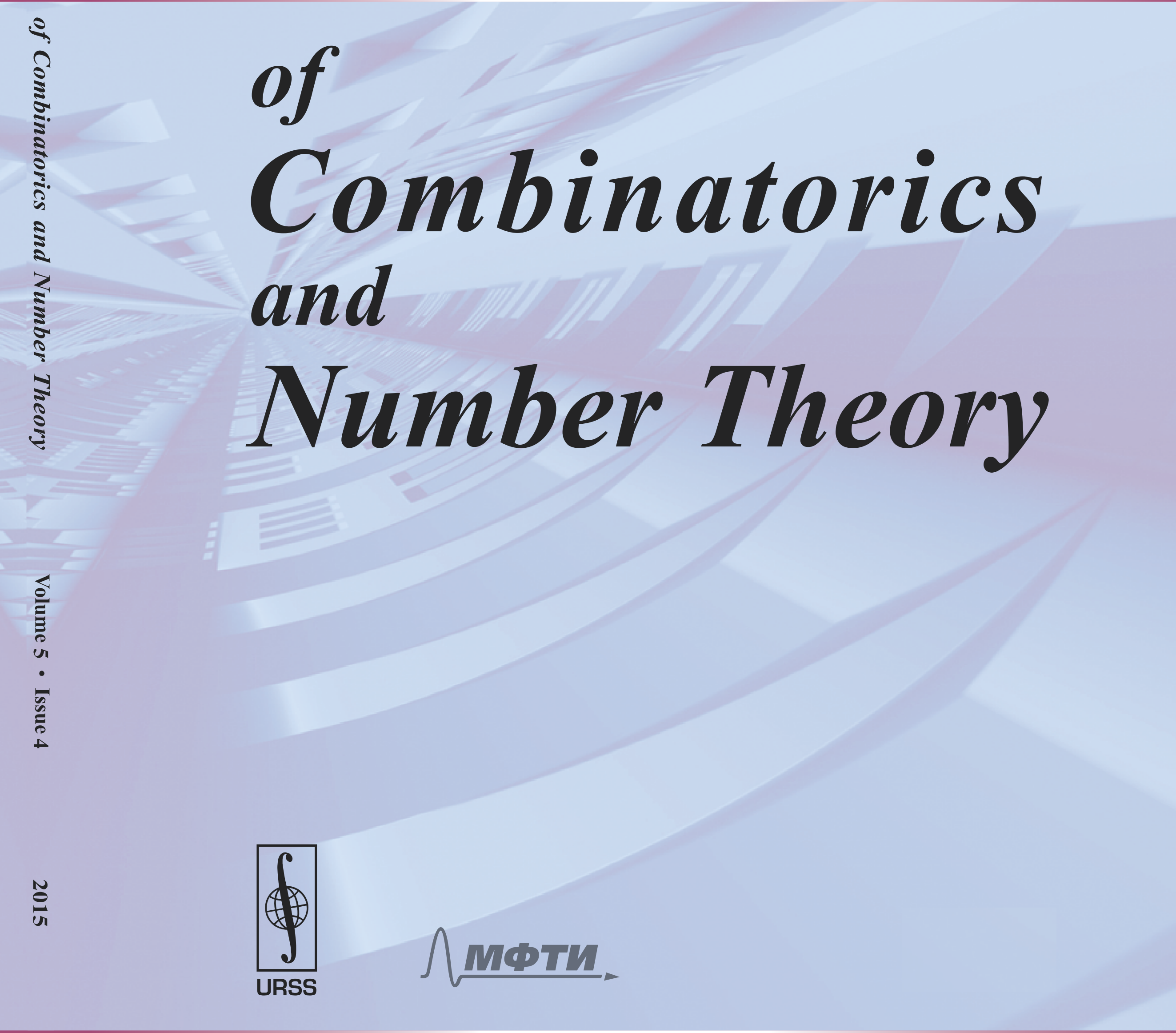
Reprint from

ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory



URSS

МФТИ

Volume 5 • Issue 4

2015

Moscow Journal of Combinatorics and Number Theory Volume 5 • Issue 4 2015

URSS

Moscow Journal
of Combinatorics
and Number Theory
2015, vol. 5, iss. 4,
pp. 53–71 [pp. 255–273]

New bounds for 3-part Sperner families

András Mészáros (Budapest)

Abstract: In this paper we give a new upper and lower bound on the size of maximum 3-part Sperner families. We prove that $1,05 < d_3 < 1,0722$. Further we disprove a conjecture of Aydinian, Czabarka, Erdős, Székely on the maximum size of k-part Sperner families for the case of equal parts of size $2^\ell - 1$.

Keywords: Extremal set theory, Sperner theorem, k-part Sperner family, Chain partition

AMS Subject classification: 05D05

Received: 04.04.2015; **revised:** 12.10.2015

1. Introduction

Let X be a finite set and let $\mathcal{P}(X)$ denote the family of its subsets. Consider a partition $X_1 \cup X_2 \cup \ldots \cup X_k$ of X into k pairwise disjoint subsets. A family $\mathcal{F} \subset \mathcal{P}(X)$ is called a k-part Sperner family on X with respect to the partition, if there are no two distinct elements F_1, F_2 of $\mathcal{F}$ such that $F_1 \subset F_2$ and $F_2 \backslash F_1 \subset X_j$ for some $1 \leq j \leq k$.

Let $|X_i| = n_i \geq 0, n = |X| = \sum_{i=1}^{k} n_i$. We denote the maximum size of a k-part Sperner family on X by $g(n_1, n_2, \ldots, n_k)$. Let $f(n,k)$ be the maximum of $g(n_1, n_2, \ldots, n_k)$ for all choices of n_i's. It is generally believed that the maximum is attained when $n \bmod k$ of the $n_i = \lceil \frac{n}{k} \rceil$ and $k - (n \bmod k)$ of the $n_i = \lfloor \frac{n}{k} \rfloor$.

With this notation, Sperner's theorem states that $f(n,1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Katona [7] and Kleitman [9] proved the two-part Sperner theorem: $g(n_1, n_2) = \binom{n_1+n_2}{\lfloor \frac{n_1+n_2}{2} \rfloor}$.

An important tool in their proof is the symmetric chain partition. If X is a finite set, then a sequence of subsets $(A_1, A_2, \dots, A_h)$ is a symmetric chain of length h if $A_i \subset A_{i+1}$, $|A_{i+1} \setminus A_i| = 1$ for all $1 \le i \le h-1$ and $|A_1| + |A_h| = |X|$. A symmetric chain partition is a partition of $\mathcal{P}(X)$ into symmetric chains. Such a partition always exists, see [2].

Symmetric chain partitions also give us a proof of Sperner's theorem. Since each chain can contain at most one element of a Sperner family, therefore the number of chains, $\binom{|X|}{\lfloor \frac{|X|}{2} \rfloor}$, is an upper bound to the size of a Sperner family. Our method to obtain an upper bound to 3-part Sperner families will be similar (see Section 2).

