + 2d_1 + 1 - \frac{d_1(d_1+1)}{2}\right)}{d_2(d_2+1)(d_2+2)} - \right.$$

$$\left. - [d_1 = m,\ d_2 = k]\frac{d_1+1}{k} + [d_1 = m+1]\frac{(k+1)(d_1-1)d_1}{2d_2(d_2+1)(d_2+2)} + O_k\left(\frac{d_2}{t}\right)\right). \tag{19}$$

PROOF. If $d_1 > 2m$ or $d_2 > 2(kt+m)$, then $h(d_1, d_2, t, m) = 0$ by the definition of h and $\widehat{G}_{kt+m}$. The bound holds in this case. Suppose now that $d_1 \leqslant 2m$ and $d_2 \leqslant 2(kt+m)$.

We use induction on m. The implied constants in $O(\cdot)$ can be different for different values of m. However, since $0 \leqslant m \leqslant k$, all those constants can be bounded by one constant depending only on k.

The induction base $m = 0$ follows immediately from (7), (13). Note that $r(d, t, 0) = 0$ for $d < k$.

The induction step. Suppose that the statement is proved for m. Let us prove it for $m+1$. Let us substitute the induction hypothesis into (6). If $d_2 < k$, all the terms on the right-hand side are zero. Consequently, the left-hand side is zero. This proves the induction step in this case. Now, let us suppose that $d_2 \geqslant k$. Then,

$$h(d_1, d_2, t, m+1) =$$

$$= \frac{d_1-1}{2kt+2m+1}\left([d_1-2=m]\frac{(k+1)(2kt+2m+1)}{d_2(d_2+1)(d_2+2)} + O_k\left(1+\frac{d_2}{t}\right)\right) +$$

$$+ \frac{d_2-1}{2kt+2m+1}\left([d_1-1=m,\ d_2-1 \geqslant k]\frac{(k+1)(2kt+2m+1)}{(d_2-1)d_2(d_2+1)} + \right.$$

$$\left. + O_k\left(1+\frac{d_2}{t}\right)\right) + \left(1 - \frac{d_1+d_2}{2kt+2m+1}\right) \times$$

$$\times\left([d_1-1=m]\frac{(k+1)\left(2kt+2d_1-1-\frac{(d_1-1)d_1}{2}\right)}{d_2(d_2+1)(d_2+2)}-[d_1-1=m,\ d_2=k]\frac{d_1}{k}+\right.$$

$$\left.+[d_1-1=m+1]\frac{(k+1)(d_1-2)(d_1-1)}{2d_2(d_2+1)(d_2+2)}+O_k\left(\frac{d_2}{t}\right)\right)=$$

$$=[d_1=m+2]\frac{(k+1)(d_1-1)}{d_2(d_2+1)(d_2+2)}+[d_1=m+1,\ d_2\geqslant k+1]\frac{k+1}{d_2(d_2+1)}+$$

$$+[d_1=m+1]\frac{(k+1)\left(2kt+2d_1-1-\frac{(d_1-1)d_1}{2}\right)}{d_2(d_2+1)(d_2+2)}-[d_1=m+1,\ d_2=k]\frac{d_1}{k}+$$

$$+[d_1=m+2]\frac{(k+1)(d_1-2)(d_1-1)}{2d_2(d_2+1)(d_2+2)}-[d_1=m+1]\frac{(k+1)(d_1+d_2)}{d_2(d_2+1)(d_2+2)}+$$

$$+O_k\left(\frac{d_2}{t}\right)=[d_1=m+1]\left(\frac{k+1}{d_2(d_2+1)}-[d_2=k]\frac{k+1}{d_2(d_2+1)}+\right.$$

$$\left.+\frac{(k+1)\left(2kt+2d_1-1-\frac{(d_1-1)d_1}{2}\right)}{d_2(d_2+1)(d_2+2)}-[d_2=k]\frac{d_1}{k}-\frac{(k+1)(d_1+d_2)}{d_2(d_2+1)(d_2+2)}\right)+$$

$$+[d_1=m+2]\left(\frac{(k+1)(d_1-1)}{d_2(d_2+1)(d_2+2)}+\frac{(k+1)(d_1-2)(d_1-1)}{2d_2(d_2+1)(d_2+2)}\right)+O_k\left(\frac{d_2}{t}\right)=$$

$$=\left(\frac{k+1}{d_2(d_2+1)(d_2+2)}\left(d_2+2+2kt+2d_1-1-\frac{(d_1-1)d_1}{2}-d_1-d_2\right)-\right.$$

$$\left.-[d_2=k]\frac{d_1+1}{k}\right)[d_1=m+1]+[d_1=m+2]\frac{(k+1)d_1(d_1-1)}{2d_2(d_2+1)(d_2+2)}+O_k\left(\frac{d_2}{t}\right)=$$

$$=[d_1=m+1]\left(\frac{(k+1)\left(2kt+2d_1+1-\frac{d_1(d_1+1)}{2}\right)}{d_2(d_2+1)(d_2+2)}-[d_2=k]\frac{d_1+1}{k}\right)+$$

$$+[d_1=m+2]\frac{(k+1)d_1(d_1-1)}{2d_2(d_2+1)(d_2+2)}+O_k\left(\frac{d_2}{t}\right).$$

