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On coloring uniform hypergraphs without 3-cycles

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Abstract: This paper deals with a combinatorial problem concerning sparse hypergraphs with high chromatic number. Let H be a hypergraph and let $\Delta(H)$ denote the maximum vertex degree of H. We study the quantity $\Delta(n, r, s)$, which is the minimum possible $\Delta(H)$, where H is an n-uniform non-r-colorable hypergraph with girth at least $s+1$. With the help of the method of random recoloring we prove that for large n and for all r

$$\Delta(n, r, 3) \geqslant r^{n-1} n^{-4\left\lfloor\sqrt{\frac{\ln n}{\ln(2\ln n)}}\right\rfloor^{-1}}.$$

This bound improves asymptotically all previously known results if $\ln r = O(n^{2n-3})$.

Keywords: sparse hypergraphs, hypergraph colorings, random recoloring method

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1. Introduction and history of the problem

The current paper deals with a combinatorial problem concerning sparse hypergraphs with high chromatic number. First of all, we recall the main definitions.

Let $H = (V, E)$ be a hypergraph. A *cycle of length* k in H is a sequence $e_0, v_0, e_1, v_1, \dots, e_{k-1}, v_{k-1}, e_k = e_0$ of distinct edges $e_0, \dots, e_{k-1}$ and vertices $v_0, \dots, v_{k-1}$ such that $v_i \in e_i \cap e_{i+1}$ for all $i = 0, \dots, k-1$. The length of the shortest cycle in a hypergraph is called *the girth* of the hypergraph. We shall use $girth(H)$ to denote the girth of H. If $girth(H) > 2$, then the hypergraph $H = (V, E)$ is called *simple*. In this case every two of its edges have at most one common vertex, i. e.

$$\forall\ e, f \in E,\ \ f \neq e : |e \cap f| \leqslant 1.$$

Let $\Delta(H)$ denote the maximum vertex degree and let $\chi(H)$ denote the chromatic number of a hypergraph H. For every $n, r \geqslant 2,\ s \geqslant 1$ let $\Delta(n, r, s)$ denote the minimum D such that there exists an n-uniform non-r-colorable hypergraph H with girth at least $s+1$ and maximum vertex degree D. In other words,

$$\Delta(n, r, s) = \min\{\Delta(H) : H \text{ is } n\text{-uniform}, \chi(H) > r,\ girth(H) > s\}.$$

First results concerning $\Delta(n, r, s)$ were obtained by P. Erdős and L. Lovász in 1972. In their famous seminal paper [3] they proved (see Theorem 2 in [3]) that for all $n, r \geqslant 2$

$$\Delta(n, r, 1) > \frac{1}{4} n^{-1} r^{n-1}. \tag{1}$$

Moreover, in Theorem 1′ (see [3]) Erdős and Lovász showed that for all sufficiently large n and all $r \geqslant 2,\ s \geqslant 2$,

$$\Delta(n, r, s) \leqslant 20 n^2 r^{n+1}. \tag{2}$$

The lower bound (1) was asymptotically improved first in the case $r = 2$. In 1990 Z. Szabó (see [10]) proved the following theorem.

Theorem 1 (Z. Szabó, [10]). *For every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for every $n \geqslant n_0$, any n-uniform simple hypergraph, which is not 2-colorable, has a vertex with degree at least $2^n n^{-\varepsilon}$.*

This theorem implies the inequality

$$\Delta(n, 2, 2) > 2^n n^{-\varepsilon}. \tag{3}$$

An alternative proof of Theorem 1 was obtained by J. Radhakrishnan and A. Srinivasan in [7]. They also proved that for large n

$$\Delta(n, 2, 1) > \frac{0{,}17}{\sqrt{n \ln n}} 2^n.$$

This bound asymptotically improves (1) in the case $r = 2$.

In 2009 A. V. Kostochka and M. Kumbhat proved (see [5]) the following theorem.

THEOREM 2 (A. V. Kostochka, M. Kumbhat, [5]). *For every $\varepsilon > 0$, $r \geqslant 2$ there exists an integer $n_0 = n_0(r, \varepsilon)$ such that for every $n \geqslant n_0$, any n-uniform simple hypergraph with maximum edge-degree $r^n n^{1-\varepsilon}$ is r-colorable.*

This theorem gives a lower bound for $\Delta(n, r, 2)$ in the case of fixed r and large n: for all $r \geqslant 2$ and $\varepsilon > 0$, there exists an integer $n_0 = n_0(r, \varepsilon)$ such that for all $n > n_0$

$$\Delta(n, r, 2) > r^{n-1} n^{-\varepsilon}. \tag{4}$$

This result generalizes Szabó's bound (3) to the case of an arbitrary fixed value of parameter r.

