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On the distribution of values of Hardy's Z-function in short intervals

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Abstract: Given a primitive Dirichlet character χ of conductor $q > 1$, Hardy's Z-function $Z(t,\chi)$ is a real-valued function with the property that $|Z(t,\chi)| = |L(1/2+it,\chi)|$ for $t \in \mathbb{R}$, so that the real zeros of $Z(t,\chi)$ correspond to the zeros of the Dirichlet L-function $L(s,\chi)$ on the critical line $\operatorname{Re} s = 1/2$. In this note, we study the distribution of positive and negative values of $Z(t,\chi)$. We show that $Z(t,\chi)$ takes both positive and negative values for a positive proportion of values of t in $[T, T+H]$ where $T^{3/4+\varepsilon} \leqslant H \leqslant T$ with $0 < \varepsilon < 1/4$. More precisely, we show that $\mu\left(I_\pm(T,H)\right) \gg H$ where μ denotes the Lebesgue measure on the line and $I_\pm(T,H)$ is the set of $t \in [T, T+H]$ such that $Z(t,\chi) > 0$ (resp. $Z(t,\chi) < 0$). This extends and improves upon a recent result of Gonek and Ivić for Hardy's Z-function corresponding to the Riemann zeta-function $\zeta(s)$ and the proof we give here, although closely following theirs, is different at a crucial point.

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1. Introduction and statement of results

Hardy's Z-function $Z(t,\chi)$ for a Dirichlet L-function $L(s,\chi)$ corresponding to a primitive Dirichlet character χ of conductor q is defined by

$$Z(t,\chi) := \Psi(1/2+it,\chi)^{-1/2} L(1/2+it,\chi)$$

where

$$\Psi(s,\chi) = \mathfrak{w}(\chi)\left(\frac{\pi}{q}\right)^{s-1/2}\frac{\Gamma((1-s+\mathfrak{a})/2)}{\Gamma((s+\mathfrak{a})/2)}$$

is the factor from the functional equation $L(s,\chi) = \Psi(s,\chi)L(1-s,\overline{\chi})$. Recall that $\mathfrak{a} = (1-\chi(-1))/2$ measures the parity of the character χ and that $\mathfrak{w}(\chi) = \tau(\chi)/\left(i^{\mathfrak{a}}\sqrt{q}\right)$ with $\tau(\chi) = \sum_{a \pmod q} \chi(a)e\left(a/q\right)$ is called the root number of χ. With this data, it is immediately seen that $Z(t,\chi)$ is a real-valued function of the real variable t and that $|Z(t,\chi)| = |L(1/2+it,\chi)|$, so that the real zeros of $Z(t,\chi)$ correspond to the zeros of $L(s,\chi)$ on the critical line. One of the main interests of Hardy's Z-functions stems from this fact. For an introduction to the theory of Hardy's Z-function and a survey of open problems regarding it, see the Cambridge Tract [6]. Besides its use as a tool for studying the zeros of the zeta and L-functions, it has its own intrinsic interest as a number-theoretic function. Some striking properties of the primitive $\int_0^T Z(t)\,\mathrm{d}t$ are proved for example in [7] (see Theorem 1). Hardy's function $Z(t)$ for $\zeta(s)$ was employed by Hardy and Littlewood to prove Hardy's theorem (which was proved previously by Hardy [4]) that $\zeta(s)$ has infinitely many zeros on the critical line $\operatorname{Re} s = 1/2$ in the following way: We have

$$\int_T^{2T} Z(t)\,\mathrm{d}t \ll T^{7/8}$$

whereas

$$\int_T^{2T} |Z(t)|\ \mathrm{d}t \gg T$$

for T big enough, showing that $Z(t)$ is not ultimately of constant sign. This method of proof of Hardy's theorem was first carried out by Landau in [8] (see the introductory remarks in [7], also Theorem 8 of Chapter II of [1] and the end-note to that chapter). A natural question to ask then is: For what proportion of $t \in [T, 2T]$ is $Z(t)$ positive? Somewhat more generally, given $2 \leqslant H \leqslant T$, what is the Lebesgue measure of the set $\{t \in [T, T+H] : Z(t) > 0\}$? Call this set $I_+(T,H)$ and let $I_-(T,H)$ denote the corresponding set for which $Z(t)$ is negative. Recently, Gonek and Ivić [3] gave a short, elegant proof that

$$\mu\,(I_+(T,T)) \gg T \qquad \text{and} \qquad \mu\,(I_-(T,T)) \gg T \tag{1}$$

where μ is the Lebesgue measure on the line.