The statement is proved. □

3.4. Asymptotic behaviour of g

STATEMENT 3. If d_1, d_2, t are positive integers and m is an integer such that $0 \leqslant m \leqslant k$, then

$$g(d_1, d_2, t, m) = [d_2 \geqslant k+1]\Bigg([d_1 = m]\frac{(k+1)d_1}{d_2(d_2+1)}\left(1 - \frac{d_1(d_1-1)}{2(2kt+2m-1)}\right) - \\ - [d_1 = m,\ d_2 = k+1]\frac{d_1^2}{2kt+2m-1} + [d_1 = m+1]\frac{(k+1)d_1(d_1-1)(d_1-2)}{2(2kt+2m-1)d_2(d_2+1)} + \\ + O_k\left(\left(\frac{d_2}{t}\right)^2\right)\Bigg). \tag{20}$$

PROOF. As in the previous subsection, if $d_1 > 2m$ or $d_2 > 2(kt+m)$, then $g(d_1,d_2,t,m) = 0$ and the statement holds. Suppose now that $d_1 \leqslant 2m$ and $d_2 \leqslant 2(kt+m)$.

We use induction on m. All the implied constants in $O(\cdot)$ for different values of m can be bounded by one constant.

The induction base $m = 0$ follows trivially from (5).

The induction step. Suppose that Statement 3 is proved for m. Let us prove it for $m+1$. Let us multiply (4) by $2kt + 2m + 1$ and make use of the induction hypothesis and (19). If $d_2 < k+1$, the induction step is trivial. Now, let us suppose that $d_2 \geqslant k$:

$$(2kt+2m+1)g(d_1,d_2,t,m+1) = \\ = (d_2-1)\Bigg([d_1-1=m]\frac{(k+1)\left(2kt+2d_1-1-\frac{(d_1-1)d_1}{2}\right)}{(d_2-1)d_2(d_2+1)} - \\ - [d_1-1=m,\ d_2-1=k]\frac{d_1}{k} + [d_1-1=m+1]\frac{(k+1)(d_1-2)(d_1-1)}{2(d_2-1)d_2(d_2+1)} + O_k\left(\frac{d_2}{t}\right)\Bigg) + \\ + (d_1-1)\left([d_1-2=m]\frac{(k+1)(d_1-2)}{d_2(d_2+1)} + O_k\left(\frac{d_2}{t}\right)\right) + \\ + (d_2-1)\left([d_1-1=m,\ d_2-1 \geqslant k+1]\frac{(k+1)(d_1-1)}{(d_2-1)d_2} + O_k\left(\frac{d_2}{t}\right)\right) +$$

$$+(2kt+2m+1-d_1-d_2)[d_1-1=m]\frac{(k+1)(d_1-1)}{d_2(d_2+1)}-$$

$$-[d_1-1=m]\frac{(k+1)(d_1-1)}{d_2(d_2+1)}\frac{(d_1-1)(d_1-2)}{2}-[d_1-1=m,d_2=k+1](d_1-1)^2+$$

$$+[d_1-1=m+1]\frac{(k+1)(d_1-1)(d_1-2)(d_1-3)}{2d_2(d_2+1)}+O_k\left(\frac{d_2^2}{t}\right)=$$

$$=[d_1=m+1]\left(\frac{(k+1)\left(2kt+2d_1-1-\frac{(d_1-1)d_1}{2}\right)}{d_2(d_2+1)}+\frac{(k+1)(d_1-1)}{d_2}+\right.$$

$$\left.+(2kt+2m+1-d_1-d_2)\frac{(k+1)(d_1-1)}{d_2(d_2+1)}-\frac{(k+1)(d_1-1)^2(d_1-2)}{2d_2(d_2+1)}\right)-$$

$$-[d_1=m+1,\ d_2=k+1]\left(d_1+\frac{(k+1)(d_1-1)}{d_2}+(d_1-1)^2\right)+$$

$$+[d_1=m+2]\left(\frac{3(k+1)(d_1-2)(d_1-1)}{2d_2(d_2+1)}+\frac{(k+1)(d_1-1)(d_1-2)(d_1-3)}{2d_2(d_2+1)}\right)+$$

$$+O_k\left(\frac{d_2^2}{t}\right)=[d_1=m+1]\frac{k+1}{d_2(d_2+1)}\left(2kt+2d_1-1-\frac{(d_1-1)d_1}{2}+(d_1-1)(d_2+1)+\right.$$

$$\left.+(d_1-1)(2kt+2m+1-d_1-d_2)-\frac{(d_1-1)^2(d_1-2)}{2}\right)-$$

$$-[d_1=m+1,\ d_2=k+1]d_1^2+[d_1=m+2]\frac{(k+1)(d_1-2)(d_1-1)d_1}{2d_2(d_2+1)}+O_k\left(\frac{d_2^2}{t}\right).$$

The term with the factor $[d_1 = m+1]$ can be transformed as follows. Note that m and $d_1 - 1$ are interchangeable in this term. So

$$2kt+2m+1-\frac{(d_1-1)d_1}{2}+(d_1-1)(d_2+1)+(d_1-1)(2kt+2m+1-d_1-d_2)-$$
$$-\frac{(d_1-1)^2(d_1-2)}{2}=2kt+2m+1-\frac{(d_1-1)d_1}{2}+$$
$$+(d_1-1)\left(d_2+1+2kt+2m+1-d_1-d_2-\frac{(d_1-1)(d_1-2)}{2}\right)=$$

$$= 2kt + 2m + 1 - \frac{(d_1-1)d_1}{2} +$$
$$+ (d_1-1)\left(2kt+2m+1-(d_1-1)-\frac{(d_1-1)(d_1-2)}{2}\right) =$$
$$= d_1\left(2kt+2m+1-\frac{(d_1-1)d_1}{2}\right).$$

