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A new proof of the theorem on Ohno relations for MZVs

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Abstract: Classical proof of the duality theorem for multiple zeta values is carried out with the help of a simple change of variables in the integral representation. We introduce another change of variables which not only equates the very same integrals but also provides a simple proof of Ohno relations for multiple zeta values.

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1. Introduction

Multiple zeta values (or MZVs) $\zeta(s_1, s_2, \dots, s_l)$ are defined for positive integers $s_1 > 1, s_2, \dots, s_l$ as sums of series

$$\zeta(s_1, s_2, \dots, s_l) = \sum_{n_1 > n_2 > \dots > n_l \geqslant 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_l^{s_l}}.$$

They have been known since L. Euler proved the famous formula

$$\zeta(2, 1) = \zeta(3), \tag{1.1}$$

or, more generally,

$$\zeta(m-1, 1) + \zeta(m-2, 2) + \dots + \zeta(2, m-2) = \zeta(m), \quad m \geqslant 3. \tag{1.2}$$

Not so long ago (1.1) was generalized by D. Zagier [7], who proved the following.

THEOREM 1.1 (DUALITY FOR MZVS). *Let $a_1, b_1, \ldots, a_d, b_d \in \mathbb{N}$. Then*

$$\zeta(a_1+1, \{1\}_{b_1-1}, \ldots, a_d+1, \{1\}_{b_d-1}) = \zeta(b_d+1, \{1\}_{a_d-1}, \ldots, b_1+1, \{1\}_{a_1-1}).$$

Here $\{1\}_k = \underbrace{1, \ldots, 1}_{k \text{ times}}$ and $\{1\}_0 = \varnothing$. As for (1.2), it is a special case of the *sum formula*.

THEOREM 1.2 (SUM FORMULA). *For all positive integers l and $m > l$ we have*

$$\sum_{\substack{s_1+\ldots+s_l=m \\ s_1>1, s_2\geqslant 1, \ldots, s_l\geqslant 1}} \zeta(s_1, s_2, \ldots, s_l) = \zeta(m).$$

The sum formula was conjectured in 1994 by C. Moen (see [2]) and independently by C. Markett [4]. The first proof was obtained three years later by A. Granville [1].

Let us denote $S_m(s_1, \ldots, s_l) = \sum\limits_{\substack{k_1+\ldots+k_l=m \\ k_1\geqslant 0, \ldots, k_l\geqslant 0}} \zeta(s_1+k_1, \ldots, s_l+k_l)$. The following theorem by Y. Ohno [5] generalizes both Theorems 1.1 and 1.2.

THEOREM 1.3 (OHNO). *For each $m \geqslant 0$ and $a_1, b_1, \ldots, a_d, b_d \in \mathbb{N}$ we have*

$$S_m(a_1+1, \{1\}_{b_1-1}, \ldots, a_d+1, \{1\}_{b_d-1}) = S_m(b_d+1, \{1\}_{a_d-1}, \ldots, b_1+1, \{1\}_{a_1-1}).$$

Clearly, setting $m = 0$ in Theorem 1.3 gives the duality theorem (Theorem 1.1), and setting $d = 1, a_1 = 1, b_1 = l, m = m - l - 1$, gives the sum formula (Theorem 1.2).

Let us denote $\Sigma_i = a_1 + b_1 + \ldots + a_i + b_i$, $\Sigma_0 = 0$, and $\Sigma_i^a = \Sigma_{i-1} + a_i$. Then due to M. Kontsevich (see [7]) we have the following integral representation

$$\zeta(a_1+1, \{1\}_{b_1-1}, \ldots, a_d+1, \{1\}_{b_d-1}) = \tag{1.3}$$
$$= \int\limits_{1>t_1>\ldots>t_{\Sigma_d}>0} \prod_{i=1}^d \left(\frac{dt_{\Sigma_i}}{1-t_{\Sigma_i}} \frac{dt_{\Sigma_i-1}}{1-t_{\Sigma_i-1}} \cdots \frac{dt_{\Sigma_i^a+1}}{1-t_{\Sigma_i^a+1}} \frac{dt_{\Sigma_i^a}}{t_{\Sigma_i^a}} \frac{dt_{\Sigma_i^a-1}}{t_{\Sigma_i^a-1}} \cdots \frac{dt_{\Sigma_{i-1}+1}}{t_{\Sigma_{i-1}+1}} \right),$$