In this paper, we propose to extend this result of Gonek and Ivić to the case of Hardy's functions for Dirichlet L-functions. Throughout this note, we fix a primitive character χ of conductor $q > 1$. For $2 \leqslant H \leqslant T$, we write

$$I_+(T,H;\chi) = \{T < t \leqslant T+H : Z(t,\chi) > 0\},$$
$$I_-(T,H;\chi) = \{T < t \leqslant T+H : Z(t,\chi) < 0\}.$$

Our result is the following:

THEOREM 1. *Fix $0 < \varepsilon < 1/4$. Let $T^{3/4+\varepsilon} \leqslant H \leqslant T$. Then we have*

$$\mu\,(I_+(T,H;\chi)) \gg H \quad \textit{and} \quad \mu\,(I_-(T,H;\chi)) \gg H. \tag{2}$$

Our method of proof, which follows, but differs at one major point from, the proof of (1) in [3], does not seem to allow us to prove a result much stronger than (2), although it does more easily allow us to replace the $\gg$ symbols by a constant. Indeed, we have

$$\mu\,(I_+(T,H;\chi)) \geqslant (1/4c)H, \tag{3}$$

where c is an upper bound for the coefficient of H in (12) with $0 < \theta < 1/8$, as in the statement of Lemma 8. In particular, it seems unlikely that this method will lead to a considerable reduction of the length H of the interval. Also, it does not distinguish between the two sets $I_+(T,H;\chi)$ and $I_-(T,H;\chi)$, in the sense that it gives the same lower bound for them. In any case, the two sets are conjectured to be roughly equal in size, at least for the case of $\zeta(s)$ (see Table 1 and Table 2 in [3] and the remarks prior to the Theorem 1 there).

We remark that for the purpose of the current note, we fix the conductor $q > 1$ and also the character χ, so that what happens in the current setting can be adapted to the case of $\zeta(s)$ and vice versa, except that $H/4c$ in (3) will have to be replaced by $H/12c$ (see the remark following (11) below). The "q-aspect" will be the subject of our subsequent investigation.

2. Proof of the theorem

The idea for proving Theorem 1 consists in the following. We will define a mollifying Dirichlet polynomial $B_X(s,\chi)$ which is roughly equal to the Dirichlet series for $L(s,\chi)^{-1/2}$ truncated at length X. We then study the average sizes of the functions $Z(t,\chi)\,|B_X(1/2+it,\chi)|^2$, $|Z(t,\chi)|\,|B_X(1/2+it,\chi)|^2$ and $Z(t,\chi)^2\,|B_X(1/2+it,\chi)|^4$ over the interval $[T,T+H]$. Since $B_X(s,\chi)$ is roughly equal to the Dirichlet series for $L(s,\chi)^{-1/2}$ truncated at length X and since $Z(t,\chi)$ is equal to $L(1/2+it,\chi)$ in absolute value, it is reasonable to expect that these three functions are close to 1 and that their respective integrals over $[T,T+H]$ are $\approx H$. Indeed, we have the following three inequalities:

1. The asymptotic inequality

$$\int_T^{T+H} Z(t,\chi)\,|B_X(1/2+it,\chi)|^2\ \mathrm{d}t = o(H)$$

   holds for $X=T^{\theta}, H=T^{\vartheta}$ with $0<\theta<1/4$ and $3/4+\theta<\vartheta\leqslant 1$.

2. We have

$$\int_T^{T+H} |Z(t,\chi)|\,|B_X(1/2+it,\chi)|^2\ \mathrm{d}t \geqslant H+o(H)$$

   for $X=T^{\theta}, H=T^{\vartheta}$ with $0<\theta<1/2$ and $1/2+\theta<\vartheta\leqslant 1$.

3. Finally,

$$\int_T^{T+H} Z(t,\chi)^2\,|B_X(1/2+it,\chi)|^4\ \mathrm{d}t \ll H$$

   for $X=T^{\theta}, H=T^{\vartheta}$ with $0<\theta<1/8$ and $3/4+2\theta\leqslant\vartheta\leqslant 1$.

We remark that for the third inequality above, Gonek and Ivić [3] require $0<\theta<1/100$. We have a better condition and our argument for proving it is different and independent of theirs.

Given the above three asymptotic inequalities, the proof of Theorem 1 runs as follows. Write for short $I_+ = I_+(T, H; \chi)$. Then clearly

$$\int_{I_+} Z(t,\chi)\,|B_X(1/2+it,\chi)|^2\ \mathrm{d}t = \frac{1}{2}\Bigg(\int_T^{T+H} Z(t,\chi)\,|B_X(1/2+it,\chi)|^2\ \mathrm{d}t + \int_T^{T+H} |Z(t,\chi)|\,|B_X(1/2+it,\chi)|^2\ \mathrm{d}t\Bigg) \geqslant \frac{1}{2}H + o(H)$$

by the first two inequalities above. Using Cauchy-Schwarz inequality and the third inequality above, we therefore have

$$\frac{1}{2}H + o(H) \leqslant \mu(I_+)^{1/2}\Bigg(\int_T^{T+H} Z(t,\chi)^2\,|B_X(1/2+it,\chi)|^4\ \mathrm{d}t\Bigg)^{1/2} \ll \mu(I_+)^{1/2}H^{1/2}$$

so that $\mu(I_+) \gg H$. This proves the first estimate in Theorem 1. The second is proved similarly.