The last known nontrivial lower bound for $\Delta(n, r, 2)$ was proved by A. Frieze and D. Mubayi. In their paper [4] they established the following interesting result.

THEOREM 3 (A. Frieze, D. Mubayi, [4]). *Let H be an n-uniform simple hypergraph with $\Delta(H) = \Delta$. Then*

$$\chi(H) < c(n) \left(\frac{\Delta}{\ln \Delta} \right)^{\frac{1}{n-1}},$$

where $c(n)$ does not depend on H.

In particular, this theorem gives the following lower bound for $\Delta(n, r, 2)$:

$$\Delta(n, r, 2) \geqslant c_0(n)\, r^{n-1} \ln r. \tag{5}$$

However, the order of the quantity $c_0(n)$ on the right-hand side of (5) is very small. If follows from the proof of Theorem 3 that $c_0(n) = O(n^{2-2n})$.

Another upper bound for $\Delta(n, r, s)$ was obtained by A. V. Kostochka and V. Rödl (see [6]): for all $n, r \geqslant 2$ and $s \geqslant 2$

$$\Delta(n, r, s) \leqslant \lceil n r^{n-1} \ln r \rceil. \tag{6}$$

It is easy to see that this bound improves the preceding result of Erdős and Lovász (2) for all n, r and s. Together with (5) this inequality shows that the quantity $\Delta(n, r, s)$ is of order $\Theta\left(r^{n-1} \ln r\right)$ if n is fixed, $r \to \infty$ and $s \geqslant 2$.

Finally, the author of this paper showed in [8] that for all $n, r \geqslant 3$

$$\Delta(n, r, 1) \geqslant \frac{1}{14} r^{n-1} n^{-1/2}. \tag{7}$$

Obviously, the estimate (7) is asymptotically better than the classical result of Erdős and Lovász (1) for all $r \geqslant 3$.

2. New lower bound for $\Delta(n, r, 3)$

Let us sum up what is known about estimating the maximum vertex degree in the class of hypergraphs with girth at least 4, i. e. the hypergraphs without 2- and 3-cycles. We have good upper bounds (2) and (6) for the quantity $\Delta(n, r, 3)$. Since $\Delta(n, r, s+1) \geqslant \Delta(n, r, s)$, the inequalities (3) and (4) also provide lower bounds for $\Delta(n, r, 3)$, but they hold only for small values of r (fixed in asymptotics). On the other hand, the result of Frieze and Mubayi (5) gives a nontrivial lower bound for $\Delta(n, r, 3)$ only if r is very large as compared with n, namely, if $\ln r \gg n^{1-2n}$. So, we do not have any nontrivial lower estimate for $\Delta(n, r, 3)$ in the «medium» area, we have to use the result (7), which holds for all n-uniform hypergraphs.

The main result of this paper is a new lower bound for $\Delta(n, r, 3)$. Besides improving the result (4) of Kostochka and Kumbhat, we give an estimate that holds for all values of r. The exact statement is formulated in the following theorem.

THEOREM 4. *There exists an integer n_0 such that for all $n \geqslant n_0$ and $r \geqslant 2$*

$$\Delta(n, r, 3) \geqslant r^{n-1} n^{-4\left\lfloor \sqrt{\frac{\ln n}{\ln(2\ln n)}} \right\rfloor^{-1}}. \tag{8}$$

Let us compare the obtained bound (8) with those previously known. It is clear that (8) asymptotically improves the lower bound (1) by Erdős and Lovász, as well as the result (7) (recall that (1) and (7) provide bounds for $\Delta(n, r, 3)$ since $\Delta(n, r, s+1) \geqslant \Delta(n, r, s)$).