In the case of k-part Sperner families $(k > 2)$ there are several results by Füredi [5] and Griggs, Odlyzko, Shearer [6]. They proved that for fixed k the limit $d_k := \lim\limits_{n \to \infty} \frac{f(n,k)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists. They also proved that $d_k \sim \frac{\sqrt{k\pi}}{\sqrt{4 \log k}}$. However, we do not know the exact value of d_k beyond $d_1 = d_2 = 1$. A good survey on this topic is [4] and [1].

In this paper we prove that $1.05 < d_3 < 1.0722$, improving previous best bounds of 1.036 and 1.131 in [4].

Griggs, Odlyzko, Shearer mentioned in their paper [6] that they managed to prove that $\lim\limits_{n \to \infty} \frac{g(n,n,n)}{\binom{3n}{\lfloor 1.5n \rfloor}} < 1.0722$, since this balanced case is believed to be extremal, they conjectured that this bound also holds for d_3 . Our results proves this conjecture. We will show stronger bounds if the ratios of n_i's are fixed.

In Section 2 we give upper bounds for 3-part Sperner families using monochromatic chain partitions. Similar arguments have already appeared before, e. g. in [5] and [8]. However, to obtain the asymptotic value of these upper bounds we need some calculations which are contained in Section 3.

In Section 4 we will present the results of Erdős, Katona [3] and Füredi, Griggs, Odlyzko, Shearer [6], [4] about homogeneous families. We will see, as a consequence, that finding the optimal k-part Sperner family is equivalent to finding the solution of a certain integer program. Solving this with the aid of a computer in the case of $k = 3, n_1 = n_2 = n_3 = 15$ we disprove the conjecture of Aydinian, Czabarka, Erdős, Székely [1] on the maximum size of k-part Sperner families for the case of equal parts of size $2^\ell - 1$. We also establish some connection with monochromatic chain partitions.

In Section 5, to obtain a better lower bound we will basically use the same method as Füredi [5] and Griggs, Odlyzko, Shearer [6]. However where they used a clever, explicit construction, we find a better one by the aid of a computer.

2. Monochromatic chain partitions

Definition 1. *Let* $X_1 \cup X_2 \cup \ldots \cup X_k$ *be a partition of* X *into* k *pairwise disjoint subsets. A sequence* $(A_1, A_2, \ldots, A_h)$ *of subsets of* X *is called a monochromatic chain of* X *if* $A_i \subset A_{i+1}$ *for* $i = 1, 2, \ldots, h-1$ *and there exist* j *such that* $1 \leq j \leq k$ *and* $A_{i+1} \backslash A_i \subset X_j$ *for* $i = 1, 2, \ldots, h-1$. *Here* h *is the length of the chain*, j *is the color of the chain.*

Definition 2. *Let* $X_1 \cup X_2 \cup \ldots \cup X_k$ *be a partition of* X *into* k *pairwise disjoint subsets. A set* $\mathcal{C}$ *of monochromatic chains of* X *is called a monochromatic chain partition if every subset of* X *is contained in exactly one monochromatic chain in* $\mathcal{C}$.

If $\mathcal{F}$ is a k-part Sperner family on X with respect to the partition $X_1 \cup X_2 \cup \ldots \cup X_k$, then every monochromatic chain can contain at most one element of $\mathcal{F}$, so the lemma below clearly follows.

Lemma 1. *If* $\mathcal{F}$ *is a* k*-part Sperner family on* $X = X_1 \cup \ldots \cup X_k$, $\mathcal{C}$ *is a monochromatic chain partition of* X, *then* $|\mathcal{F}| \leq |\mathcal{C}|$.

Let $X = X_1 \cup \ldots \cup X_k$ and $Y = Y_1 \cup \ldots \cup Y_m$ be disjoint sets with monochromatic chain partitions $\mathcal{C}_1$ and $\mathcal{C}_2$. Our aim is to construct a monochromatic chain partition $\mathcal{C}_1 \square \mathcal{C}_2$ on $Z = X_1 \cup \ldots \cup X_k \cup Y_1 \cup \ldots \cup Y_m$. Let $(A_1, A_2, \ldots, A_h)$ and $(B_1, B_2, \ldots, B_g)$ be chains in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We will show there are $\min(h, g)$ monochromatic chains with length $\max(h, g)$, such that they cover exactly those $g \cdot h$ subsets of Z which have the form $A_i \cup B_j$. If $h \geq g$, take the following monochromatic chains for $1 \leq i \leq g$: $(A_1 \cup B_i, A_2 \cup B_i, \ldots, A_h \cup B_i)$. If $h < g$, take the following monochromatic chains for $1 \leq i \leq h$: $(A_i \cup B_1, A_i \cup B_2, \ldots, A_i \cup B_g)$. By choosing these monochromatic chains for all possible $|\mathcal{C}_1| \cdot |\mathcal{C}_2|$ pairs of monochromatic chains, we obtain a monochromatic chain partition $\mathcal{C}_1 \square \mathcal{C}_2$ of Z.

Definition 3. *Let* $\mathcal{C}$ *be a monochromatic chain partition. Its profile vector is defined as* $p(\mathcal{C}) = (k_1, k_2, k_3, \ldots)$ *where* k_i *is the number of chains in* $\mathcal{C}$ *with length* i. *We will also use the following form*: $p(\mathcal{C}) = \sum_{i=1}^{\infty} k_i \mathbf{b}_i$, *where* $\mathbf{b}_i = (0, 0, \ldots, 0, \overset{i}{\check{1}}, 0, \ldots)$.

Definition 4. *Let A be an algebra over $\mathbb{R}$ with linearly independent generators $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \dots$. We define the product $\times$ on the generators in the following way: $\mathbf{b}_i \times \mathbf{b}_j = \min(i, j)\mathbf{b}_{\max(i,j)}$. There is a unique way to extend this to an algebra.*

For $u = \sum_{i=1}^{\infty} k_i\mathbf{b}_i$, we will sometimes use the notation $(u)_i = k_i$. It is easy to see that A will be a commutative, associative algebra. Moreover the following statement holds.

Lemma 2. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are monochromatic chain partitions then $p(\mathcal{C}_1 \Box \mathcal{C}_2) = p(\mathcal{C}_1) \times p(\mathcal{C}_2)$.*

Definition 5. *The linear map $L : A \to \mathbb{R}$ is defined in the following way: for $u = \sum_{i=1}^{\infty} k_i\mathbf{b}_i$, $L(u) = \sum_{i=1}^{\infty} k_i$.*

Note that if u is a profile vector of a monochromatic chain partition, then it gives us the number of chains.

Suppose that X is partitioned into 3 parts: $X = X_1 \cup X_2 \cup X_3$. Take a symmetric chain partition $\mathcal{C}_i$ of $\mathcal{P}(X_i)$. (See Section 1.) Then $L(p(\mathcal{C}_1) \times p(\mathcal{C}_2) \times p(\mathcal{C}_3))$ gives us an upper bound for the size of any 3-part Sperner family on X by Lemma 1. Denote this upper bound by $U(m_1, m_2, m_3)$ for the sizes $|X_i| = m_i$. First we try to determine $U(2n_1, 2n_2, 2n_3)$.