The three inequalities above are derived easily from Lemmas 6, 7 and 8 respectively in the next section.

3. Lemmas

In this section, we give several lemmas essential for proving the main theorem. The three key inequalities stated in the preceding section are also restated and proved as lemmas.

First, we quote [11, 12] (see also [2]) for the following form of the approximate functional equation for $L(s,\chi)$.

LEMMA 2. *Let H and K be positive constants. Let us write $2\pi xy = q\,|t|$. Then the following approximate functional equation holds uniformly for $-H \leqslant \sigma \leqslant H, x > K$, $y > K$:*

$$L(s,\chi) = \sum_{n\leqslant x} \frac{\chi(n)}{n^s} + \mathfrak{w}(\chi)\left(\frac{\pi}{q}\right)^{s-1/2} \frac{\Gamma\left((1-s+\mathfrak{a})/2\right)}{\Gamma\left((s+\mathfrak{a})/2\right)} \sum_{n\leqslant y} \frac{\overline{\chi}(n)}{n^{1-s}} + O(x^{-\sigma}) + O\left(y^{\sigma-1}\,|t|^{1/2-\sigma}\right).$$

Using this, the following lemma is immediate (take $x = y$).

LEMMA 3. *We have the following approximate functional equation for* $Z(t, \chi)$:

$$Z(t, \chi) = \Theta(t, t, \chi) + \overline{\Theta}(t, t, \chi) + O(|t|^{-\frac{1}{4}})$$

where

$$\Theta(t, u, \chi) = \mathfrak{w}(\chi)^{-1/2} e^{(\pi i \mathfrak{a}/4)-(\pi i/8)} \left(\frac{qt}{2\pi e}\right)^{it/2} \sum_{n \leqslant \sqrt{q|u|/(2\pi)}} \frac{\chi(n)}{n^{1/2+it}}. \tag{4}$$

In particular, for $T \leqslant t \leqslant 2T$, *we have*

$$Z(t, \chi) = \Theta(t, \chi) + \overline{\Theta}(t, \chi) + O(T^{-1/4}) \tag{5}$$

where $\Theta(t, \chi) = \Theta(t, T, \chi)$.

We shall also need the following simple approximate equation.

LEMMA 4. *We have*

$$L(s, |\chi|) = \sum_{n \leqslant x} \frac{|\chi(n)|}{n^s} - \frac{\varphi(q)}{q} \frac{x^{1-s}}{1-s} + O(x^{-\sigma})$$

uniformly for $\sigma \geqslant \frac{1}{2}, |t| \leqslant x$. *Also, as* $s \to 1$, *we have*

$$(s-1)L(s, |\chi|) = \frac{\varphi(q)}{q} + O(|s-1|).$$

Following [9] and using the notation of [3], we write

$$L(s, \chi)^{-1/2} = \sum_{n=1}^{\infty} \alpha_n(\chi) n^{-s} \quad (\mathrm{Re}\, s > 1).$$

Making use of the Euler product representation for $L(s, \chi)$, we obtain

$$\alpha_n(\chi) = (-1)^{e_1+\cdots+e_k} \binom{1/2}{e_1} \cdots \binom{1/2}{e_k} \chi(n) \tag{6}$$

where $n = p_1^{e_1} \cdots p_k^{e_k}$ is the factorisation of n. We see that $\alpha_n = \alpha_n(\chi)$ is multiplicative, that is, $\alpha_{mn} = \alpha_m \alpha_n$ provided $(m,n) = 1$, and that $|\alpha_n| \leqslant 1$ for all $n \geqslant 1$. For $1 \leqslant n \leqslant X$ let

$$\beta_n \equiv \beta_n(\chi) = \alpha_n(\chi)\left(1 - \frac{\log n}{\log X}\right);$$

it is clear that $|\beta_n| \leqslant 1$. Define the mollifying function

$$B_X(s,\chi) = \sum_{n \leqslant X} \beta_n n^{-s}$$

which corresponds to the sum $\eta(t)$ used by Selberg [9] (see also [10], pp. 85–141). Write

$$B_X(s,\chi)^2 = \sum_{n \leqslant X^2} b_n(\chi) n^{-s} \tag{7}$$

where $b_n(\chi) = \sum_{d|n} \beta_d(\chi)\beta_{n/d}(\chi)$ with the sum running through those $d \leqslant X, d|n$ such that $n/d \leqslant X$; note that $|b_n(\chi)| \leqslant d(n)$ where $d(n)$ is the usual divisor function.

The following lemma gives an estimate for a weighted Dirichlet polynomial involving $\alpha_n(|\chi|)$ as coefficients.