This proves the induction step and the statement. □

3.5. Asymptotic behaviour of f and proof of Theorem 1

STATEMENT 4. If d_1, d_2, t are positive integers and m is an integer such that $0 \leqslant m \leqslant k$, then

$$f(d_1,d_2,t,m) = ([d_1 \geqslant k, d_2 \geqslant k] - [d_1 = k, d_2 = k])\times$$
$$\times\left(\frac{k(k+1)(2kt+2m+1)}{d_1(d_1+1)d_2(d_2+1)}\left(1-\frac{C_{2k+2}^{k+1}C_{d_1+d_2-2k}^{d_1-k}}{C_{d_1+d_2+2}^{d_1+1}}\right)-\right.$$
$$-\sum_{n=1}^{k}\frac{C_{d_1+d_2-2n}^{d_1-n}}{d_1 d_2 C_{d_1+d_2}^{d_1}}\left(\frac{(2n)!}{n!(n+1)!}\frac{k+1}{2k}+[n=k]\frac{(2k)!}{2(k-1)!^2}\right)-$$
$$\left.-\left(m+\frac{k-1}{2k}\right)\left([d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}\right)+O_{k,d_1,d_2}\left(\frac{1}{t}\right)\right).$$

Theorem 1 is the special case of this statement with $m = 0$.

PROOF. If $d_1 < k$ or $d_2 < k$, or $d_1 = d_2 = k$, then it follows from (20), (2), (1) and (3) that $g(d_1,d_2,t,m) = g(d_2,d_1,t,m) = 0$ and $f(d_1,d_2,t,m) = 0$. This proves Statement 4 in this case. Further we suppose that

$$(d_1,d_2) \in S = \{d_1 \geqslant k+1,\ d_2 \geqslant k\} \cup \{d_1 \geqslant k,\ d_2 \geqslant k+1\}.$$

Let

$$\widetilde{f}_1(d_1,d_2,t,m) = \frac{k(k+1)}{d_1(d_1+1)d_2(d_2+1)}\left(1-\frac{C_{2k+2}^{k+1}C_{d_1+d_2-2k}^{d_1-k}}{C_{d_1+d_2+2}^{d_1+1}}\right)(2kt+2m+1),$$

$$\widetilde{f}_2(d_1,d_2)=\sum_{n=1}^{k}\frac{C_{d_1+d_2-2n}^{d_1-n}}{d_1d_2C_{d_1+d_2}^{d_1}}\left(\frac{(2n)!}{n!(n+1)!}\frac{k+1}{2k}+[n=k]\frac{(2k)!}{2(k-1)!^2}\right),$$

$$\widetilde{f}_3(d_1,d_2,t,m)=\left(m+\frac{k-1}{2k}\right)\left([d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}\right)$$

for $(d_1,d_2)\in S$. If $(d_1,d_2)\notin S$, let $\widetilde{f}_i=0$.

LEMMA 1. *If* $(d_1,d_2)\in S$, *then*

$$\widetilde{f}_1(d_1,d_2,t,m+1)-\widetilde{f}_1(d_1-1,d_2,t,m)\frac{d_1-1}{2kt+2m+1}-$$
$$-\widetilde{f}_1(d_1,d_2-1,t,m)\frac{d_2-1}{2kt+2m+1}-\widetilde{f}_1(d_1,d_2,t,m)\left(1-\frac{d_1+d_2}{2kt+2m+1}\right)=$$
$$=[d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}.$$

LEMMA 2. *If* $(d_1,d_2)\in S$, *then*

$$\widetilde{f}_2(d_1,d_2)-\widetilde{f}_2(d_1-1,d_2)\frac{d_1-1}{2kt+2m+1}-$$
$$-\widetilde{f}_2(d_1,d_2-1)\frac{d_2-1}{2kt+2m+1}-\widetilde{f}_2(d_1,d_2)\left(1-\frac{d_1+d_2}{2kt+2m+1}\right)=$$
$$=\frac{[d_1=k]}{2kt+2m+1}\left(\frac{(k+1)(k-1)}{2kd_2(d_2+1)}+[d_2=k+1]\frac{k^2+1}{2k}\right)+$$
$$+\frac{[d_2=k]}{2kt+2m+1}\left(\frac{(k+1)(k-1)}{2kd_1(d_1+1)}+[d_1=k+1]\frac{k^2+1}{2k}\right).$$

LEMMA 3. *If* $(d_1,d_2)\in S$, *then*

$$\widetilde{f}_3(d_1,d_2,t,m+1)-\widetilde{f}_3(d_1-1,d_2,t,m)\frac{d_1-1}{2kt+2m+1}-$$
$$-\widetilde{f}_3(d_1,d_2-1,t,m)\frac{d_2-1}{2kt+2m+1}-\widetilde{f}_3(d_1,d_2,t,m)\left(1-\frac{d_1+d_2}{2kt+2m+1}\right)=$$
$$=[d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}+\frac{1}{2kt+2m+1}\left(m+\frac{k-1}{2k}\right)\times$$

$$\times\left([d_1=k]\frac{(k-1)(k+1)}{d_2(d_2+1)}+[d_2=k]\frac{(k-1)(k+1)}{d_1(d_1+1)}-[d_1=k+1]\frac{k(k+1)}{d_2(d_2+1)}-\right.$$

$$\left.-[d_2=k+1]\frac{k(k+1)}{d_1(d_1+1)}+2[d_1=k,d_2=k+1]+2[d_1=k+1,d_2=k]\right).$$