A symmetric chain partition on a $2n$-element set has the following profile vector:

$$p_{2n} = \sum_{i=0}^{n-1} \left(\binom{2n}{n+i} - \binom{2n}{n+i+1} \right) \mathbf{b}_{2i+1} + \mathbf{b}_{2n+1} =$$
$$= \binom{2n}{n}\mathbf{b}_1 + \sum_{i=1}^{n} \binom{2n}{n+i} (\mathbf{b}_{2i+1} - \mathbf{b}_{2i-1}) = \sum_{i=0}^{n} \binom{2n}{n+i} \mathbf{d}_i,$$

where $\mathbf{d}_0 = \mathbf{b}_1$ and $\mathbf{d}_i = \mathbf{b}_{2i+1} - \mathbf{b}_{2i-1}$. Using the distributivity of $\times$ and the linearity of L we obtain

$$U(2n_1, 2n_2, 2n_3) = L(p_{2n_1} \times p_{2n_2} \times p_{2n_3}) =$$
$$= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{2n_1}{n_1+i} \binom{2n_2}{n_2+j} \binom{2n_3}{n_3+k} L(\mathbf{d}_i \times \mathbf{d}_j \times \mathbf{d}_k).$$

Now we determine the value of $L(\mathbf{d}_i \times \mathbf{d}_j \times \mathbf{d}_k)$. Due to symmetry we may assume that $i \geq j \geq k$. The following statements are easy consequences:

$$\begin{aligned}
L(\mathbf{d}_0 \times \mathbf{d}_0 \times \mathbf{d}_0) &= 1, \\
L(\mathbf{d}_i \times \mathbf{d}_i \times \mathbf{d}_i) &= 6 - 4i \quad \text{if } i > 0, \\
L(\mathbf{d}_i \times \mathbf{d}_i \times \mathbf{d}_0) &= 2 \quad \text{if } i > 0, \\
L(\mathbf{d}_i \times \mathbf{d}_i \times \mathbf{d}_k) &= 4 \quad \text{if } i > k > 0, \\
L(\mathbf{d}_i \times \mathbf{d}_j \times \mathbf{d}_k) &= 0 \quad \text{if } i > j \geq k.
\end{aligned}$$

Now the following notations are introduced:

$$\begin{aligned}
A(n_1, n_2, n_3) &= \sum_{i=0}^{\min(n_1,n_2)} \sum_{j=0}^{\min(n_3,i)} \binom{2n_1}{n_1+i}\binom{2n_2}{n_2+i}\binom{2n_3}{n_3+j}, \\
B(n_1, n_2, n_3) &= \sum_{i=0}^{\min(n_1,n_2,n_3)} \binom{2n_1}{n_1+i}\binom{2n_2}{n_2+i}\binom{2n_3}{n_3+i} i, \\
E(n_1, n_2, n_3) &= 11\binom{2n_1}{n_1}\binom{2n_2}{n_2}\binom{2n_3}{n_3} + \\
&\quad + 6\sum_{i=1}^{\min(n_1,n_2,n_3)} \binom{2n_1}{n_1+i}\binom{2n_2}{n_2+i}\binom{2n_3}{n_3+i} + \\
&\quad + 2\sum_{i=1}^{\min(n_1,n_2)} \binom{2n_1}{n_1+i}\binom{2n_2}{n_2+i}\binom{2n_3}{n_3} + \\
&\quad + 2\sum_{i=1}^{\min(n_1,n_3)} \binom{2n_1}{n_1+i}\binom{2n_2}{n_2}\binom{2n_3}{n_3+i} + \\
&\quad + 2\sum_{i=1}^{\min(n_2,n_3)} \binom{2n_1}{n_1}\binom{2n_2}{n_2+i}\binom{2n_3}{n_3+i}.
\end{aligned}$$

Using these notations we obtain

$$\begin{aligned}
U(2n_1, 2n_2, 2n_3) = 4A(n_1, n_2, n_3) + 4A(n_2, n_3, n_1) + 4A(n_3, n_1, n_2) - \\
- 4B(n_1, n_2, n_3) - E(n_1, n_2, n_3).
\end{aligned} \tag{2.1}$$

To obtain a common upper bound we prove the following theorem.

THEOREM. *If* $3n = n_1 + n_2 + n_3$ *then* $U(2n_1, 2n_2, 2n_3) \leq U(2n, 2n, 2n)$.

We need further lemmas to prove this theorem.

DEFINITION 6. *The linear map* $\nu : A \to A$ *is defined first on the generators*: $\nu(\mathbf{b}_1) = \mathbf{b}_2$ *and* $\nu(\mathbf{b}_i) = \mathbf{b}_{i-1} + \mathbf{b}_{i+1}$ *for* $i > 1$, *and then linearly extended.*

LEMMA 3. $\nu(\nu(p_{2n})) = p_{2n+2}$.

PROOF. Recall how we constructed the symmetric chain partition of $\mathcal{P}(X \cup \{t\})$ from the symmetric chain partition of $\mathcal{P}(X)$. See [7]. $\square$

LEMMA 4. *For* $u = \sum_{i=1}^{\infty} k_i \mathbf{b}_i$ *and* $w = \sum_{i=1}^{\infty} \ell_i \mathbf{b}_i$ *we have*

$$L(\nu(u) \times w) = 2L(u \times w) - \sum_{i=1}^{\infty} k_i \ell_i.$$

PROOF. Observe that $L(\nu(\mathbf{b}_i) \times \mathbf{b}_i) = 2L(\mathbf{b}_i \times \mathbf{b}_i) - 1$ and $L(\nu(\mathbf{b}_i) \times \mathbf{b}_j) = 2L(\mathbf{b}_i \times \mathbf{b}_j)$ if $i \neq j$. Now the statement can be obtained using distributivity and linearity. $\square$

To simplify the calculations, define $\binom{m}{m+1}$ to be 0.

LEMMA 5. *For fixed* $0 \leq j < k$ *the following function is monotone decreasing in* m *where* $m \geq k$:

$$\frac{\binom{2m}{m+j} - \binom{2m}{m+k}}{\binom{2m}{m+k} - \binom{2m}{m+k+1}}.$$

PROOF. For fixed $a > 0$ the function below is monotone decreasing in m if $m \geq a$:

$$\frac{\binom{2m}{m+a-1} - \binom{2m}{m+a}}{\binom{2m}{m+a} - \binom{2m}{m+a+1}} = \frac{(2a-1)(m+a+1)}{(2a+1)(m-a+1)}.$$

(Note that the equality above also holds in the degenerate case of $a = m$.)