which can be proved by writing down each fraction $\frac{1}{1-t}$ as the sum of a geometrical progression and further consecutive integration on each variable. Theorem 1.1 can now easily be deduced. We put

$$u_j = 1 - t_{\Sigma_d+1-j}, \quad j = 1, \dots, \Sigma_d \tag{1.4}$$

and obtain an integral which gives exactly $\zeta(b_d+1, \{1\}_{a_d-1}, \dots, b_1+1, \{1\}_{a_1-1})$.

As M. Igarashi reports (see [3]), H. Ochiai has discovered that the same technique allows proving the sum formula. Denote

$$S(s_1, \dots, s_l; x) = \sum_{m=0}^{\infty} S_m(s_1, \dots, s_l) x^m.$$

Decomposing $\dfrac{1}{n-x} = \dfrac{1}{n\left(1-\frac{x}{n}\right)} = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{n^{k+1}}$ we get

$$S(s_1, \dots, s_l; x) = \sum_{n_1 > \dots > n_l \geqslant 1} \frac{1}{n_1^{s_1-1}(n_1-x)\cdots n_l^{s_l-1}(n_l-x)}.$$

Same as in (1.3), we can thus obtain

$$S(a_1+1, \{1\}_{b_1-1}, \dots, a_d+1, \{1\}_{b_d-1}; x) \; = \tag{1.5}$$

$$= \int\limits_{1>t_1>\dots>t_{\Sigma_d}>0} \prod_{i=1}^{d} \left(\left(\frac{t_{\Sigma_i^a}}{t_{\Sigma_i}} \right)^x \frac{\mathrm{d}t_{\Sigma_i}}{1-t_{\Sigma_i}} \cdots \frac{\mathrm{d}t_{\Sigma_i^a+1}}{1-t_{\Sigma_i^a+1}} \frac{\mathrm{d}t_{\Sigma_i^a}}{t_{\Sigma_i^a}} \cdots \frac{\mathrm{d}t_{\Sigma_{i-1}+1}}{t_{\Sigma_{i-1}+1}} \right).$$

Due to the uniqueness of Taylor series expansion it suffices to prove that $S(2, \{1\}_{l-1}; x) = S(l+1; x)$. This would imply Theorem 1.2. To this end let us consider the integral representation of the left-hand side of this equality, which is

$$\int\limits_{1>t_1>\dots>t_{l+1}>0} \left(\frac{t_1}{t_{l+1}} \right)^x \frac{\mathrm{d}t_{l+1}}{1-t_{l+1}} \cdots \frac{\mathrm{d}t_2}{1-t_2} \frac{\mathrm{d}t_1}{t_1},$$

and, same as in (1.4), put $u_1 = 1 - t_{l+1}, \dots, u_{l+1} = 1 - t_1$. The resulting integral

$$\int\limits_{1>u_1>\dots>u_{l+1}>0} \left(\frac{1-u_{l+1}}{1-u_1} \right)^x \frac{\mathrm{d}u_{l+1}}{1-u_{l+1}} \frac{\mathrm{d}u_l}{u_l} \cdots \frac{\mathrm{d}u_1}{u_1}$$

is not exactly what we wanted to get (which is indicated in the Appendix below) but, luckily, it also equals $S(l+1; x)$. Such an effect can be observed only in this particular case, so the change of variables (1.4) does not allow proving a more general result like Theorem 1.3 on Ohno relations, which is equivalent to

$$S(a_1+1,\{1\}_{b_1-1},\ldots,a_d+1,\{1\}_{b_d-1};x) \;=\; \tag{1.6}$$
$$=\; S(b_d+1,\{1\}_{a_d-1},\ldots,b_1+1,\{1\}_{a_1-1};x).$$

This equivalence was also used by J-i. Okuda and K. Ueno in [6], where they gave another proof of that theorem. In the next section we propose a more sophisticated change of variables which equates exactly the integrals representing both sides of (1.6).