LEMMA 5. *Let* $1 \leqslant d \leqslant X$, $0 \leqslant \gamma \leqslant 1/\sqrt{\log X}$ *and* ϱ *be a positive integer. Then, for* $r = 2,3$,

$$\sum_{\substack{n \leqslant X/d \\ (n,\varrho)=1}} \frac{\alpha_n(|\chi|)}{n^{1+i\gamma}} \left(\log \frac{X}{dn}\right)^{r-1} = c_r \prod_{p|\varrho} \left(1 - |\chi(p)|\, p^{-1-i\gamma}\right)^{-1/2} \sqrt{\gamma} \left(\log \frac{X}{d}\right)^{r-1} + O\left(\prod_{p|\varrho} (1 + |\chi(p)|\, p^{-3/4})(\log X)^{r-3/2}\right)$$

where c_r *is a constant. Here and in the following*, (m,n) *denotes the* gcd *of two integers* m *and* n.

This lemma is analogous to Lemma 12 in [9] and can be proved in the same way.

The following three lemmas are at the heart of the proof of the main theorem.

LEMMA 6. *Let* $X = T^\theta$ *with* $\theta > 0$ *and* $H = T^\vartheta$ *with* $0 < \vartheta < 1$. *Then*

$$\int_T^{T+H} Z(t,\chi)\,|B_X(1/2+it,\chi)|^2\,\mathrm{d}t = o(T^{3/4+\theta}\log^2 T) \quad (T\to\infty).$$

PROOF. By the definition of $Z(t,\chi)$, this integral can be rewritten as

$$\frac{1}{i}\int_{1/2+iT}^{1/2+i(T+H)} L(s,\chi)\Psi(s,\chi)^{-1/2}B_X(s,\chi)B_X(1-s,\overline{\chi})\,\mathrm{d}s.$$

Cauchy's theorem allows us to replace the path of integration by the rectangular path going from $\frac{1}{2}+iT$ to $c+iT$, then to $c+i(T+H)$ and then to $\frac{1}{2}+i(T+H)$, where $c = 1+1/\log T$. We first estimate the contribution of the integrals along the two horizontal paths. Now, for $\sigma \geqslant -1$, we have

$$B_X(s,\chi) \ll \max\{X^{1-\sigma},\log X\} \quad \text{and} \quad B_X(1-s,\overline{\chi}) \ll X^\sigma. \tag{8}$$

Also, by Stirling's formula, we have

$$\Psi(\sigma+it,\chi) = \mathfrak{w}(\chi)\left(\frac{2\pi}{qt}\right)^{\sigma+it-1/2} e^{it+\pi i(1-2\mathfrak{a})/4}\big(1+O(1/t)\big) \quad (t\geqslant t_0). \tag{9}$$

As for the $L(s,\chi)$ factor, we have the convexity bound

$$L(\sigma+it,\chi) \ll \left(t^{(1-\sigma)/3}\log t+1\right)\log t \quad (\sigma\geqslant A, t\geqslant t_0).$$

It follows from these estimates that the overall contribution of the integral along the horizontal segments is

$$\ll \int_{\frac{1}{2}}^{c} \left(T^{(1-\sigma)/3}\log T+1\right)T^{(\sigma-1/2)/2}\max\{X^{1-\sigma},\log X\}X^\sigma\log T\,\mathrm{d}\sigma \ll XT^{1/4}\log T.$$

On the vertical segment, the Dirichlet series for $L(s,\chi)$ converges absolutely; using this fact together with (8) and (9), we see that the integral along the vertical segment is

$$\mathfrak{w}(\chi)^{-\frac{1}{2}} \int_T^{T+H} \left(\sum_{n \geqslant 1} \chi(n) n^{-c-it} \right) \left(\sum_{k \leqslant X} \beta_k(\chi) k^{-c-it} \right) \left(\sum_{l \leqslant X} \beta_l(\overline{\chi}) l^{c+it-1} \right) \times \left(\frac{qt}{2\pi} \right)^{(c+it-1/2)/2} e^{-i(t+\pi(1-2\mathfrak{a})/4)/2} \big(1 + O(1/t)\big)\, \mathrm{d}t,$$

of which the O-term has a contribution

$$\ll \int_T^{T+H} (\log^2 T) X^{1-(1/\log T)} t^{(c-1/2)/2-1}\, \mathrm{d}t \ll XT^{1/4} \log^2 T.$$

Omitting the constant coefficients, the remaining expression can be rewritten as

$$\sum_{n \geqslant 1} \sum_{k \leqslant X} \sum_{l \leqslant X} \frac{\chi(n)\beta_k(\chi)\beta_l(\overline{\chi}) l^{c-1}}{(nk)^c} \int_T^{T+H} \left(\frac{qt}{2\pi} \right)^{(c-1/2)/2} \exp\left(\frac{it}{2} \log\left(\frac{qtl^2}{2\pi e n^2 k^2} \right) \right) \mathrm{d}t.$$

As in the proof of Lemma 2 in [3], using the fact that $|\chi(n)| \leqslant 1, |\beta_m| \leqslant 1$, we conclude that this last expression is $\ll T^{3/4} X \log^2 T$. This completes the proof of the lemma. □

Next, we have the following lemma.