PROOF OF LEMMA 1. Note that the formula defining $\widetilde{f}_1$ gives the correct result for $d_1=d_2=k$, so we may consider the special case $d_1=k+1,\ d_2=k$ together with the main case $d_1\geqslant k+1,\ d_2\geqslant k$. We have

$$\widetilde{f}_1(d_1,d_2,t,m+1)-\widetilde{f}_1(d_1-1,d_2,t,m)\frac{d_1-1}{2kt+2m+1}-$$

$$-\widetilde{f}_1(d_1,d_2-1,t,m)\frac{d_2-1}{2kt+2m+1}-\widetilde{f}_1(d_1,d_2,t,m)\left(1-\frac{d_1+d_2}{2kt+2m+1}\right)=$$

$$=\frac{k(k+1)}{d_1(d_1+1)d_2(d_2+1)}\left(1-\frac{C_{2k+2}^{k+1}C_{d_1+d_2-2k}^{d_1-k}}{C_{d_1+d_2+2}^{d_1+1}}\right)(2+(d_1+d_2))-$$

$$-[d_1-1\geqslant k]\frac{k(k+1)}{d_1d_2(d_2+1)}\left(1-\frac{C_{2k+2}^{k+1}C_{d_1+d_2-2k-1}^{d_1-k-1}}{C_{d_1+d_2+1}^{d_1}}\right)-$$

$$-[d_2-1\geqslant k]\frac{k(k+1)}{d_1(d_1+1)d_2}\left(1-\frac{C_{2k+2}^{k+1}C_{d_1+d_2-2k-1}^{d_1-k}}{C_{d_1+d_2+1}^{d_1+1}}\right). \tag{21}$$

Suppose first that $d_1=k$. Then $d_2>k$ and

$$\frac{1}{d_2(d_2+1)}\left(1-\frac{C_{2k+2}^{k+1}}{C_{k+d_2+2}^{k+1}}\right)(2+k+d_2)-\frac{1}{d_2}\left(1-\frac{C_{2k+2}^{k+1}}{C_{k+d_2+1}^{k+1}}\right)=$$

$$=\frac{2+k+d_2}{d_2(d_2+1)}-\frac{1}{d_2}=\frac{k+1}{d_2(d_2+1)}.$$

The expression (21) is symmetric in d_1 and d_2, so the case $d_1>k,\ d_2=k$ is symmetric to the previous case. Suppose finally that $d_1>k$ and $d_2>k$. Then

$$\frac{k(k+1)(2+d_1+d_2)}{d_1(d_1+1)d_2(d_2+1)}-\frac{k(k+1)C_{2k+2}^{k+1}}{d_1d_2(d_1+d_2+1)C_{d_1+d_2}^{d_1}}C_{d_1+d_2-2k}^{d_1-k}-\frac{k(k+1)}{d_1d_2(d_2+1)}+$$

$$+\frac{k(k+1)C_{2k+2}^{k+1}}{d_1d_2(d_1+d_2+1)C_{d_1+d_2}^{d_1}}C_{d_1+d_2-2k-1}^{d_1-k-1}-\frac{k(k+1)}{d_1(d_1+1)d_2}+$$

$$+\frac{k(k+1)C_{2k+2}^{k+1}}{d_1d_2(d_1+d_2+1)C_{d_1+d_2}^{d_1}}C_{d_1+d_2-2k-1}^{d_1-k}=0.$$

Lemma 1 is proved. □

Proof of Lemma 2. First we simplify the expression for $\widetilde{f}_2$ in the special case $d_1=k,\ d_2\geqslant k+1$:

$$\widetilde{f}_2(k,d_2)=\frac{k+1}{2k^2d_2C_{k+d_2}^k}\sum_{n=1}^{k}C_{k+d_2-2n}^{k-n}\frac{(2n)!}{n!(n+1)!}+\frac{(2k)!}{2(k-1)!^2kd_2C_{k+d_2}^k}.$$

Now, we apply the identity (26) from [10, 1.2.6]. If k is an integer and r,s,t are real numbers, then

$$\sum_{n\geqslant 0}C_{r-tn}^{n}C_{s-t(k-n)}^{k-n}\frac{r}{r-tn}=C_{r+s-tk}^{k}.$$

Let $t=-2,\ r=1,\ s=d_2-k$ and let k be the parameter of the model:

$$\sum_{n\geqslant 0}\frac{C_{2n+1}^{n}}{2n+1}C_{(d_2-k)+2(k-n)}^{k-n}=C_{1+(d_2-k)+2k}^{k}.$$

Since $C_a^{k-n}=0$ for $k-n<0$ by the definition of binomial coefficients,

$$\sum_{n=0}^{k}\frac{C_{2n+1}^{n}}{2n+1}C_{k+d_2-2n}^{k-n}=C_{d_2+k+1}^{k}.$$

Let us move the term with $n=0$ to the right-hand side:

$$\sum_{n=1}^{k}\frac{C_{2n+1}^{n}}{2n+1}C_{k+d_2-2n}^{k-n}=C_{d_2+k+1}^{k}-C_{d_2+k}^{k}=C_{d_2+k}^{k-1}.$$

Therefore,

$$\widetilde{f}_2(k,d_2)=\frac{(k+1)C_{d_2+k}^{k-1}}{2k^2d_2C_{k+d_2}^k}+\frac{(2k)!}{2(k-1)!^2kd_2C_{k+d_2}^k}=\frac{k+1}{2kd_2(d_2+1)}+\frac{(2k)!}{2(k-1)!(k+1)!C_{k+d_2}^{k+1}}.$$

The proof consists of 3 cases depending on whether d_1 or d_2 is equal to k. Since $(d_1, d_2) \in S$, the case $d_1 = d_2 = k$ is impossible.