Our bound (8) also slightly improves the estimate (4) of Kostochka and Kumbhat (as well as Szabó's result (3), which is a particular case of (4)) for the quantity $\Delta(n, r, 3)$. Indeed, from (4) we get a lower bound for $\Delta(n, r, 3)$ of the type $r^n n^{-\varepsilon(n)}$ for fixed r, where $\varepsilon(n) = o(1)$. But the order of the quantity $\varepsilon(n)$ remains unclear. In their final comment, Kostochka and Kumbhat said that the parameter ε in (4) could be considered as a function of n and chosen to be equal to $\dfrac{\ln\ln\ln n}{\ln\ln n}$. Our bound (8) gives a clear order of this infinitesimal quantity: $\sqrt{(\ln\ln n)/(\ln n)}$. So, our bound is asymptotically better. Moreover, the inequality (8) holds for all possible r.

Furthermore, our lower bound (8) is better than the result (5) of Frieze and Mubayi when r is not very large as compared with n:

$$\ln r \leqslant (c_0(n))^{-1} n^{-4\left\lfloor\sqrt{\frac{\ln n}{\ln(2\ln n)}}\right\rfloor^{-1}}.$$

This relation holds, for instance, when $\ln r = O\big(n^{2n-2}e^{-O(\sqrt{\ln n \ln\ln n})}\big)$ (recall that $c_0(n) = O(n^{2-2n})$).

Thus, we have obtained a new lower bound for $\Delta(n, r, 3)$, which asymptotically improves all previously known results if $\ln r = O(n^{2n-3})$. Note that the upper bound (6) is only $n^{1+o(1)} \ln r$ times greater than (8).

Theorem 4 is a simple corollary of the following multiparametric theorem, which, in fact, provides a new lower bound for the maximum edge-degree in a non-r-colorable hypergraph of large girth.

THEOREM 5. *Let $n \geqslant 3$, $r \geqslant 2$ be integers, let k, α be positive numbers. Let us denote*

$$t = \left\lfloor\sqrt{\frac{\ln n}{\ln(\alpha \ln n)}}\right\rfloor, \quad q = \frac{\alpha \ln n}{n}. \tag{9}$$

Let $H = (V, E)$ be an n-uniform simple hypergraph without 3-cycles (i. e. girth$(H) > 3$) such that every edge of H intersects at most d other edges of H, where

$$d \leqslant r^{n-1} n^{1-k/t} - 1. \tag{10}$$

If the following inequalities hold

$$k \leqslant t < n, \tag{11}$$

$$\frac{2}{n} \leqslant q \leqslant \frac{1}{2}, \tag{12}$$

$$\frac{n^2}{2^n} + (t+1)n^{1-\alpha}e^{\alpha(\ln n)t/n}(\alpha \ln n)^t + \frac{(t+1)^2}{t!}n^{2-k} + (t+1)te^{t-1}n^{1+\alpha-k} < \frac{1}{4}, \tag{13}$$

then $\chi(H) \leqslant r$.

The proof of Theorem 5 is based on the method of random recoloring. This method in the case of two colors was developed in the papers by J. Beck [2], J. Spencer [9], Radhakrishnan and Srinivasan [7]. In this paper we generalize their techniques to the case of arbitrary number of colors r.

3. Proofs

3.1. Proof of Theorem 4

We shall use Theorem 5. Let us choose the parameters k and α:

$$k = 4, \quad \alpha = 2.$$

By this choice of k and α, there exists an integer n_1 such that for all $n \geqslant n_1$ the inequalities (11) and (12) hold. Let us consider the left-hand side of (13). We have $t = O\left(\sqrt{\ln n / \ln\ln n}\right)$ (see (9)), so

$$(t+1)n^{1-\alpha}e^{\alpha(\ln n)t/n} = e^{O(\ln\ln n)}n^{-1}e^{o(1)} = o(1), \quad n \to \infty,$$
$$\frac{(t+1)^2}{t!}\,n^{2-k} = O\big(n^{-2}\big) = o(1), \quad n \to \infty,$$
$$(t+1)te^{t-1}n^{1+\alpha-k} = e^{O(\sqrt{\ln n})}n^{-1} = o(1), \quad n \to \infty.$$

These relations imply the existence of an integer n_2 such that the inequality (13) holds for all $n \geqslant n_2$.