Multiplying functions of this form we obtain that for fixed $0 \leq j < k$ the following function is monotone decreasing in m, $(m > j)$:

$$\frac{\binom{2m}{m+j} - \binom{2m}{m+j+1}}{\binom{2m}{m+k} - \binom{2m}{m+k+1}}.$$

Summing these kinds of functions we obtain the statement of the lemma. ☐

LEMMA 6. *If* $n_1 < n_2$, *then* $U(2n_1+2, 2n_2, 2n_3) \geq U(2n_1, 2n_2+2, 2n_3)$.

PROOF.

We denote p_{2n_i} by q_i. By Lemma 3 we need to compare $L(\nu(\nu(q_1)) \times q_2 \times q_3)$ and $L(\nu(\nu(q_2)) \times q_1 \times q_3)$. Further $q_2 \times q_3$ has 0's in all its even coordinates, and $\nu(q_1)$ has 0's in all its odd coordinates, so $L(\nu(\nu(q_1)) \times q_2 \times q_3) = 2L(\nu(q_1) \times q_2 \times q_3)$. Similarly, $L(\nu(\nu(q_2)) \times q_3 \times q_3) = 2L(\nu(q_2) \times q_1 \times q_3)$. So we have to compare $L(\nu(q_1) \times q_2 \times q_3) = 2L(q_1 \times q_2 \times q_3) - \sum_{i=1}^{\infty}(q_1)_i(q_2 \times q_3)_i$ and $L(\nu(q_1) \times q_2 \times q_3) =$ $= 2L(q_1 \times q_2 \times q_3) - \sum_{i=1}^{\infty}(q_2)_i(q_1 \times q_3)_i$.

It is sufficient to prove that $(q_1)_i(q_2 \times q_3)_i \leq (q_2)_i(q_1 \times q_3)_i$ for every i. Expanding the two sides of the inequality the following formulas are obtained:

$$(q_1)_i(q_2 \times q_3)_i = i(q_1)_i(q_2)_i(q_3)_i + (q_1)_i(q_2)_i \sum_{j=1}^{i-1} j(q_3)_j + (q_1)_i(q_3)_i \sum_{j=1}^{i-1} j(q_2)_j,$$

$$(q_2)_i(q_1 \times q_3)_i = i(q_1)_i(q_2)_i(q_3)_i + (q_1)_i(q_2)_i \sum_{j=1}^{i-1} j(q_3)_j + (q_2)_i(q_3)_i \sum_{j=1}^{i-1} j(q_1)_j.$$

Some terms cancel out and we can divide by $(q_3)_i$, so we only have to prove the following inequality:

$$(q_1)_i \sum_{j=1}^{i-1} j(q_2)_j \leq (q_2)_i \sum_{j=1}^{i-1} j(q_1)_j.$$

If i is even, both sides are zero, assume that $i = 2k+1$. If $k > n_1$ then the left hand side is 0, because $(q_1)_i = 0$, the other side is nonnegative thus we are done. So we may assume that $k \leq n_1 < n_2$. Using the equations

$$\sum_{j=1}^{i-1} j(p_{2m+1})_j = \sum_{j=0}^{k-1}(2j+1)\left(\binom{2m}{m+j} - \binom{2m}{m+j+1}\right) =$$

$$= \binom{2m}{m} + 2\sum_{j=1}^{k-1}\binom{2m}{m+j} - (2k-1)\binom{2m}{m+k} =$$
$$= \left(\binom{2m}{m} - \binom{2m}{m+k}\right) + 2\sum_{j=1}^{k-1}\left(\binom{2m}{m+j} - \binom{2m}{m+k}\right)$$

our inequality reduces to

$$\left(\binom{2n_1}{n_1+k} - \binom{2n_1}{n_1+k+1}\right)\times$$
$$\times\left(\left(\binom{2n_2}{n_2} - \binom{2n_2}{n_2+k}\right) + 2\sum_{j=1}^{k-1}\left(\binom{2n_2}{n_2+j} - \binom{2n_2}{n_2+k}\right)\right) \leq$$
$$\leq \left(\binom{2n_2}{n_2+k} - \binom{2n_2}{n_2+k+1}\right)\times$$
$$\times\left(\left(\binom{2n_1}{n_1} - \binom{2n_1}{n_1+k}\right) + 2\sum_{j=1}^{k-1}\left(\binom{2n_1}{n_1+j} - \binom{2n_1}{n_1+k}\right)\right).$$

Therefore it is sufficient to show that

$$\left(\binom{2n_1}{n_1+k} - \binom{2n_1}{n_1+k+1}\right)\left(\binom{2n_2}{n_2+j} - \binom{2n_2}{n_2+k}\right) \leq$$
$$\leq \left(\binom{2n_2}{n_2+k} - \binom{2n_2}{n_2+k+1}\right)\left(\binom{2n_1}{n_1+j} - \binom{2n_1}{n_1+k}\right).$$

holds for every $0 \leq j \leq k-1$. However, this follows from rearranging Lemma 5 and using the fact that $n_1 < n_2$. □

Proof. (Theorem 2.) Lemma 6 can be written in the following form. If $n_1 < n_2$ then $U(2n_1+2, 2n_2-2, 2n_3) \geq U(2n_1, 2n_2, 2n_3)$. Suppose $n_1 < n$ and $n_2 > n$. Then $U(2n_1, 2n_2, 2n_3) \leq U(2n_1+2, 2n_2-2, 2n_3)$ and with this transformation we can reach $U(2n, 2n, 2n)$ which proves the statement. □

3. Asymptotic calculations

We will determine the value of $\lim\limits_{n\to\infty} \frac{U(2p_n,2q_n,2r_n)}{\binom{2s_n}{s_n}}$, where p_n, q_n, r_n are sequences of positive integers, $s_n = p_n + q_n + r_n$, $\lim\limits_{n\to\infty} s_n = \infty$, $\lim\limits_{n\to\infty} \frac{p_n}{s_n} = \alpha$, $\lim\limits_{n\to\infty} \frac{p_n}{s_n} = \beta$ and $\lim\limits_{n\to\infty} \frac{p_n}{s_n} = \gamma$ for fixed positive numbers α, β, γ such that $\alpha + \beta + \gamma = 1$.