LEMMA 7. *Let $X = T^{\theta}$ with $0 < \theta < 1/2$ and $H = T^{\vartheta}$ with $1/2 + \theta < \vartheta \leqslant 1$. Then we have*

$$\int_T^{T+H} |Z(t, \chi)|\, |B_X(1/2 + it, \chi)|^2\, \mathrm{d}t \geqslant H + O(\sqrt{T} X \log X) \quad (T \to \infty).$$

PROOF. We have

$$\int_T^{T+H} |Z(t, \chi)|\, |B_X(1/2 + it, \chi)|^2\, \mathrm{d}t = \int_T^{T+H} |L(1/2 + it, \chi)|\, |B_X(1/2 + it, \chi)|^2\, \mathrm{d}t \geqslant \left| \int_T^{T+H} L(1/2 + it, \chi) B_X(1/2 + it, \chi)^2\, \mathrm{d}t \right|.$$

Using the approximate formula

$$L(1/2+it,\chi)=\sum_{n\leqslant T}\chi(n)n^{-1/2-it}+O(T^{-1/2})\quad (T\leqslant t\leqslant 2T),$$

(which is easily derived from the method of proof of Theorem 1.8 in [5] or from Lemma 2 above), we have

$$\left|\int_T^{T+H} L(1/2+it,\chi)B_X(1/2+it,\chi)^2\mathrm{d}t\right|\geqslant\left|\int_T^{T+H}\sum_{m\leqslant T}\chi(m)m^{-1/2-it}\sum_{n\leqslant X^2}b_n(\chi)n^{-1/2-it}\mathrm{d}t\right| +O\left(T^{-1/2}\int_T^{T+H}\left|\sum_{n\leqslant X}\beta_n(\chi)n^{-1/2-it}\right|^2\mathrm{d}t\right).$$

Using the mean-value theorem for Dirichlet polynomials (see Theorem 5.2 of [5]), the integral inside the O-term is easily seen to be

$$H\sum_{n\leqslant X}\frac{|\beta_n(\chi)|^2}{n}+O\left(\sum_{n\leqslant X}|\beta_n(\chi)|^2\right)=O\left(H\log X\right)+O(X)=O(H\log X)$$

so that the contribution of the O-term is $O\left(T^{-1/2}H\log X\right)$.

It is easy to see that

$$\begin{aligned}\int_T^{T+H}\sum_{m\leqslant T}\chi(m)m^{-1/2-it}\sum_{n\leqslant X^2}b_n(\chi)n^{-1/2-it}\,\mathrm{d}t&=H+\sum_{\substack{m\leqslant T,n\leqslant X^2\\ mn>1}}\frac{\chi(m)b_n(\chi)}{\sqrt{mn}}\int_T^{T+H}(mn)^{-it}\,\mathrm{d}t\\ &=H+O\left(\sum_{\substack{m\leqslant T,n\leqslant X^2\\ mn>1}}\frac{d(n)}{\sqrt{mn}\log(mn)}\right)\\ &=H+O\left(\sqrt{T}X\log X\right).\end{aligned}$$ □

The result follows.

Finally, we have the following lemma.

LEMMA 8. *Let $X = T^{\theta}$ with $0 < \theta < 1/8$ and $H = T^{\vartheta}$ with $3/4 + 2\theta \leqslant \vartheta < 1$. Then*

$$\int_T^{T+H} Z(t,\chi)^2 \left|B_X(1/2+it,\chi)\right|^4 \, \mathrm{d}t \ll H \quad (T \to \infty).$$

PROOF. Let us consider the integral

$$\int_T^{T+H} Z(t,\chi)^2 \left|B_X(1/2+it,\chi)\right|^4 \, \mathrm{d}t.$$

Using the approximate equation (5) of Lemma 3, we have

$$\begin{aligned}\int_T^{T+H} Z(t,\chi)^2 \left|B_X(1/2+it,\chi)\right|^4 \, \mathrm{d}t &= \int_T^{T+H} \Theta(t,\chi)^2 \left|B_X(1/2+it,\chi)\right|^4 \, \mathrm{d}t \qquad (10)\\ + \int_T^{T+H} \overline{\Theta}(t,\chi)^2 \left|B_X(1/2+it,\chi)\right|^4 \, \mathrm{d}t &+ 2\int_T^{T+H} \left|\Theta(t,\chi)\right|^2 \left|B_X(1/2+it,\chi)\right|^4 \, \mathrm{d}t \\ &+ O\left(T^{-1/4} \int_T^{T+H} \left|\Theta(t,\chi)\right| \left|B_X(1/2+it,\chi)\right|^4 \, \mathrm{d}t\right) \\ &+ O\left(T^{-1/2} \int_T^{T+H} \left|B_X(1/2+it,\chi)\right|^4 \, \mathrm{d}t\right).\end{aligned}$$