Let $H = (V, E)$ be an n-uniform hypergraph with $girth(H) > 3$ and $\chi(H) > r$. In the case $n \geqslant n_0 = \max(n_1, n_2)$ the hypergraph H satisfies all the conditions of Theorem 5, except (10). But H is not r-colorable, and so there exists an edge $e \in E$ with edge-degree at least $\lfloor r^{n-1}n^{1-k/t} \rfloor$. So, e contains a vertex with degree at least

$$\frac{\lfloor r^{n-1}n^{1-k/t} \rfloor}{n+1} \geqslant \frac{\left(r^{n-1}n^{1-k/t} - 1\right)}{n+1} = r^{n-1}n^{-k/t} + 1 - \frac{1}{n}.$$

Thus, we have established the inequality $\Delta(H) > r^{n-1}n^{-k/t}$ and, consequently,

$$\Delta(n, r, 3) \geqslant r^{n-1}n^{-k/t} = r^{n-1}n^{-4\left\lfloor\sqrt{\frac{\ln n}{\ln(2\ln n)}}\right\rfloor^{-1}}.$$

Theorem 4 is proved.

3.2. Proof of Theorem 5

The proof is based on the method of vertex random coloring. To prove Theorem 5 we have to show the existence of a proper vertex r-coloring for H. We shall construct some random r-coloring and estimate the probability that this coloring is proper for H. If this probability is greater than 0, then we prove the existence of a required coloring, and the theorem follows.

3.2.1. Algorithm of random recoloring

We follow the ideas of Radhakrishnan and Srinivasan from [7] concerning random recoloring. Let $V = \{v_1, \dots, v_w\}$. The algorithm consists of two phases.

Phase 1. We color all the vertices randomly and uniformly with r colors, independently of one another. Let us denote the generated random coloring by χ_0.

The obtained coloring χ_0 can contain monochromatic edges and «almost monochromatic» edges. An edge $e \in E$ is said to be *almost monochromatic* in χ_0 if there is a color u such that

$$n - t + 2 < |\{v \in e: \ v \text{ is colored with } u \text{ in } \chi_0\}| < n,$$

where t is a parameter, whose value will be chosen later. In this case, the color u is called *dominating* in e. For every $v \in V$, $u = 1, \dots, r$, let us denote

$$\mathcal{M}(v) = \{e \in E : v \in e, \ e \text{ is monochromatic in } \chi_0\},$$
$$\mathcal{AM}(v, u) = \{e \in E : v \in e, \ e \text{ is almost monochromatic in } \chi_0 \text{ with dominating color } u\}.$$

Phase 2. At this phase we want to recolor some vertices in the edges that are monochromatic in χ_0. We consider the vertices according to an arbitrary fixed order $v_1, \dots, v_w$. Let $\{\eta_1, \dots, \eta_w\}$ be mutually independent equally distributed random variables taking the values $1, \dots, r$ with the same probability p (the value of the parameter p will be chosen later) and the value 0 with probability $1 - rp$. The recoloring procedure consists of w steps.

Step 1. Assume that $\mathcal{M}(v_1) \neq \varnothing$ and, moreover, that there is no $u = 1, \dots, r$ and $e \in \mathcal{AM}(v_1, u)$ such that

1. $\eta_1 = u$,
2. v_1 is the only vertex in e not colored by u in χ_0.

Then we try to recolor v_1 according to the value of the random variable η_1:

- if $\eta_1 = 0$, then we do not recolor v_1,
- if $\eta_1 \neq 0$, then we recolor v_1 with the color η_1.

In all the other situations, we do not change the color of v_1. Let χ_1 be the coloring after v_1 is considered.

Suppose now that the vertices $v_1, \dots, v_{i-1}$ are considered, and the coloring χ_{i-1} is obtained.

Step ***i***. Assume that some $f \in \mathcal{M}(v_i)$ is still monochromatic in χ_{i-1} and, moreover, there is no $u = 1, \dots, r$ and $e \in \mathcal{AM}(v_i, u)$ such that

1. $\eta_i = u$,
2. v_i is the only vertex in e not colored with u in χ_{i-1}.

Then we try to recolor v_i according to the value of the random variable η_i:

- if $\eta_i = 0$, then we do not recolor v_i,
- if $\eta_i \neq 0$, then we recolor v_i with the color η_i.

In all the other situations, we do not change the color of v_i. Let the resulting coloring be χ_i.

Let $\widetilde{\chi}$ be the coloring obtained after considering all the vertices.

Let us describe the structure of the proof. We give a more formal construction of $\widetilde{\chi}$ using the technique of random variables. This is useful for the further proof. We analyze the event $\mathcal{F}$ that $\widetilde{\chi}$ is not a proper coloring of H. We divide $\mathcal{F}$ into some «local» events and estimate their probabilities. Finally, we use the Local Lemma to show that all these events do not occur simultaneously with positive probability. This implies that $\widetilde{\chi}$ is a proper coloring of H with positive probability, whence we get that H is r-colorable.