From the de Moivre–Laplace theorem we have

$$\binom{2m}{m+k} \sim \frac{1}{\sqrt{\pi m}} 2^{2m} \exp\left(-\frac{k^2}{m}\right).$$

So

$$\binom{2p_n}{p_n+i}\binom{2q_n}{q_n+i}\binom{2r_n}{r_n+j} \sim \frac{1}{\pi^{\frac{3}{2}}\sqrt{p_n q_n r_n}} 2^{2s_n} \exp\left(-\frac{i^2}{p_n} - \frac{i^2}{q_n} - \frac{j^2}{r_n}\right) =$$

$$= \frac{2^{2s_n}}{\pi^{\frac{3}{2}}\sqrt{s_n}} \sqrt{\frac{s_n}{p_n}\frac{s_n}{q_n}\frac{s_n}{r_n}} \left(\frac{1}{\sqrt{s_n}}\right)^2 \times$$

$$\times \exp\left(-\frac{s_n}{p_n}\left(\frac{i}{\sqrt{s_n}}\right)^2 - \frac{s_n}{q_n}\left(\frac{i}{\sqrt{s_n}}\right)^2 - \frac{s_n}{r_n}\left(\frac{j}{\sqrt{s_n}}\right)^2\right) \sim$$

$$\sim \frac{2^{2s_n}}{\pi^{\frac{3}{2}}\sqrt{s_n}\sqrt{\alpha\beta\gamma}} \int\limits_{x=\frac{i}{\sqrt{s_n}}}^{\frac{i+1}{\sqrt{s_n}}} \int\limits_{y=\frac{j}{\sqrt{s_n}}}^{\frac{j+1}{\sqrt{s_n}}} \exp\Big(-\frac{1}{\alpha}x^2 - \frac{1}{\beta}x^2 - \frac{1}{\gamma}y^2\Big) dydx.$$

Thus

$$A(p_n, q_n, r_n) = \sum_{i=0}^{\min(p_n,q_n)} \sum_{j=0}^{\min(r_n,i)} \binom{2p_n}{p_n+i}\binom{2q_n}{q_n+i}\binom{2r_n}{r_n+j} \sim$$

$$\sim \frac{2^{2s_n}}{\pi^{\frac{3}{2}}\sqrt{s_n}\sqrt{\alpha\beta\gamma}} \int\limits_D \exp\left(-\frac{1}{\alpha}x^2 - \frac{1}{\beta}x^2 - \frac{1}{\gamma}y^2\right) dxdy,$$

where $D = \{(x,y) | x \geq 0, x \geq y \geq 0\}$. With $\varphi(x,y) = \left(\frac{\sqrt{\alpha+\beta}}{\sqrt{\alpha\beta}}x, \frac{1}{\sqrt{\gamma}}y\right)$ and $F(x,y) = \exp(-x^2 - y^2)$, using integration with substitution we obtain

$$\int_{\varphi(D)} F = \int_{D} F \circ \varphi |\det D\varphi|.$$

Therefore we have

$$\int_{D} \exp\left(-\frac{\alpha+\beta}{\alpha\beta}x^2 - \frac{1}{\gamma}y^2\right)dxdy = \frac{\sqrt{\alpha\beta\gamma}}{\sqrt{\alpha+\beta}} \int_{\varphi(D)} \exp(-x^2 - y^2)dxdy.$$

The transformation of D by φ can be seen on Figure 1. The integral of $\exp(-x^2 - y^2)$ on the whole plane is π, so the integral on $\varphi(D)$ is $\frac{1}{2}\arctan\frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}}$.

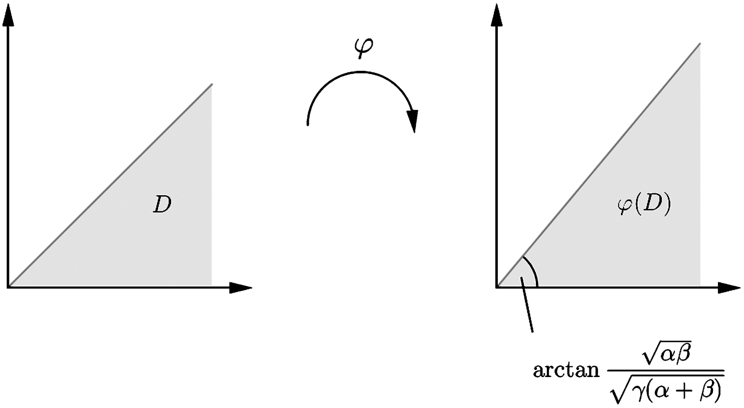


Fig. 1

The well-known asymptotic formula

$$\binom{2s_n}{s_n} \sim \frac{2^{2s_n}}{\sqrt{\pi s_n}}$$

implies

$$\lim_{n\to\infty} \frac{A(p_n, q_n, r_n)}{\binom{2s_n}{s_n}} = \frac{1}{2\pi\sqrt{\alpha+\beta}} \arctan\frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}}.$$

The same method leads to

$$\lim_{n\to\infty}\frac{B(p_n,q_n,r_n)}{\binom{2s_n}{s_n}}=\frac{\sqrt{\alpha\beta\gamma}}{2\pi(\alpha\beta+\beta\gamma+\gamma\alpha)}\ \text{and}\ \lim_{n\to\infty}\frac{E(p_n,q_n,r_n)}{\binom{2s_n}{s_n}}=0.$$

From these equations and from equation 2.1 we obtain that $\lim\limits_{n\to\infty}\frac{U(2p_n,2q_n,2r_n)}{\binom{2s_n}{s_n}}=$ $=u(\alpha,\beta,\gamma)$, where $u(\alpha,\beta,\gamma)$ is defined as:

$$\frac{2}{\pi}\left(\frac{\arctan\frac{\sqrt{\alpha\beta}}{\sqrt{\gamma(\alpha+\beta)}}}{\sqrt{\alpha+\beta}}+\frac{\arctan\frac{\sqrt{\alpha\gamma}}{\sqrt{\beta(\alpha+\gamma)}}}{\sqrt{\alpha+\gamma}}+\frac{\arctan\frac{\sqrt{\beta\gamma}}{\sqrt{\alpha(\beta+\gamma)}}}{\sqrt{\beta+\gamma}}-\frac{\sqrt{\alpha\beta\gamma}}{\alpha\beta+\beta\gamma+\gamma\alpha}\right).$$

If we extend the definition of u to the case of $\alpha\beta\gamma=0$ as $u(\alpha,\beta,\gamma)=1$ we have the following theorem.

THEOREM. *Let a_n, b_n, c_n be sequences of nonnegative integers, $m_n=a_n+b_n+c_n$, $\lim\limits_{n\to\infty} m_n=\infty$, $\lim\limits_{n\to\infty}\frac{a_n}{m_n}=\alpha$, $\lim\limits_{n\to\infty}\frac{b_n}{m_n}=\beta$, $\lim\limits_{n\to\infty}\frac{c_n}{m_n}=\gamma$. Then*

$$\limsup_{n\to\infty}\frac{g(a_n,b_n,c_n)}{\binom{m_n}{\lfloor\frac{m_n}{2}\rfloor}}\le u(\alpha,\beta,\gamma).$$

PROOF. In the case of $\alpha,\beta,\gamma>0$ the calculation above shows that the theorem is true if all the numbers a_n, b_n, c_n are even, but we can easily eliminate the odd numbers using the following simple lemma.

LEMMA 7 (GRIGGS, ODLYZKO, SHEARER [6]). *The following inequality holds for all $n_1>0$:*

$$g(n_1,n_2,n_3)\le 2g(n_1-1,n_2,n_3).$$

The case of $\alpha\beta\gamma=0$ can also be easily handled by this lemma. □

From Lemma 2 we obtain that the maximum of u is attained at $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$. Thus the following statement is proved.

THEOREM. *The following bound holds for d_3:*

$$d_3\le u\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),$$

that is $d_3\le 1,072\ldots$

4. Homogeneous families and related integer programs

For sake of simplicity, all the theorems in this section will be stated for 3-part Sperner families, but one can easily generalize them for k-parts.