Using the trivial bound

$$\left|B_X(1/2+it,\chi)\right| \leqslant \sum_{n \leqslant X} \frac{1}{\sqrt{n}} \ll \sqrt{X},$$

we see that the last O-term is $O\left(T^{-1/2} H X^2\right)$, and the first O-term is

$$O\left(T^{-1/4} X^2 \int_T^{T+H} \left|\Theta(t,\chi)\right| \, \mathrm{d}t\right).$$

By the Cauchy-Schwartz inequality and the mean-value theorem for Dirichlet series, the expression (5) gives

$$\begin{aligned}\int_T^{T+H} |\Theta(t,\chi)|\ \mathrm{d}t &\leqslant \sqrt{H}\left(\int_T^{T+H}\left|\sum_{n\leqslant\sqrt{qT/2\pi}}\frac{\chi(n)}{n^{1/2+it}}\right|^2\ \mathrm{d}t\right)^{1/2}\\ &= \sqrt{H}\left(H\sum_{n\leqslant\sqrt{qT/2\pi}}\frac{|\chi(n)|}{n}+O\left(\sum_{n\leqslant\sqrt{qT/2\pi}}|\chi(n)|\right)\right)^{1/2}\\ &= O\left(H\sqrt{\log T}\right).\end{aligned}$$

Hence, the first O-term in (3) is $O\left(T^{-1/4}HX^2\sqrt{\log T}\right)$.

Since the first two integrals in (3) are bounded in absolute value by the third integral, it is enough to look at the third integral, namely

$$J=\int_T^{T+H}|\Theta(t,\chi)|^2\,|B_X(1/2+it,\chi)|^4\ \mathrm{d}t. \tag{11}$$

(One may in fact prove that the other two integrals are $o(H)$ when $q>1$, using the Pólya-Vinogradov inequality, but not for the case of $\zeta(s)$.) Write $\tau=\sqrt{qT/2\pi}$. Using the expressions (7) and (5), we have

$$\begin{aligned}J &= \sum_{\substack{k,l\leqslant\tau\\ m,n\leqslant X^2}}\frac{\chi(k)\overline{\chi}(l)b_m(\chi)b_n(\overline{\chi})}{\sqrt{klmn}}\int_T^{T+H}\left(\frac{kn}{lm}\right)^{it}\ \mathrm{d}t\\ &= H\sum_{\substack{k,l\leqslant\tau;m,n\leqslant X^2\\ kn=lm}}\frac{\chi(k)\overline{\chi}(l)b_m(\chi)b_n(\overline{\chi})}{\sqrt{klmn}}\\ &\quad+\sum_{\substack{k,l\leqslant\tau;m,n\leqslant X^2\\ kn\neq lm}}\frac{\chi(k)\overline{\chi}(l)b_m(\chi)b_n(\overline{\chi})}{i\sqrt{klmn}\log(kn/lm)}\left((kn/lm)^{i(T+H)}-(kn/lm)^{iT}\right).\end{aligned} \tag{12}$$

Using the fact that $|b_m(\chi)|\leqslant d(n), |b_n(\overline{\chi})|\leqslant d(n)$, we see that the second sum above is $O\left(T^{3/4}X^2\log^2 T\right)$. We will now estimate the first sum. We have to show that it is $\ll 1$. Note first that

$$\sum_{\substack{k,l\leqslant\tau;m,n\leqslant X^2\\ kn=lm}}\frac{\chi(k)\overline{\chi}(l)b_m(\chi)b_n(\overline{\chi})}{\sqrt{klmn}}=\sum_{m,n\leqslant X^2}\frac{b_m(\chi)b_n(\overline{\chi})}{mn}\chi(m)\overline{\chi}(n)g\sum_{k\leqslant\tau_0}\frac{|\chi(k)|}{k} \quad (13)$$

where $\tau_0=g\tau/\max(m,n)$ and $g=(m,n)$ in each term. From the expression (6) for $\alpha_n(\chi)$, we see that $\alpha_n(\chi)\overline{\chi}(n)=\alpha_n(|\chi|)$ and that the sum in (12) is real. Expanding the b_m, we define, for $\gamma\geqslant 0$,

$$S(\gamma)=\mathrm{Re}\left\{\tau^{i\gamma}\sum_{n_j\leqslant X}\frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_1n_2n_3n_4}\frac{g^{1+i\gamma}}{(n_1n_3)^{i\gamma}}\right.$$
$$\left.\times\sum_{k\leqslant\tau_1}\frac{|\chi(k)|}{k^{1+i\gamma}}\right\}.$$