2. Main result

Denote $\sigma_i = b_d + a_d + \ldots + b_{i+1} + a_{i+1}$, $\sigma_d = 0$, and $\sigma_i^b = \sigma_i + b_i$, and set $t_0 = u_0 = 1$. Obviously, $\sigma_0 = \Sigma_d$.

Now let us examine the following change of variables.

$$i = d,\ldots,1,\quad \begin{cases} u_{\sigma_i+j} = \dfrac{1-t_{\Sigma_i-j}}{1-t_{\Sigma_i}}\cdot u_{\sigma_i}, & j=1,\ldots,b_i,\\ (1-u_{\sigma_i^b+k}) = \dfrac{t_{\Sigma_i^a-k}}{t_{\Sigma_i^a}}\cdot(1-u_{\sigma_i^b}), & k=1,\ldots,a_i. \end{cases} \tag{2.1}$$

We begin with computing the Jacobian. For $i = d,\ldots,1$ we have

$$\frac{\partial u_{\sigma_i+j}}{\partial t_r} = u_{\sigma_i}\frac{\partial}{\partial t_r}\left(\frac{1-t_{\Sigma_i-j}}{1-t_{\Sigma_i}}\right) + \frac{1-t_{\Sigma_i-j}}{1-t_{\Sigma_i}}\frac{\partial u_{\sigma_i}}{\partial t_r},\;\; j=1,\ldots,b_i,\;\; r=1,\ldots,\Sigma_d.$$

According to the properties of a determinant we can erase the summand

$$\frac{1-t_{\Sigma_i-j}}{1-t_{\Sigma_i}}\frac{\partial u_{\sigma_i}}{\partial t_r}$$

and expel u_{σ_i} from b_i rows of the Jacobian matrix. In the same way we treat

$$\frac{\partial u_{\sigma_i^b+k}}{\partial t_r} = -\frac{\partial(1-u_{\sigma_i^b+k})}{\partial t_r} = -(1-u_{\sigma_i^b})\frac{\partial}{\partial t_r}\left(\frac{t_{\Sigma_i^a-k}}{t_{\Sigma_i^a}}\right) + \frac{t_{\Sigma_i^a-k}}{t_{\Sigma_i^a}}\frac{\partial u_{\sigma_i^b}}{\partial t_r}$$

for $k = 1, \dots, a_i$. This leads us to the determinant of the following matrix

$$
\left(\begin{array}{cc|cccccccc|cc}
.. & \cdots & \cdots & .. & \cdots & \cdots & \cdots & .. & \cdots & \cdots & \cdots & .. \\
.. & 0 & 0 & .. & 0 & 0 & 0 & .. & 0 & \dfrac{-1}{t_{\Sigma^a_{i+1}}} & 0 & .. \\
\hline
.. & 0 & 0 & .. & 0 & 0 & 0 & .. & \dfrac{-1}{1-t_{\Sigma_i}} & \dfrac{1-t_{\Sigma_i-1}}{(1-t_{\Sigma_i})^2} & 0 & .. \\
.. & \cdots & \cdots & .. & \cdots & \cdots & \cdots & .. & \cdots & \cdots & \cdots & .. \\
.. & 0 & 0 & .. & 0 & 0 & \dfrac{-1}{1-t_{\Sigma_i}} & .. & 0 & \dfrac{1-t_{\Sigma^a_i+1}}{(1-t_{\Sigma_i})^2} & 0 & .. \\
.. & 0 & 0 & .. & 0 & \dfrac{-1}{1-t_{\Sigma_i}} & 0 & .. & 0 & \dfrac{1-t_{\Sigma^a_i}}{(1-t_{\Sigma_i})^2} & 0 & .. \\
.. & 0 & 0 & .. & \dfrac{-1}{t_{\Sigma^a_i}} & \dfrac{t_{\Sigma^a_i-1}}{t^2_{\Sigma^a_i}} & 0 & .. & 0 & 0 & 0 & .. \\
.. & \cdots & \cdots & .. & \cdots & \cdots & \cdots & .. & \cdots & \cdots & \cdots & .. \\
.. & 0 & \dfrac{-1}{t_{\Sigma^a_i}} & .. & 0 & \dfrac{t_{\Sigma_{i-1}+1}}{t^2_{\Sigma^a_i}} & 0 & .. & 0 & 0 & 0 & .. \\
.. & \dfrac{-1}{t_{\Sigma^a_i}} & 0 & .. & 0 & \dfrac{t_{\Sigma_{i-1}}}{t^2_{\Sigma^a_i}} & 0 & .. & 0 & 0 & 0 & .. \\
\hline
.. & \dfrac{1-t_{\Sigma_{i-1}-1}}{(1-t_{\Sigma_{i-1}})^2} & 0 & .. & 0 & 0 & 0 & .. & 0 & 0 & 0 & .. \\
.. & \cdots & \cdots & .. & \cdots & \cdots & \cdots & .. & \cdots & \cdots & \cdots & ..
\end{array}\right),
$$