Case 1. Let $d_1 = k$, $d_2 \geqslant k+1$. The left-hand side of the formula in Lemma 2 is equal to

$$\widetilde{f}_2(k, d_2)\frac{d_1 + d_2}{2kt + 2m + 1} - [d_2 - 1 \geqslant k + 1]\widetilde{f}_2(k, d_2 - 1)\frac{d_2 - 1}{2kt + 2m + 1} =$$
$$= \left(\frac{k+1}{2kd_2(d_2+1)} + \frac{(2k)!}{2(k-1)!(k+1)!C_{k+d_2}^{k+1}}\right)\frac{k+d_2}{2kt+2m+1} -$$
$$- \left(\frac{k+1}{2kd_2} + \frac{(2k)!(k+d_2)}{2(k-1)!(k+1)!C_{k+d_2}^{k+1}}\right)\frac{1}{2kt+2m+1} +$$
$$+ \frac{[d_2 = k+1]}{2kt+2m+1}\left(\frac{1}{2k} + \frac{(2k+1)!}{2(k-1)!(k+1)!C_{2k+1}^{k+1}}\right) =$$
$$= \frac{1}{2kt+2m+1}\frac{(k+1)(k-1)}{2kd_2(d_2+1)} + \frac{[d_2 = k+1]}{2kt+2m+1}\frac{k^2+1}{2k}.$$

Case 2. Let $d_2 = k$. Since the function $\widetilde{f}_2$ is symmetric, Lemma 2 follows from the case 1.

Case 3. Let $d_1 > k$, $d_2 > k$. The expression for $\widetilde{f}_2$ is a linear combination of the functions $\dfrac{C_{d_1+d_2-2n}^{d_1-n}}{d_1 d_2 C_{d_1+d_2}^{d_1}}$. The coefficients depend only on n and k. We have

$$\frac{C_{d_1+d_2-2n}^{d_1-n}}{d_1 d_2 C_{d_1+d_2}^{d_1}}\frac{d_1+d_2}{2kt+2m+1} - \frac{C_{d_1-1+d_2-2n}^{d_1-1-n}}{(d_1-1)d_2 C_{d_1-1+d_2}^{d_1-1}}\frac{d_1-1}{2kt+2m+1} -$$
$$- \frac{C_{d_1+d_2-1-2n}^{d_1-n}}{d_1(d_2-1)C_{d_1+d_2-1}^{d_1}}\frac{d_2-1}{2kt+2m+1} =$$
$$= \frac{C_{d_1+d_2-2n}^{d_1-n}(d_1+d_2)}{d_1 d_2 C_{d_1+d_2}^{d_1}(2kt+2m+1)}\left(1 - \frac{d_1-n}{d_1+d_2-2n} - \frac{d_2-n}{d_1+d_2-2n}\right) = 0.$$

Lemma 2 follows in this case due to linearity. □

PROOF OF LEMMA 3. The left-hand side of the formula in Lemma 3 is equal to

$$[d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}+\frac{1}{2kt+2m+1}\left(m+\frac{k-1}{2k}\right)\times$$

$$\times\left(\left([d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}\right)(d_1+d_2)-\right.$$

$$-(d_1-1)\left([d_1-1=k,\ d_2\geqslant k+1]\frac{k+1}{d_2(d_2+1)}+[d_1\geqslant k+2,\ d_2=k]\frac{k+1}{(d_1-1)d_1}\right)-$$

$$\left.-(d_2-1)\left([d_1=k,\ d_2\geqslant k+2]\frac{k+1}{(d_2-1)d_2}+[d_1\geqslant k+1,\ d_2-1=k]\frac{k+1}{d_1(d_1+1)}\right)\right)=$$

$$=[d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}+\frac{1}{2kt+2m+1}\left(m+\frac{k-1}{2k}\right)\times$$

$$\times\left([d_1=k]\frac{(k-1)(k+1)}{d_2(d_2+1)}+[d_1=k,\ d_2=k+1]+[d_2=k]\frac{(k-1)(k+1)}{d_1(d_1+1)}+\right.$$

$$+[d_1=k+1,\ d_2=k]-[d_1=k+1,\ d_2\geqslant k+1]\frac{k(k+1)}{d_2(d_2+1)}-$$

$$\left.-[d_1\geqslant k+1,\ d_2=k+1]\frac{k(k+1)}{d_1(d_1+1)}\right)=$$

$$=[d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}+\frac{1}{2kt+2m+1}\left(m+\frac{k-1}{2k}\right)\times$$

$$\times\left([d_1=k]\frac{(k-1)(k+1)}{d_2(d_2+1)}+[d_2=k]\frac{(k-1)(k+1)}{d_1(d_1+1)}-[d_1=k+1]\frac{k(k+1)}{d_2(d_2+1)}-\right.$$

$$\left.-[d_2=k+1]\frac{k(k+1)}{d_1(d_1+1)}+2[d_1=k,\ d_2=k+1]+2[d_1=k+1,\ d_2=k]\right).$$

Lemma 3 is proved. □

Let

$$\delta(d_1,d_2,t,m)=f(d_1,d_2,t,m)-(\widetilde{f}_1(d_1,d_2,t,m)-\widetilde{f}_2(d_1,d_2)-\widetilde{f}_3(d_1,d_2,t,m)).$$