3.2.2. Formal construction of the random coloring from section 3.2.1

Without loss of generality we may assume that $V = \{1, 2, 3, \dots, w\}$. Let us also fix an arbitrary ordering of the edges of H. Consider on some probability space the following set of mutually independent random elements:

1) equally distributed random variables $\xi_1, \dots, \xi_w$ taking the values $1, 2, \dots, r$ with equal probability $1/r$;
2) equally distributed random variables $\eta_1, \dots, \eta_w$ taking the values $1, 2, \dots, r$ with equal probability p and the value 0 with probability $1 - rp$. We take p equal to $p = q/(r-1)$. By (12) such choice of p is correct, i. e. for every $r \geqslant 2$ we have $rp \leqslant r/(2(r-1)) \leqslant 1$;

3) random subsets $\Theta(e, Y)$ with uniform distribution on the set $\binom{Y}{t-1}$, where $e \in E$ is an edge of the hypergraph, and $Y \subset e$ is a subset of e satisfying $|Y| \geqslant t-1$.

Note that the random sets $\Theta(e, Y)$ are correctly defined, since $t-1 \leqslant n$ (see (11)).

Let $e \in E$ be an edge of H. For every $u = 1, \dots, r$, let $\mathcal{M}(e, u)$ and $\mathcal{AM}(e, u)$ denote the following events:

$$\mathcal{M}(e, u) = \bigcap_{s \in e} \{\xi_s = u\}, \quad \mathcal{AM}(e, u) = \Big\{0 < \sum_{s \in e} I\{\xi_s \neq u\} \leqslant t-2\Big\}. \tag{14}$$

We shall introduce successively random variables ζ_i, $i = 1, \dots, w$. Let $\mathcal{D}_1$ and $\mathcal{A}_1$ denote the following events:

$$\mathcal{D}_1 = \bigcup_{e \in E: 1 \in e} \bigcup_{u=1}^{r} \mathcal{M}(e, u),$$

$$\mathcal{A}_1 = \bigcup_{f \in E: 1 \in f} \bigcup_{u=1}^{r} \Big(\Big\{\xi_1 \neq u,\ \eta_1 = u,\ \sum_{s \in f:\ s>1} I\{\xi_s = u\} = n-1\Big\} \cap \mathcal{AM}(f, u)\Big),$$

and let

$$\zeta_1 = \xi_1\, I\{\overline{\mathcal{D}_1} \cup \{\eta_1 = 0\} \cup \mathcal{A}_1\} + \eta_1\, I\{\mathcal{D}_1 \cap \{\eta_1 \neq 0\} \cap \overline{\mathcal{A}_1}\}.$$

For every $i > 1$, let $\mathcal{D}_i$ and $\mathcal{A}_i$ denote the events

$$\mathcal{D}_i = \bigcup_{e \in E: i \in e} \bigcup_{u=1}^{r} \Big\{\mathcal{M}(e, u) \cap \bigcap_{s \in e: s<i} \{\zeta_s = u\}\Big\},$$

$$\mathcal{A}_i = \bigcup_{f \in E:\ i \in f} \bigcup_{u=1}^{r} \Big(\Big\{\xi_i \neq u,\ \eta_i = u,\ \sum_{s \in f:\ s<i} I\{\zeta_s = u\} + \sum_{s \in f:\ s>i} I\{\xi_s = u\} = n-1\Big\} \cap \mathcal{AM}(f, u)\Big).$$

We define ζ_i in the following way:

$$\zeta_i = \xi_i\, I\{\overline{\mathcal{D}_i} \cup \{\eta_i = 0\} \cup \mathcal{A}_i\} + \eta_i\, I\{\mathcal{D}_i \cap \{\eta_i \neq 0\} \cap \overline{\mathcal{A}_i}\}.$$

It is easy to see that the random variables ζ_i take values only from $\{1, 2, \dots, r\}$. So, we may interpret the random vector $\vec{\zeta} = (\zeta_1, \dots, \zeta_w)$ as a random r-coloring of the vertex set V (we assign the color ζ_i to the vertex i). Let $\mathcal{F}$ denote the event that $\vec{\zeta}$ is not a proper coloring of H, i. e.

$$\mathcal{F} = \bigcup_{e\in E} \bigcup_{u=1}^{r} \bigcap_{s\in e} \{\zeta_s = u\}. \tag{15}$$

Our task is to prove that $\mathsf{P}(\mathcal{F}) < 1$ under the hypothesis of Theorem 5.