which is equal to the product of determinants of antidiagonal blocks having similar appearance for every i. Eventually, for the absolute value of the Jacobian we have

$$
|J| = \prod_{i=1}^{d} \left(\frac{t_{\Sigma_{i-1}}}{t_{\Sigma^a_i}^{a_i+1}} \frac{1-t_{\Sigma^a_i}}{(1-t_{\Sigma_i})^{b_i+1}} (1-u_{\sigma^b_i})^{a_i} u_{\sigma_i}^{b_i} \right).
$$

Applying (2.1) we get

$$
|J| \cdot \prod_{i=1}^{d} \frac{1}{(1-u_{\sigma_{i-1}})(1-u_{\sigma_{i-1}-1}) \cdots (1-u_{\sigma^b_i+1}) u_{\sigma^b_i} u_{\sigma^b_i-1} \cdots u_{\sigma_i+1}} =
$$

$$
= \prod_{i=1}^{d} \frac{1}{(1-t_{\Sigma_i})(1-t_{\Sigma_i-1}) \cdots (1-t_{\Sigma^a_i+1}) t_{\Sigma^a_i} t_{\Sigma^a_i-1} \cdots t_{\Sigma_{i-1}+1}}. \tag{2.2}
$$

In the Appendix we give a little example of computing the Jacobian, that may look clearer than in the general case.

Now, it immediately follows from $1 = t_0 > t_1 > \ldots > t_{\Sigma_d} > 0$ and (2.1) that $1 = u_0 > u_1 > \ldots > u_{\sigma_0}$. Let us additionally prove that $u_{\sigma_0} > 0$. Then integration of both parts of (2.2) according to (1.3) will lead us directly to the proof of Theorem 1.1.

LEMMA 2.1. *For $0 \leqslant s < r \leqslant d$ we have*

$$u_{\sigma_s} \geqslant u_{\sigma^b_{s+1}} \cdot \frac{u_{\sigma^b_{s+2}}}{u_{\sigma_{s+1}}} \cdots \frac{u_{\sigma^b_r}}{u_{\sigma_{r-1}}} \cdot \frac{t_{\Sigma_s}}{t_{\Sigma^a_{s+1}}} \cdots \frac{t_{\Sigma_{r-1}}}{t_{\Sigma^a_r}} \cdot \left(1 - \frac{1 - t_{\Sigma^a_r}}{u_{\sigma^b_r}}\right),$$

where the equality holds if and only if $s = 0$.

PROOF. We will perform induction on r. The choice $i = s+1, k = a_{s+1}$ in (2.1) gives

$$u_{\sigma_s} = 1 - \frac{t_{\Sigma_s}}{t_{\Sigma^a_{s+1}}} \cdot (1 - u_{\sigma^b_{s+1}}) = u_{\sigma^b_{s+1}} \cdot \frac{t_{\Sigma_s}}{t_{\Sigma^a_{s+1}}} \cdot \left(1 - \frac{1 - \frac{t_{\Sigma^a_{s+1}}}{t_{\Sigma_s}}}{u_{\sigma^b_{s+1}}}\right).$$

Obviously,

$$1 - \frac{1 - \frac{t_{\Sigma^a_{s+1}}}{t_{\Sigma_s}}}{u_{\sigma^b_{s+1}}} \geqslant 1 - \frac{1 - t_{\Sigma^a_{s+1}}}{u_{\sigma^b_{s+1}}}$$

with an equality possible only in the case $t_{\Sigma_s} = 1$, which happens if and only if $s = 0$. This gives us the statement of Lemma for $r = s+1$.