where we have written $\tau_1=g\tau/\max(n_1n_2,n_3n_4)$ with g now standing for (n_1n_2,n_3n_4). Note that the right side of (13) is equal to $S(0)$. We will show that $S(\gamma)=O(1)$ for $0<\gamma\leqslant 1/\log T$; it will follow by continuity that $S(0)=O(1)$ as well. The proof is essentially contained in Selberg's proof that $K(\gamma)=O(1)$ in [9, pp. 27–31]. However, there are major differences in our treatment of the sums. First of all, we have

$$\sum_{k\leqslant\tau_1}\frac{|\chi(k)|}{k^{1+i\gamma}}=\sum_{k\leqslant\tau_2}\frac{|\chi(k)|}{k^{1+i\gamma}}+O(\log T)$$

with $\tau_2=g\tau/n_1n_2$. So

$$S(\gamma)=\mathrm{Re}\left\{\tau^{i\gamma}\sum_{n_j\leqslant X}\frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_1n_2n_3n_4}\frac{g^{1+i\gamma}}{(n_1n_3)^{i\gamma}}\sum_{k\leqslant\tau_2}\frac{|\chi(k)|}{k^{1+i\gamma}}\right\} \quad (14)$$
$$+O\left((\log T)\left|\sum_{n_j\leqslant X}\frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_1n_2n_3n_4}\frac{g^{1+i\gamma}}{(n_1n_3)^{i\gamma}}\right|\right).$$

Let us call the first term above $S_1(\gamma)$. Using Lemma 4, we can write

$$S_1(\gamma)=\mathrm{Re}\left\{\tau^{i\gamma}L(1+i\gamma,|\chi|)\sum_{n_j\leqslant X}\frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_1n_2n_3n_4}\frac{g^{1+i\gamma}}{(n_1n_3)^{i\gamma}}\right\}$$

$$+ \operatorname{Re}\left\{\frac{i\varphi(q)}{\gamma q}\sum_{n_j\leqslant X} g\frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_1n_2n_3n_4}\left(\frac{n_2}{n_3}\right)^{i\gamma}\right\}$$

$$+ O\left(T^{-1/2}X^2\left(\sum_{n\leqslant X}\frac{|\beta_n(|\chi|)|}{n}\right)^4\right).$$

In the second sum, when n_1, n_2, n_3, n_4 is permuted to n_4, n_3, n_2, n_1 the sign of the corresponding term is reversed (note that the terms corresponding to $n_2 = n_3$ do not matter, as there is a factor i and we take the real part). Hence the second term must vanish identically. Since $|\beta_n| \leqslant 1$, we see that the O-term is $O\left(T^{-1/2}X^2\log^4 T\right)$, so

$$S_1(\gamma) = \operatorname{Re}\left\{\tau^{i\gamma}L(1+i\gamma,|\chi|)\sum_{n_j\leqslant X}\frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_1n_2n_3n_4}\frac{g^{1+i\gamma}}{(n_1n_3)^{i\gamma}}\right\}$$

$$+ O(T^{-1/2}X^2\log^4 T).$$

Note that the sum in the first term above is the same as the sum inside the O-term in (14); let us call it $S_2(\gamma)$. Admit for the moment the truth of

$$S_2(\gamma) \ll \gamma. \tag{15}$$

Then, using the fact that $L(1+i\gamma,|\chi|) = \varphi(q)/(qi\gamma) + O(1)$ (see Lemma 4), it will follow that $S_1(\gamma) \ll 1$ and that the O-term in (14) is also $\ll 1$ for $0 < \gamma \leqslant 1/\log T$, which will imply that we indeed have $S(\gamma) \ll 1$ for $0 < \gamma \leqslant 1/\log T$. It therefore remains to prove (15).

As in [9, (4.25)], for any positive integer n, write

$$\varphi_{i\gamma}(n) = n^{1+i\gamma}\sum_{d|n}\frac{\mu(d)}{d^{1+i\gamma}} = n^{1+i\gamma}\prod_{p|n}\left(1 - p^{-1-i\gamma}\right).$$

This is a generalisation of Euler's totient $\varphi(n)$ and analogously to the identity $n = \sum_{d|n}\varphi(d)$, we have $n^{1+i\gamma} = \sum_{d|n}\varphi_{i\gamma}(d)$. Using this fact, $S_2(\gamma)$ may be written as

$$S_2(\gamma) = \sum_{\substack{n_j\leqslant X\\ d|n_1n_2,\, d|n_3n_4}}\varphi_{i\gamma}(d)\frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_2n_4(n_1n_3)^{1+i\gamma}}$$

$$
\begin{aligned}
&= \sum_{d \leqslant X^2} \varphi_{i\gamma}(d) \sum_{\substack{n_j \leqslant X \\ d|n_1n_2, d|n_3n_4}} \frac{\beta_{n_1}(|\chi|)\beta_{n_2}(|\chi|)\beta_{n_3}(|\chi|)\beta_{n_4}(|\chi|)}{n_2n_4(n_1n_3)^{1+i\gamma}} \\
&= \sum_{d \leqslant X^2} \varphi_{i\gamma}(d) \Bigg(\sum_{\substack{m,n \leqslant X \\ d|mn}} \frac{\beta_m(|\chi|)\beta_n(|\chi|)}{mn^{1+i\gamma}} \Bigg)^2 .
\end{aligned}
$$