Definition 1. *The set F has type (r_1, r_2, r_3) if $|X_i \cap F| = r_i$ for $i = 1, 2, 3$.*

Definition 2. *A family $\mathcal{F} \subset \mathcal{P}(X)$ is homogeneous if $F \in \mathcal{F}$ implies that all sets of the same type are in $\mathcal{F}$.*

A homogeneous family $\mathcal{F}$ will be a Sperner family, if there are no two sets in $\mathcal{F}$ with two different types (r_1, r_2, r_3) and (s_1, s_2, s_3) such that there are $i \neq j \in \{1, 2, 3\}$, $r_i = s_i$ and $r_j = s_j$.

Lemma 8 (Griggs, Odlyzko, Shearer [6] and Erdős, Katona [3]).

If $\mathcal{F}_0$ is a maximally sized 3-part Sperner family then there is a homogeneous Sperner family $\mathcal{F}$ with the same size.

Therefore, it is enough to search for the maximum among the homogeneous families. It has been shown in previous work (Griggs, Odlyzko, Shearer [6]) that this is equivalent to solving the following integer program: $x(i, j, k)$ are binary variables $(0 \leq i \leq n_1, 0 \leq j \leq n_2, 0 \leq k \leq n_3)$, $x(i, j, k) = 1$ if the sets of type (i, j, k) are in $\mathcal{F}$, 0 otherwise. The constraints are the following:

$$\forall\, j, k \quad (0 \leq j \leq n_2, 0 \leq k \leq n_3) \qquad \sum_{i=0}^{n_1} x(i, j, k) \leq 1,$$

$$\forall\, i, k \quad (0 \leq i \leq n_1, 0 \leq k \leq n_3) \qquad \sum_{j=0}^{n_2} x(i, j, k) \leq 1,$$

$$\forall\, i, j \quad (0 \leq i \leq n_1, 0 \leq j \leq n_2) \qquad \sum_{k=0}^{n_3} x(i, j, k) \leq 1.$$

Determine $\max \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \binom{n_1}{i}\binom{n_2}{j}\binom{n_3}{k} x(i, j, k)$.

Conjecture 1 (Aydinian, Czabarka, Erdős, Székely [1, Conjecture 7.1]). For every $k \geq 1$ and every set of the form $n_1 = n_2 = \ldots = n_k = 2^\ell - 1$ for some $\ell \in \mathbb{N}$, $\mathcal{F}$ is a maximally sized homogeneous k-part Sperner family if and only if $\mathcal{F} \in \mathfrak{F}_0^{(k)}$, where a family $\mathcal{F}$ is in $\mathfrak{F}_0^{(k)}$ if and only if it can be constructed in the following way. Take a permutation π of $\{0, 1, \ldots, 2^\ell - 1\}$ such that the sequence $\binom{2^\ell - 1}{\pi(i)}$ is monotone decreasing. Then the family $\mathcal{F}$ which consists of those sets whose type is from the set $\{(x_1, x_2, \ldots, x_k) | \pi^{-1}(x_1) \oplus \pi^{-1}(x_2) \oplus \ldots \oplus \pi^{-1}(x_k) = 0\}$ is a k-part Sperner family. (Here $\oplus$ means bitwise exclusive or, also known as nim sum.)

In the case of $k = 3$ and $n_1 = n_2 = n_3 = 15$ the solution of the integer program shows that the conjecture is not true, since we can find a family of larger size than the size of the families in $\mathfrak{F}_0^{(k)}$. See [10] for details. (The authors of the conjecture verified it in the cases of $k = 3$, $\ell = 1, 2, 3$ with checking all the homogeneous families, but in the case of $\ell = 4$ this is too slow, however the integer program can be solved in a few minutes.)

One of the reasons we can solve this integer program this quickly is that the optimum value of the objective function of the integer program and the LP relaxation happen to be the same. And the same happens in all the cases we have seen. Although we checked it in all the cases of $n_1, n_2, n_3 \le 15$ and they were equal, we believe that there must be a counter example for larger n_i's.

We may formulate Lemma 1 in the following way:

$$\max\{|\mathcal{F}| : \mathcal{F} \text{ is a 3-part Sperner family}\} \le$$
$$\le \min\{|\mathcal{C}| : \mathcal{C} \text{ is a monochromatic chain partition}\}. \quad (4.1)$$

It is a natural question whether the inequality can be replaced by equality. We will show that if the LP and integer optimum are not the same, the inequality strictly holds.

Consider the LP relaxation of the integer program above, its optimum value is at least the optimum value of the integer program. But the LP optimum is equal to the optimum of the dual problem:

$$\forall\, i, j, k \quad (0 \le i \le n_1, \quad 0 \le j \le n_2, \quad 0 \le k \le n_3)$$
$$y(*, j, k) + y(i, *, k) + y(i, j, *) \ge \binom{n_1}{i}\binom{n_2}{j}\binom{n_3}{k} x(i, j, k),$$
$$y(*, j, k) \ge 0, \quad y(i, *, k) \ge 0, \quad y(i, j, *) \ge 0,$$

$$\min \sum_{j=0}^{n_2}\sum_{k=0}^{n_3} y(*,j,k) + \sum_{i=0}^{n_1}\sum_{k=0}^{n_3} y(i,*,k) + \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} y(i,j,*).$$

Therefore if we have a feasible solution to the dual problem we obtain an upper bound for the size of the three-part Sperner families. Now we show a method for obtaining a feasible solution to the dual problem from a monochromatic chain partition.

Lemma 9. *If $\mathcal{C}$ is a monochromatic chain partition, there is a feasible solution to the dual problem where the value of the objective function is $|\mathcal{C}|$.*

Proof. A monochromatic chain $C = (F_1, F_2, \ldots, F_m)$ has type $(*, j, k)$ if its color is 1, $|F_1 \cap X_2| = j$ and $|F_1 \cap X_3| = k$. (Note that $F_h \cap X_2$ are the same for $h = 1, 2, \ldots, m$.) A monochromatic chain $C = (F_1, F_2, \ldots, F_m)$ has type $(i, *, k)$ if its color is 2, $|F_1 \cap X_1| = i$ and $|F_1 \cap X_3| = k$. A monochromatic chain $C = (F_1, F_2, \ldots, F_m)$ has type $(i, j, *)$ if its color is 3, $|F_1 \cap X_1| = i$ and $|F_1 \cap X_2| = j$. (If $C = (F_1)$ then it has type $(*, |F_1 \cap X_2|, |F_1 \cap X_3|)$).