Choosing $i = r-1, j = b_{r-1}$ in (2.1) we get

$$1 - \frac{1 - t_{\Sigma^a_{r-1}}}{u_{\sigma^b_{r-1}}} = 1 - \frac{1 - t_{\Sigma_{r-1}}}{u_{\sigma_{r-1}}} = \frac{u_{\sigma_{r-1}} - 1 + t_{\Sigma_{r-1}}}{u_{\sigma_{r-1}}} = \frac{t_{\Sigma_{r-1}}}{u_{\sigma_{r-1}}} \left(1 - \frac{1 - u_{\sigma_{r-1}}}{t_{\Sigma_{r-1}}}\right),$$

which (with $i = r, k = a_r$ in (2.1)) is equal to

$$\frac{t_{\Sigma_{r-1}}}{u_{\sigma_{r-1}}} \left(1 - \frac{1 - u_{\sigma^b_r}}{t_{\Sigma^a_r}}\right) = \frac{t_{\Sigma_{r-1}}}{u_{\sigma_{r-1}}} \cdot \frac{t_{\Sigma^a_r} - 1 + u_{\sigma^b_r}}{t_{\Sigma^a_r}} = \frac{u_{\sigma^b_r}}{u_{\sigma_{r-1}}} \cdot \frac{t_{\Sigma_{r-1}}}{t_{\Sigma^a_r}} \left(1 - \frac{1 - t_{\Sigma^a_r}}{u_{\sigma^b_r}}\right).$$

This provides the induction step from $r-1$ to r. □

Choosing $i = d, j = b_d$ in (2.1) we see that

$$1 - \frac{1 - t_{\Sigma_d^a}}{u_{\sigma_d^b}} = 1 - (1 - t_{\Sigma_d}) = t_{\Sigma_d}.$$

Thus, Lemma 2.1 with $r = d$ gives

$$u_{\sigma_s} \geqslant u_{\sigma_{s+1}^b} \cdot \frac{u_{\sigma_{s+2}^b}}{u_{\sigma_{s+1}}} \cdots \frac{u_{\sigma_r^b}}{u_{\sigma_{r-1}}} \cdot \frac{t_{\Sigma_s}}{t_{\Sigma_{s+1}^a}} \cdots \frac{t_{\Sigma_{r-1}}}{t_{\Sigma_r^a}} \cdot t_{\Sigma_d}.$$

If $u_{\sigma_{s+1}} > 0$, then by (2.1) we have $u_{\sigma_{s+1}^b} > 0$. Hence $u_{\sigma_s} > 0$. Particularly, this is true for $s = 0$, so, we obtain a new proof of Theorem 1.1, the one we mentioned above. On the other hand, Lemma 2.1 with $s = 0, r = d$ results in

$$\frac{t_{\Sigma_1^a}}{t_{\Sigma_1}} \cdots \frac{t_{\Sigma_d^a}}{t_{\Sigma_d}} = \frac{u_{\sigma_1^b}}{u_{\sigma_0}} \cdots \frac{u_{\sigma_d^b}}{u_{\sigma_{d-1}}}.$$

Together with (2.2) this means that the integral representation (1.5) of

$$S(a_1 + 1, \{1\}_{b_1 - 1}, \dots, a_d + 1, \{1\}_{b_d - 1}; x)$$

under the change of variables (2.1) converts to

$$\int\limits_{1 > u_1 > \ldots > u_{\sigma_0} > 0} \prod_{i=1}^{d} \left(\left(\frac{u_{\sigma_i^b}}{u_{\sigma_{i-1}}} \right)^x \frac{\mathrm{d}u_{\sigma_{i-1}}}{1 - u_{\sigma_{i-1}}} \cdots \frac{\mathrm{d}u_{\sigma_i^b + 1}}{1 - u_{\sigma_i^b + 1}} \frac{\mathrm{d}u_{\sigma_i^b}}{u_{\sigma_i^b}} \cdots \frac{\mathrm{d}u_{\sigma_i + 1}}{u_{\sigma_i + 1}} \right)$$

which is an integral representation of $S(b_d + 1, \{1\}_{a_d - 1}, \dots, b_1 + 1, \{1\}_{a_1 - 1}; x)$, according to (1.5). Thus, we have also proved the Ohno theorem in its equivalent form (1.6).