In the following, d_1 and d_2 will always denote positive integers only divisible by primes which divide d. Since $\beta_n(|\chi|) = \alpha_n(|\chi|)\log(X/n)/\log X$ and α_n is multiplicative, we can write

$$
\begin{aligned}
\sum_{\substack{m,n \leqslant X \\ d|mn}} \frac{\beta_m(|\chi|)\beta_n(|\chi|)}{mn^{1+i\gamma}} = \frac{1}{\log^2 X} \sum_{\substack{d_1,d_2 \leqslant X^2 \\ d|d_1d_2}} \frac{\alpha_{d_1}(|\chi|)\alpha_{d_2}(|\chi|)}{d_1 d_2^{1+i\gamma}} \Bigg\{ \sum_{\substack{n \leqslant X/d_1 \\ (n,d)=1}} \frac{\alpha_n(|\chi|)}{n} \log \frac{X}{d_1 n} \Bigg\} \\
\times \Bigg\{ \sum_{\substack{n \leqslant X/d_2 \\ (n,d)=1}} \frac{\alpha_n(|\chi|)}{n^{1+i\gamma}} \log \frac{X}{d_2 n} \Bigg\}.
\end{aligned}
$$

Using Lemma 5, we have

$$
\begin{aligned}
\sum_{\substack{n \leqslant X/d_1 \\ (n,d)=1}} \frac{\alpha_n(|\chi|)}{n} \log \frac{X}{d_1 n} &= O\left(\sqrt{\log X} \prod_{p|d} \left(1 + |\chi(p)|\, p^{-3/4}\right) \right), \\
\sum_{\substack{n \leqslant X/d_2 \\ (n,d)=1}} \frac{\alpha_n(|\chi|)}{n^{1+i\gamma}} \log \frac{X}{d_2 n} &= O\left(\sqrt{\gamma}(\log X) \prod_{p|d} \left(1 + |\chi(p)|\, p^{-3/4}\right) \right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{\substack{m,n \leqslant X \\ d|mn}} \frac{\beta_m(|\chi|)\beta_n(|\chi|)}{mn^{1+i\gamma}} &\ll \sqrt{\frac{\gamma}{\log X}} \sum_{\substack{d_1,d_2 \leqslant X^2 \\ d|d_1d_2}} \frac{\left|\alpha_{d_1}(|\chi|)\alpha_{d_2}(|\chi|)\right|}{d_1 d_2} \prod_{p|d} \left(1+|\chi(p)|\, p^{-3/4}\right)^2 \\
&\ll \sqrt{\frac{\gamma}{\log X}} \sum_{\substack{d_1,d_2 \leqslant X^2 \\ d|d_1d_2}} \frac{1}{d_1 d_2} \prod_{p|d} \left(1+|\chi(p)|\, p^{-3/4}\right)^2 \\
&\ll \frac{1}{d} \sqrt{\frac{\gamma}{\log X}} \prod_{p|d} \left(1+|\chi(p)|\, p^{-3/4}\right)^2 (1-\frac{1}{p})^{-1}
\end{aligned}
$$

since

$$\sum_{\substack{d_1,d_2\leqslant X^2\\ d|d_1d_2}} \frac{1}{d_1d_2} \leqslant \sum_{\substack{d_1,d_2\\ d|d_1d_2}} \frac{1}{d_1d_2} = \frac{1}{d}\prod_{p|d}(1-\frac{1}{p})^{-1}.$$

Now, $(1-1/p)^{-1} \ll (1+p^{-3/4})$, and

$$\left|\varphi_{i\gamma}(d)\right| = d\prod_{p|d}\left|1-p^{-1-i\gamma}\right| \ll d\prod_{p|d}\left(1+p^{-3/4}\right)$$

so we finally get

$$S_2(\gamma) \ll \frac{\gamma}{\log X}\sum_{d\leqslant X^2}\frac{1}{d}\prod_{p|d}\left(1+p^{-3/4}\right)^7.$$

But we have

$$\begin{aligned}\sum_{d\leqslant X^2}\frac{1}{d}\prod_{p|d}\left(1+p^{-3/4}\right)^7 &\ll \sum_{d\leqslant X^2}\frac{1}{d}\prod_{p|d}\left(1+\frac{1}{\sqrt{p}}\right)\\ &\ll \sum_{d\leqslant X^2}\frac{1}{d}\sum_{n|d}\frac{1}{\sqrt{n}} = \sum_{n\leqslant X^2}\frac{1}{n^{3/2}}\sum_{d\leqslant X^2/n}\frac{1}{d}\\ &\ll \log X.\end{aligned}$$

This completes the proof of (15). □

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