Let $y(*, j, k), y(i, *, k)$ and $y(i, j, *)$ be the number of chains in $\mathcal{C}$ with type $(*, j, k)$, $(i, *, k)$ and $(i, j, *)$ respectively. Consider an F which has type (i, j, k). This F must be covered by a chain from $\mathcal{C}$, but it can only be covered with chains that have type $(*, j, k)$, $(i, *, k)$ or $(i, j, *)$. There are $\binom{n_1}{i}\binom{n_2}{j}\binom{n_3}{k}$ sets with type (i, j, k), so $y(*, j, k) + y(i, *, k) + y(i, j, *) \geq \binom{n_1}{i}\binom{n_2}{j}\binom{n_3}{k}$. □

Remark 4.1. This means that if the LP and integer optimum is not the same somewhere, then we have strict inequality in (4.1). If this is the case, we can not get the best bound by partitioning into monochromatic chains.

5. Lower bounds

Our method is a refinement of that of Füredi [5] and Griggs, Odlyzko, Shearer [6]. In this section, we always assume that $X = X_1 \cup X_2 \cup X_3$, $|X_i| = n$ for $i = \{1, 2, 3\}$.

Definition 1. *The set*

$$R_n(a_1, a_2, a_3, t) = \{F \in \mathcal{P}(X) | a_i \leq |X_i \cap F| < a_i + t, \quad (i = 1, 2, 3)\}$$

is called a cube of side t.

DEFINITION 2. *The cubes* $R_n(a_1, a_2, a_3, t)$ *and* $R_n(b_1, b_2, b_3, u)$ *are compatible if there exist* $i \neq j \in \{1, 2, 3\}$ *such that* $[a_i, a_i + t) \cap [b_i, b_i + u) = \varnothing$ *and* $[a_j, a_j + t) \cap \cap [b_j, b_j + u) = \varnothing$.

The proof of the following lemma is straightforward.

LEMMA 10. *If* $R_n(a_1(i), a_2(i), a_3(i), t(i))$, $i = 1, 2, \ldots, N$ *are pairwise compatible cubes and* $\mathcal{F}_i \subset R_n(a_1(i), a_2(i), a_3(i), t(i))$ *are three-part Sperner families then* $\mathcal{F} = \cup_{i=1}^{N} \mathcal{F}_i$ *is also a three-part Sperner family.*

Now we give a general method for finding large three-part Sperner families. Consider some Sperner families $\mathcal{F}_i \subset R_n(a_1(i), a_2(i), a_3(i), t(i))$, (i=1,2,...,N) and construct a graph G on the vertices $\{1, 2, \ldots, N\}$. A pair of vertices (i, j) will be an edge of G if $R_n(a_1(i), a_2(i), a_3(i), t(i))$ and $R_n(a_1(j), a_2(j), a_3(j), t(j))$ are not compatible. We assign vertex i with the weight $|\mathcal{F}_i|$.. Let H be an independent set in G. There will be a Sperner family with the size of the total weight of H. Namely $\cup_{i \in H} \mathcal{F}_i$ is a Sperner family with the size of $\sum\limits_{i \in H} |\mathcal{F}_i|$. So our aim is to find an independent set of weight near the maximum.

We will choose the cubes corresponding to the vertices of G in the following way. Consider a large cube, L, defined as $L := R_n(\lfloor \frac{n}{2} \rfloor - kt, \lfloor \frac{n}{2} \rfloor - kt, \lfloor \frac{n}{2} \rfloor - kt, 2kt)$. Consider dividing the cube L into $(2k)^3$ smaller cubes S_i, where each

$$S_i = C_n^t(a, b, c) = R_n\Big(\lfloor \frac{n}{2} \rfloor + at, \lfloor \frac{n}{2} \rfloor + bt, \lfloor \frac{n}{2} \rfloor + ct, t\Big)$$

where $-k \leq a, b, c \leq k - 1$. The triplet (a, b, c) will identify the corresponding vertex of G. Two different cubes $C_n^t(a_1, b_1, c_1)$ and $C_n^t(a_2, b_2, c_2)$ are not compatible if and only if at least two of the following equalities hold, $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$. For a subset H of the vertices of G its characteristic vector is the vector $(x(a, b, c))_{-k \leq a,b,c \leq k-1}$, where $x(a, b, c) = 1$ if $(a, b, c) \in H$ and 0 otherwise.

It is easy to see that the binary solutions of the following integer program I_k are exactly the characteristic vectors of the independent sets of G.

$$\forall\, b, c \quad (-k \leq b, c \leq k - 1) \qquad \sum_{a=-k}^{k-1} x(a, b, c) \leq 1,$$

$$\forall\, a, c \quad (-k \le a, c \le k-1) \qquad \sum_{b=-k}^{k-1} x(a,b,c) \le 1,$$

$$\forall\, a, b \quad (-k \le a, b \le k-1) \qquad \sum_{c=-k}^{k-1} x(a,b,c) \le 1,$$

$$\forall\, a, b, c \quad (-k \le a, b, c \le k-1) \qquad x(a,b,c) \ge 0.$$

LEMMA 11. *Suppose $\mathcal{F}_n(a,b,c) \subset C_n^t(a,b,c)$. Let the weight of the vertex (a,b,c) be $w_n^t(a,b,c) = |\mathcal{F}_n(a,b,c)|$. The characteristic vector of the maximum weighted independent set can be determined as the optimum solution of the integer program I_k with the objective function* $\max \sum_{a=-k}^{k-1} \sum_{b=-k}^{k-1} \sum_{c=-k}^{k-1} w_n^t(a,b,c) \cdot x(a,b,c)$. *If the value of the objective function is M, we have a Sperner family of size M.*

What is the largest $w_n^t(a,b,c)$ that can be chosen? The following simple lemma gives us a good bound.

LEMMA 12. *For a given cube $R_n(a_1,a_2,a_3,t)$ there exist a three-part Sperner family $\mathcal{F}$ such that $\mathcal{F} \subset R_n(a_1,a_2,a_3,t)$ and $|\mathcal{F}| \ge \frac{1}{t}|R_n(a_1,a_2,a_3,t)|$.*

PROOF. Define $\mathcal{F}_i = \{F \in R_n(a_1,a_2,a_3,t) \mid \ |F| \equiv i \pmod t\}$ for $i = 1, 2, \ldots, t$. Clearly $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t$ are pairwise disjoint, their union is $R_n(a_1,a_2,a_3,t)$ and all of them are three-part Sperner families. So the largest of them will be proper. ☐

Therefore in Lemma 11 $w_n^t(a,b,c)$ can be chosen to be $\frac{|C_n^t(a,b,c)|}{t}$.