We shall divide the event $\bigcap_{s\in e}\{\zeta_s = u\}$ into three parts depending on the behavior of the random variables $\{\xi_s : \ s \in e\}$. Let $\mathcal{C}_0(e, u)$, $\mathcal{C}_1(e, u)$, $\mathcal{C}_2(e, u)$ be the following events:

$$\mathcal{C}_0(e, u) = \bigcup_{a=1,\, a\neq u}^{r} \bigcap_{s\in e} \{\zeta_s = u,\ \xi_s = a\}, \quad \mathcal{C}_1(e, u) = \bigcap_{s\in e} \{\zeta_s = u,\ \xi_s = u\},$$

$$\mathcal{C}_2(e, u) = \bigcap_{s\in e} \{\zeta_s = u\} \cap \bigcap_{a=1}^{r} \overline{\mathcal{M}(e, a)}. \tag{16}$$

We shall consider these events separately. But first we establish a simple inequality we shall use later. It follows from (12) that

$$\alpha \ln n = qn \geqslant 2. \tag{17}$$

Note that the latter inequality implies that the parameter t in (9) is correctly defined (there is no negative number under the square root).

3.2.3. First part of $\mathcal{F}$: event $\mathcal{C}_0(e, u)$

If the event $\mathcal{C}_0(e, u)$ occurs, then for every $s \in e$, one has $\zeta_s = \eta_s$, since $\zeta_s \neq \xi_s$. We get the relation

$$\bigcup_{u=1}^{r} \mathcal{C}_0(e, u) \subset \bigcup_{u=1}^{r} \bigcup_{a=1,\, a\neq u}^{r} \bigcap_{s\in e} \{\eta_s = u,\ \xi_s = a\} = \mathcal{Q}_0(e). \tag{18}$$

The probability of the event $\mathcal{Q}_0(e)$ can be easily calculated:

$$\mathsf{P}(\mathcal{Q}_0(e)) = \sum_{u=1}^{r} \sum_{a=1,\, a\neq u}^{r} \prod_{s\in e} \mathsf{P}(\eta_s = u,\ \xi_s = a) = r(r-1)\left(\frac{p}{r}\right)^n. \tag{19}$$

3.2.4. *Second part of $\mathcal{F}$: event $\mathcal{C}_1(e, u)$*

Suppose that the event $\mathcal{C}_1(e, u)$ occurs. This event, obviously, implies all the events $\mathcal{D}_s$, $s \in e$. Then the equality $\xi_s = \zeta_s = u$ for a vertex $s \in e$ can happen in two cases: either $\eta_s \in \{0, u\}$, or $\eta_s \notin \{0, u\}$ and the event $\mathcal{A}_s$ occurs. Consider the following partition of $\mathcal{C}_1(e, u)$:

$$\mathcal{C}_1(e, u) = \mathcal{S}_0(e, u) \cup \mathcal{S}_1(e, u), \tag{20}$$

where

$$\mathcal{S}_0(e, u) = \mathcal{C}_1(e, u) \cap \Big\{ \sum_{s \in e} I\{\eta_s \notin \{0, u\}\} \leqslant t - 1 \Big\},$$

$$\mathcal{S}_1(e, u) = \mathcal{C}_1(e, u) \cap \Big\{ \sum_{s \in e} I\{\eta_s \notin \{0, u\}\} > t - 1 \Big\}.$$

Consider the event $\mathcal{S}_0(e, u)$. By the definition (16) of $\mathcal{C}_1(e, u)$ the following relation holds:

$$\mathcal{S}_0(e, u) \subset \bigcap_{s \in e} \{\xi_s = u\} \cap \Big\{ \sum_{s \in e} I\{\eta_s \notin \{0, u\}\} \leqslant t - 1 \Big\}.$$

Let $\mathcal{Q}_1(e)$ denote the union of the latter events:

$$\bigcup_{u=1}^{r} \mathcal{S}_0(e, u) \subset \bigcup_{u=1}^{r} \Big\{ \bigcap_{s \in e} \{\xi_s = u\} \cap \Big\{ \sum_{s \in e} I\{\eta_s \notin \{0, u\}\} \leqslant t - 1 \Big\} \Big\} = \mathcal{Q}_1(e). \tag{21}$$