It follows from (1) and from Lemmas 1, 2, 3, that

$$\delta(d_1,d_2,t,m+1)-\delta(d_1-1,d_2,t,m)\frac{d_1-1}{2kt+2m+1}-\delta(d_1,d_2-1,t,m)\frac{d_2-1}{2kt+2m+1}-$$
$$-\delta(d_1,d_2,t,m)\left(1-\frac{d_1+d_2}{2kt+2m+1}\right)=\frac{\alpha(d_1,d_2,m)}{2kt+2m+1},$$

where

$$\alpha(d_1,d_2,m)=[d_1=k]\left(\frac{(k+1)(k-1)}{2kd_2(d_2+1)}+[d_2=k+1]\frac{k^2+1}{2k}\right)+$$
$$+[d_2=k]\left(\frac{(k+1)(k-1)}{2kd_1(d_1+1)}+[d_1=k+1]\frac{k^2+1}{2k}\right)+\left(m+\frac{k-1}{2k}\right)\times$$
$$\times\left([d_1=k]\frac{(k-1)(k+1)}{d_2(d_2+1)}+[d_2=k]\frac{(k-1)(k+1)}{d_1(d_1+1)}-[d_1=k+1]\frac{k(k+1)}{d_2(d_2+1)}-\right.$$
$$\left.-[d_2=k+1]\frac{k(k+1)}{d_1(d_1+1)}+2[d_1=k,\ d_2=k+1]+2[d_1=k+1,\ d_2=k]\right).$$

It follows from (2), (20) and from the definitions of the functions $\widetilde{f}_i$ that

$$\delta(d_1,d_2,t+1,0)-\delta(d_1,d_2,t,k)=[d_2\geqslant k+1]\times$$
$$\times\left([d_1=k]\frac{k(k+1)}{d_2(d_2+1)}\left(1-\frac{k(k-1)}{2(2kt+2k-1)}\right)-[d_1=k,\ d_2=k+1]\frac{k^2}{2kt+2k-1}+\right.$$
$$\left.+[d_1=k+1]\frac{(k+1)^2k(k-1)}{2(2kt+2k-1)d_2(d_2+1)}+O_k\left(\frac{d_2^2}{t^2}\right)\right)+[d_1\geqslant k+1]\times$$
$$\times\left([d_2=k]\frac{k(k+1)}{d_1(d_1+1)}\left(1-\frac{k(k-1)}{2(2kt+2k-1)}\right)-[d_2=k,\ d_1=k+1]\frac{k^2}{2kt+2k-1}+\right.$$
$$\left.+[d_2=k+1]\frac{(k+1)^2k(k-1)}{2(2kt+2k-1)d_1(d_1+1)}+O_k\left(\frac{d_1^2}{t^2}\right)\right)-$$
$$-k\left([d_1=k]\frac{k+1}{d_2(d_2+1)}+[d_2=k]\frac{k+1}{d_1(d_1+1)}\right).$$

It suffices to prove the bound $\delta = O_{k,d_1,d_2}(1/t)$. We use induction on $d_1 + d_2$.

The induction base. If $d_1 + d_2 \leqslant 2k$, then $(d_1, d_2) \notin S$ and $\delta(d_1, d_2, t, m) = 0$.

The induction step. Further we suppose that the bound holds for $(d_1 - 1,\ d_2)$ and $(d_1,\ d_2 - 1)$. We have

$$\delta(d_1, d_2, t, m+1) - \delta(d_1, d_2, t, m)\left(1 - \frac{d_1 + d_2}{2kt + 2m + 1}\right) = $$
$$= \frac{\alpha(d_1, d_2, m)}{2kt + 2m + 1} + O_{k,d_1,d_2}\left(\frac{1}{t^2}\right),$$
$$\delta(d_1, d_2, t, s) - \delta(d_1, d_2, t, 0)\prod_{m=0}^{s-1}\left(1 - \frac{d_1 + d_2}{2kt + 2m + 1}\right) =$$
$$= \frac{\sum\limits_{m=0}^{s-1} \alpha(d_1, d_2, m)}{2kt} + O_{k,d_1,d_2}\left(\frac{1}{t^2}\right). \tag{22}$$

Note that $\sum\limits_{s=0}^{m-1} 1 = m$ and $\sum\limits_{s=0}^{m-1} s = \frac{m(m-1)}{2}$. This allows to calculate the sum $\sum\limits_{m=0}^{s-1} \alpha(d_1, d_2, m)$. Now we substitute $s = k$ in (22) and add the result to the second recurrent equation for δ. The left-hand side is

$$\delta(d_1, d_2, t+1, 0) - \delta(d_1, d_2, t, 0)\prod_{s=0}^{k-1}\left(1 - \frac{d_1 + d_2}{2kt + 2s + 1}\right).$$

The right-hand side is $\frac{1}{2kt}$ multiplied by some coefficient with an error of order $O_{k,d_1,d_2}\left(\frac{1}{t^2}\right)$. We investigate several cases to calculate the coefficient.