Fix k. Now for a given n and various choices of t we obtain large Sperner families by applying Lemma 11 with $w_n^t(a,b,c) = \frac{|C_n^t(a,b,c)|}{t}$. But we want a bound for d_3, so we have to find some kind of limit of these integer programs. Our aim is to chose the parameter $t = t(n)$ such that the following limit exists:

$$w(a,b,c) = \lim_{n\to\infty} \frac{w_n^{t(n)}(a,b,c)}{\binom{3n}{\lfloor 1.5n \rfloor}} = \lim_{n\to\infty} \frac{|C_n^{t(n)}(a,b,c)|}{t(n)\binom{3n}{\lfloor 1.5n \rfloor}}. \tag{5.1}$$

LEMMA 13. *Consider the optimum solution of I_k with respected to the objective function* $\max \sum_{a=-k}^{k-1} \sum_{b=-k}^{k-1} \sum_{c=-k}^{k-1} w(a,b,c) \cdot x(a,b,c)$. *This gives us an independent set H with total weight W. W is a lower bound for d_3.*

PROOF. For each n we have a Sperner family with size

$$\sum_{(a,b,c)\in H} w_n^{t(n)}(a,b,c) = \sum_{(a,b,c)\in H}(1+o(1))w(a,b,c)\binom{3n}{\lfloor 1.5n\rfloor} =$$

$$= (1+o(1))\binom{3n}{\lfloor 1.5n\rfloor}\sum_{(a,b,c)\in H} w(a,b,c) = (1+o(1))\binom{3n}{\lfloor 1.5n\rfloor}W,$$

which means that W is a lower bound for d_3. □

LEMMA 14. *The limit* $w(a_1,a_2,a_3)$ *in* (5.1) *exists if we choose* $t(n)$ *to be* $\lfloor \frac{s}{2}\sqrt{n}\rfloor$, *where* $s\in\mathbb{R}^+$ *is constant, and then*

$$w(a_1,a_2,a_3) = \frac{2\sqrt{1.5\pi}}{s}\prod_{i=1}^{3}(\Phi((a_i+1)s)-\Phi(a_i s)).$$

(*Here* $\Phi(t)$ *is the cumulative density function of standard normal distribution, that is* $\Phi(t) = \int\limits_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$.)

PROOF. We have to find the following limit:

$$\lim_{n\to\infty}\frac{\frac{1}{\lfloor\frac{s}{2}\sqrt{n}\rfloor}|R_n(\lfloor\frac{n}{2}\rfloor + a_1\lfloor\frac{s}{2}\sqrt{n}\rfloor, \lfloor\frac{n}{2}\rfloor + a_2\lfloor\frac{s}{2}\sqrt{n}\rfloor, \lfloor\frac{n}{2}\rfloor + a_3\lfloor\frac{s}{2}\sqrt{n}\rfloor, \lfloor\frac{s}{2}\sqrt{n}\rfloor)|}{\binom{3n}{\lfloor 1.5n\rfloor}}.$$

From the de Moivre–Laplace theorem

$$\sum_{j=\lfloor\frac{n}{2}\rfloor + a_i\lfloor\frac{s}{2}\sqrt{n}\rfloor}^{\lfloor\frac{n}{2}\rfloor + a_i\lfloor\frac{s}{2}\sqrt{n}\rfloor + \lfloor\frac{s}{2}\sqrt{n}\rfloor - 1}\binom{n}{j} = (1+o(1))(\Phi((a_i+1)s)-\Phi(a_i s))2^n$$

is obtained. This equation and the fact that $\lim\limits_{m\to\infty}\frac{\binom{m}{\lfloor\frac{m}{2}\rfloor}}{\frac{2^m}{\sqrt{\pi m}}} = 1$ imply the statement. □

The following lemma is an easy consequence of Lemma 13.

Lemma 15. *For any* $s \in \mathbb{R}^+, k \in \mathbb{Z}^+$, *the optimum value of the integer program* I_k *with objective function*

$$\max \sum_{a=-k}^{k-1} \sum_{b=-k}^{k-1} \sum_{c=-k}^{k-1} w(a,b,c) \cdot x(a,b,c)$$

is a lower bound for d_3. ☐

Lemma 16. *The optimum value of the previous integer program is at least* 4 *times the optimum value of the following integer program*:

$$\forall\, b,c \quad (0 \le b,c \le k-1) \qquad \sum_{a=0}^{k-1} z(a,b,c) \le 1,$$

$$\forall\, a,c \quad (0 \le a,c \le k-1) \qquad \sum_{b=0}^{k-1} z(a,b,c) \le 1,$$

$$\forall\, a,b \quad (0 \le a,b \le k-1) \qquad \sum_{c=0}^{k-1} z(a,b,c) \le 1,$$

Maximize

$$\sum_{a=0}^{k-1} \sum_{b=0}^{k-1} \sum_{c=0}^{k-1} w(a,b,c) \cdot z(a,b,c),$$

where $z(a,b,c)$ *are binary variables* $(0 \le a,b,c \le k-1)$.

Proof. Let us define $|x|^*$ as x if x is nonnegative, and $-x-1$ if x is negative.

If z a feasible binary solution , then a feasible binary solution x of I_k can be defined in the following way: $x(a,b,c) = 1$ if and only if there are 0 or 2 negative numbers among a,b,c, and $z(|a|^*, |b|^*, |c|^*) = 1$. It is easy to check that this is really a feasible solution and the lemma follows from the fact that $w(a,b,c) = w(|a|^*, |b|^*, |c|^*)$. ☐

Solving the integer program in Lemma 16 by the aid of a computer using the parameters $s = 0.1$ and $k = 32$ we obtain a feasible solution with the value 0.262725 of the objective function. We have obtained that $d_3 > 1.0509$ [10].

6. Acknowledgment

The author would like to thank Gyula Katona for his useful suggestions throughout the writing of this paper. The author is also grateful for the anonymous referee's comments.

Partially supported by the ERC Consolidator Grant 648017.

Bibliography

1. ***H. Aydinian, É. Czabarka, P. L. Erdős, L. A. Székely***, A tour of M-part L-Sperner families, J. Comb. Theory (A) **118**, (2011) 702–725.
2. ***N. G. de Bruijn, C. A. van E. Tengbergen, D. Kruyswij***, On the set of divisors of a number, Nieuw Arch. Wisk. (2) **23** (1949–51), 191–193.
3. ***P. L. Erdős, G. O. H. Katona***, Convex hulls of more-part Sperner families, Graphs and Combinatorics **2** (1986), 123–134.
4. ***Z. Furedi, J. R. Griggs, A. M. Odlyzko, J. B. Shearer***, Ramsey-Sperner theory, Discrete Mathematics **63** (1987), 143–152.
5. ***Z. Füredi***, A Ramsey-Sperner theorem, Graphs and Combinatorics **1** (1985), 51–56.
6. ***J. R. Griggs, A. M. Odlyzko, J. B. Shearer***, k-Color Sperner theorems, J. Comb. Theory (A) **42** (1986), 31–54.
7. ***G. Katona***, On a conjecture of Erdős and a stronger form of Sperner's theorem, Studia Sci. Math. Hungar. **1** (1966), 059–063.
8. ***G. Katona***, A three part Sperner theorem, Studia Sci. Math. Hungar. **8** (1973), 379–390.
9. ***D. J. Kleitman***, On a lemma of Littlewood and Offord on the distribution of certain sums, Math. Z. **90** (1965), 251–259.
10. www.cs.elte.hu/mesza/morepart.html (Details of computer calculations.)

András Mészáros

Alfréd Rényi Institute,
13-15 Reáltanoda u.,
1053 Budapest,
Hungary
meszaandras@gmail.com