3. Appendix

Let us compute the Jacobian of the change of variables (2.1) in the simple case $d = 1$, $a_1 = 1$, $b_1 = l$, which corresponds to the sum formula $S(2, \{1\}_{l-1}; x) = S(l+1; x)$.

We have

$$u_1 = \frac{1-t_l}{1-t_{l+1}},\ u_2 = \frac{1-t_{l-1}}{1-t_{l+1}}, \dots,\ u_l = \frac{1-t_1}{1-t_{l+1}},\ 1-u_{l+1} = \frac{1}{t_1}(1-u_l).$$

The Jacobian is equal to the determinant of the following matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & \frac{-1}{1-t_{l+1}} & \frac{1-t_l}{(1-t_{l+1})^2} \\ 0 & 0 & \dots & \frac{-1}{1-t_{l+1}} & 0 & \frac{1-t_{l-1}}{(1-t_{l+1})^2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{-1}{1-t_{l+1}} & \dots & 0 & 0 & \frac{1-t_2}{(1-t_{l+1})^2} \\ \frac{-1}{1-t_{l+1}} & 0 & \dots & 0 & 0 & \frac{1-t_1}{(1-t_{l+1})^2} \\ \frac{1}{t_1^2}(1-u_l) + \frac{1}{t_1}\cdot\frac{-1}{1-t_{l+1}} & 0 & \dots & 0 & 0 & \frac{1}{t_1}\cdot\frac{1-t_1}{(1-t_{l+1})^2} \end{pmatrix}.$$

After subtracting the l-th row multiplied by $\frac{1}{t_1}$ from the $(l+1)$-th row we get

$$\begin{vmatrix} 0 & 0 & \dots & 0 & \frac{-1}{1-t_{l+1}} & \frac{1-t_l}{(1-t_{l+1})^2} \\ 0 & 0 & \dots & \frac{-1}{1-t_{l+1}} & 0 & \frac{1-t_{l-1}}{(1-t_{l+1})^2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{-1}{1-t_{l+1}} & \dots & 0 & 0 & \frac{1-t_2}{(1-t_{l+1})^2} \\ \frac{-1}{1-t_{l+1}} & 0 & \dots & 0 & 0 & \frac{1-t_1}{(1-t_{l+1})^2} \\ \frac{1}{t_1^2}(1-u_l) & 0 & \dots & 0 & 0 & 0 \end{vmatrix}.$$

So,

$$|J| = (1-u_l)\frac{1}{t_1^2}\cdot\frac{1-t_1}{(1-t_{l+1})^{l+1}} = \frac{1-u_{l+1}}{t_1}\frac{u_l}{1-t_2}\cdots\frac{u_2}{1-t_l}\frac{u_1}{1-t_{l+1}}.$$

Furthermore, since $1 > t_1 > \ldots > t_{l+1} > 0$ and

$$u_{l+1} = \frac{t_{l+1}}{t_1} \cdot \frac{1 - t_1}{1 - t_{l+1}},$$

we have $1 > u_1 > \ldots > u_{l+1} > 0$ as well, and

$$\frac{t_{l+1}}{t_1} = \frac{u_{l+1}}{u_l}.$$

This means that

$$\int\limits_{1>t_1>\ldots>t_{l+1}>0} \frac{\left(\frac{t_1}{t_{l+1}}\right)^x \mathrm{d}t_{l+1} \cdots \mathrm{d}t_2 \mathrm{d}t_1}{t_1(1-t_2)\cdots(1-t_{l+1})} = \int\limits_{1>u_1>\ldots>u_{l+1}>0} \frac{\left(\frac{u_l}{u_{l+1}}\right)^x \mathrm{d}u_{l+1} \mathrm{d}u_l \cdots \mathrm{d}u_1}{u_1 \cdots u_l(1-u_{l+1})},$$

whence, according to (1.5), we get $S(2, \{1\}_{l-1}; x) = S(l+1; x)$.

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