Case 1. Let $d_1 = k$ and $d_2 \geqslant k + 1$. The coefficient is equal to

$$-\frac{k(k+1)}{d_2(d_2+1)}\frac{k(k-1)}{2} - [d_2 = k+1]k^2 +$$
$$+k\left(\frac{(k+1)(k-1)}{2kd_2(d_2+1)} + [d_2 = k+1]\frac{k^2+1}{2k} + \frac{k-1}{2k}\left(\frac{(k-1)(k+1)}{d_2(d_2+1)} + [d_2 = k+1]\right)\right) +$$

$$+\frac{k(k-1)}{2}\left(\frac{(k-1)(k+1)}{d_2(d_2+1)}+[d_2=k+1]\right)=$$
$$=\frac{(k-1)(k+1)}{2d_2(d_2+1)}\left(-k^2+1+(k-1)+k(k-1)\right)+$$
$$+[d_2=k+1]\left(-k^2+\frac{k^2+1}{2}+\frac{k-1}{2}+\frac{k(k-1)}{2}\right)=0.$$

Case 2. Let $d_2=k$. Since the function δ is symmetric in d_1 and d_2, the coefficient is zero, as in the previous case.

Case 3. Let $d_1>k$, $d_2>k$. The coefficient is equal to

$$[d_1=k+1]\frac{(k+1)^2k(k-1)}{2d_2(d_2+1)}+[d_2=k+1]\frac{(k+1)^2k(k-1)}{2d_1(d_1+1)}+\left(\frac{k(k-1)}{2}+k\frac{k-1}{2k}\right)\times$$
$$\times\left(-[d_1=k+1]\frac{k(k+1)}{d_2(d_2+1)}-[d_2=k+1]\frac{k(k+1)}{d_1(d_1+1)}\right)=0.$$

Finally, we obtain that

$$\delta(d_1,d_2,t+1,0)-\delta(d_1,d_2,t,0)\prod_{s=0}^{k-1}\left(1-\frac{d_1+d_2}{2kt+2s+1}\right)=O_{k,d_1,d_2}\left(\frac{1}{t^2}\right).$$

Let $\varepsilon(d_1,d_2,t)$ denote the left-hand side of this equality. Let us apply this equality consecutively to $\delta(d_1,d_2,T,0)$, $\delta(d_1,d_2,T-1,0),\ldots,\delta(d_1,d_2,2,0)$:

$$\delta(d_1,d_2,T,0)=\delta(d_1,d_2,1,0)\prod_{s=k}^{kT-1}\left(1-\frac{d_1+d_2}{2s+1}\right)+$$
$$+\sum_{t=1}^{T-1}\varepsilon(d_1,d_2,t,0)\prod_{s=k(t+1)}^{kT-1}\left(1-\frac{d_1+d_2}{2s+1}\right). \qquad (23)$$

The number of s such that $1-\dfrac{d_1+d_2}{2s}\leqslant 0$ depends only on d_1 and d_2. Thus, the part of the product corresponding to all such values of s is $O_{k,d_1,d_2}(1)$. We already obtained a bound for the part of the product corresponding to positive multipliers in (17):

$$\prod_{s=s_0}^{kT-1}\left(1-\frac{d_1+d_2}{2s+1}\right)\leqslant\left(\frac{2s_0+1}{2kT+1}\right)^{\frac{d_1+d_2}{2}}$$

if $s_0 > (d_1+d_2)/2$. First, let us consider the terms of (23) with the product starting from $(d_1+d_2)/2$ or below. Each of these terms is $O_{k,d_1,d_2}(T^{-\frac{d_1+d_2}{2}}) = O_{k,d_1,d_2}(1/T)$. The number of all these terms is $O_{k,d_1,d_2}(1)$, so the sum of all these terms is $O(1/T)$. Now, let us consider the other terms of (23):

$$\sum_{t=t_0}^{T-1}\varepsilon(d_1,d_2,t,0)\prod_{s=k(t+1)}^{kT-1}\left(1-\frac{d_1+d_2}{2s+1}\right)\leqslant\sum_{t=t_0}^{T-1}\varepsilon(d_1,d_2,t,0)\left(\frac{2k(t+1)+1}{2kT+1}\right)^{\frac{d_1+d_2}{2}}=$$

$$=O_{k,d_1,d_2}\left(\sum_{t=t_0}^{T-1}\frac{1}{t^2}\left(\frac{t}{T}\right)^{\frac{d_1+d_2}{2}}\right)=O_{k,d_1,d_2}\left(T^{-\frac{d_1+d_2}{2}}\sum_{t=t_0}^{T-1}t^{\frac{d_1+d_2}{2}-2}\right).$$

The function $t^{\frac{d_1+d_2}{2}-2}$ is either monotonically increasing or monotonically decreasing. In the first case, the value at the point t does not exceed the integral over $[t,t+1]$. In the second case, the same value does not exceed the integral over $[t-1,t]$. The sum of the values at the consecutive integer points does not exceed an integral with the upper bound $T-1$ or T. Note that $d_1+d_2 \geqslant 2k+1 \geqslant 3$ and $(d_1+d_2)/2-2 \neq -1$. Thus, the indefinite integral is proportional to the power function with the exponent equal to $(d_1+d_2)/2-1$. Finally, we obtain the following bound for $\delta(d_1,d_2,T,0)$:

$$O_{k,d_1,d_2}\left(T^{-\frac{d_1+d_2}{2}}T^{\frac{d_1+d_2}{2}-1}\right)=O_{k,d_1,d_2}\left(T^{-1}\right).$$

The case $s \neq 0$ follows from this bound and from (22).

Statement 4 is proved. □

3.6. Proof of tight concentration for X

We follow the proof of the analogous statement for the number of nodes of fixed degree from the paper [6].

We use the inequality of Azuma—Hoeffding.

Statement 5. ([1], [8]) Let $(X_s)_{s=0}^n$ be a martingale such that $|X_{s+1} - X_s| \leqslant \delta$ for $s = 0, \dots, n-1$. Let $x > 0$. Then

$$P(|X_n - X_0| \geqslant x) \leqslant 2 \exp\left(-\frac{x^2}{2\delta^2 n}\right).$$

Let d_1, d_2, k, t be fixed. Let us investigate the random sequence $X_s = E(X|G^{(s)})$, $s = 0, \dots, kt$. Obviously, $X_0 = EX$, $X_{kt} = X$. The sequence X_s is a martingale due to the definition of G_{kt}. We now prove a bound for the difference of adjacent terms.

Fix a value of s, $0 \leqslant s \leqslant kt-1$. Let v be the conglomerate in the graph $G^{(s+1)}$ connected to the last node of this graph. Thus, v is a random quantity depending on G. By definition

$$X_s = \sum_\gamma P(v = \gamma) E\big(X|G^{(s)},\ v = \gamma\big),$$
$$X_{s+1} = E\big(X|G^{(s)},\ v = v(G^{(s+1)})\big),$$

where γ runs over all conglomerates of $G^{(s+1)}$. Therefore,

$$\min_\gamma E\big(X|G^{(s)},\ v = \gamma\big) \leqslant X_s,\ X_{s+1} \leqslant \max_\gamma E\big(X|G^{(s)},\ v = \gamma\big),$$
$$|X_s - X_{s+1}| \leqslant \max_\gamma E\big(X|G^{(s)},\ v = \gamma\big) - \min_\gamma E\big(X|G^{(s)},\ v = \gamma\big). \qquad (24)$$

Let $\gamma_1 \in \arg\min_\gamma E\big(X|G^{(s)},\ v = \gamma\big)$ and $\gamma_2 \in \arg\max_\gamma E\big(X|G^{(s)},\ v = \gamma\big)$. It is sufficient to obtain a bound for

$$E(X|G^{(s)},\ v = \gamma_2) - E(X|G^{(s)},\ v = \gamma_1).$$

Note that

$$X = \sum_{\alpha=1}^{t} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{t} [\deg \alpha = d_1,\ \deg \beta = d_2] N(\alpha, \beta). \qquad (25)$$

Let us replace the condition $v = \gamma_1$ with the condition $v = \gamma_2$. This changes the distributions of the degrees of the nodes γ_i. This also changes the distributions of $N(\gamma_i, *) = N(*, \gamma_i)$. The probability of a node to be one of the ends of a new edge depends only on the degree of this node. Thus, the distributions of the other values of N do not change. Therefore, the distributions of most terms in the sum (25) are the same for $v = \gamma_1$ and $v = \gamma_2$. The exceptions are the terms with $\{\gamma_1, \gamma_2\} \cap \{\alpha, \beta\} \neq \varnothing$. Let

$$X' = \sum_{\alpha=1}^{t} \sum_{\substack{\beta=1 \\ \beta\neq\alpha \\ \{\alpha,\beta\}\cap\{\gamma_1,\gamma_2\}\neq\varnothing}}^{t} [\deg\alpha = d_1,\ \deg\beta = d_2] N(\alpha, \beta).$$

Then

$$E\big(X - X'|G^{(s)},\ v = \gamma_1\big) = E\big(X - X'|G^{(s)},\ v = \gamma_2\big). \tag{26}$$

Obviously, $X' \geqslant 0$. Let us deduce an upper bound:

$$X' \leqslant \sum_{\alpha=1}^{t} [\deg\alpha = d_1,\ \deg\gamma_1 = d_2] N(\alpha, \gamma_1) + \sum_{\alpha=1}^{t} [\deg\alpha = d_1,\ \deg\gamma_2 = d_2] N(\alpha, \gamma_2) +$$

$$+ \sum_{\beta=1}^{t} [\deg\gamma_1 = d_1,\ \deg\beta = d_2] N(\gamma_1, \beta) + \sum_{\beta=1}^{t} [\deg\gamma_2 = d_1, \deg\beta = d_2] N(\gamma_2, \beta) \leqslant$$

$$\leqslant [\deg\gamma_1 = d_2] \sum_{\alpha=1}^{t} N(\alpha, \gamma_1) + [\deg\gamma_2 = d_2] \sum_{\alpha=1}^{t} N(\alpha, \gamma_2) +$$

$$+ [\deg\gamma_1 = d_1] \sum_{\beta=1}^{t} N(\gamma_1, \beta) + [\deg\gamma_2 = d_1] \sum_{\beta=1}^{t} N(\gamma_2, \beta) =$$

$$= [\deg\gamma_1 = d_2] d_2 + [\deg\gamma_2 = d_2] d_2 + [\deg\gamma_1 = d_1] d_1 + [\deg\gamma_2 = d_1] d_1 \leqslant$$

$$\leqslant 2(d_1 + d_2).$$

Therefore,

$$0 \leqslant E\big(X'|G^{(s)}, v = \gamma_1\big),\ E\big(X'|G^{(s)},\ v = \gamma_2\big) \leqslant 2(d_1 + d_2),$$

$$\big|E\big(X'|G^{(s)},\ v = \gamma_1\big) - E\big(X'|G^{(s)},\ v = \gamma_2\big)\big| \leqslant 2(d_1 + d_2).$$

Combining this inequality with (24) and (26), we get

$$\big|X_s - X_{s+1}\big| \leqslant 2(d_1 + d_2).$$

Thus, the sequence (X_s) satisfies the conditions of Statement 5 with $n = kt$ and $\delta = 2(d_1 + d_2)$. Theorem 3 follows from Statement 5 with $x = c(d_1 + d_2)\sqrt{kt}$.

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