For the probability of $\mathcal{Q}_1(e)$ we have the following estimate:

$$\begin{aligned} \mathsf{P}\Big(\mathcal{Q}_1(e) \Big) &= r^{1-n} \sum_{j=0}^{t-1} \binom{n}{j} q^j (1 - q)^{n-j} \leqslant r^{1-n} (1 - q)^{n-t} \sum_{j=0}^{t-1} (nq)^j \leqslant \\ &\leqslant r^{1-n} (1 - q)^{n-t} (nq)^t. \end{aligned} \tag{22}$$

The latter inequality follows from the bound (17): $nq = \alpha \ln n \geqslant 2$.

Consider now the event $\mathcal{S}_1(e, u)$. Let us fix $v \in e$ satisfying $\eta_v \notin \{0, u\}$. As it was noted above, the event $\mathcal{A}_v$ should happen for every such vertex. This event implies that for some edge f, $v \in f$, and some color $a \neq u$, the following event has to occur:

$$\mathcal{W}(v, f, u, a) = \Big\{\xi_v = u,\ \eta_v = a,\ \sum_{s\in f:\ s<v} I\{\zeta_s = a\} + \\ + \sum_{s\in f:\ s>v} I\{\xi_s = a\} = n-1\Big\} \cap \mathcal{AM}(f, a).$$

It is easy to show that $f \neq e$. Indeed, for all $s \in e$ we have $\xi_s = \zeta_s = u$, but for all $s \in f\backslash\{v\}$ we have either $\zeta_s = a$, or $\xi_s = a$.

The event $\mathcal{S}_1(e, u)$ implies that the number of vertices $v \in e$ satisfying $\eta_v \notin \{0, u\}$ is at least t. Let us denote $E(e) = \{f \in E\backslash\{e\} :\ e\cap f \neq \varnothing\}$. Then $|e\cap f| = 1$ for every $f \in E(e)$, since $girth(H) > 3$. By the above argument we get

$$\mathcal{S}_1(e,u) \subset \bigcap_{s\in e}\{\xi_s = u\} \cap \bigcup_{\substack{a_1,\ldots,a_t=1\\ a_i\neq u}}^{r} \ \bigcup_{\substack{\{f_1,\ldots,f_t\}\in\binom{E(e)}{t}\\ f_i\cap e\neq f_j\cap e}} \ \bigcap_{i=1}^{t} \mathcal{W}(e\cap f_i, f_i, u, a_i), \quad (23)$$

where the set of edges $\{f_1, \ldots, f_t\}$ is assumed to be ordered according to the originally selected ordering of E, i. e. the number of the edge f_i is less than the number of the edge f_j, if $i < j$. Let us denote $\widehat{f_i} = f_i\backslash e$ and $v_i = e \cap f_i$, $i = 1, \ldots, t$. Since H is a simple hypergraph without 3-cycles, the sets $\widehat{f_i}$, $i = 1, \ldots, t$, do not have common vertices, i. e. $\widehat{f_i} \cap \widehat{f_j} = \varnothing$, if $i \neq j$. Furthermore, $|\widehat{f_i}| = n - 1$.

If the event $\mathcal{W}(e \cap f_i, f_i, u, a_i)$ happens, then by $\mathcal{AM}(f_i, a_i)$ the edge f_i contains at most $t - 2$ vertices s satisfying $\xi_s \neq a_i$. Moreover, for all such vertices $\zeta_s = a_i$, and so $\zeta_s = \eta_s = a_i$. The set $\widehat{f_i}$ contains at most $t - 3$ such vertices, since v_i does not belong to $\widehat{f_i}$ and $\xi_{v_i} = u \neq a_i$. Thus, we obtain the relation

$$\bigcap_{s\in e}\{\xi_s = u\} \cap \bigcap_{i=1}^{t} \mathcal{W}(e\cap f_i, f_i, u, a_i) \subset \bigcap_{s\in e}\{\xi_s = u\} \cap \bigcap_{i=1}^{t}\{\eta_{v_i} = a_i\} \cap \\ \cap \bigcap_{i=1}^{t}\Big\{\bigcap_{s\in\widehat{f_i}} \big(\{\xi_s \neq a_i,\ \eta_s = a_i\} \cup \{\xi_s = a_i\}\big)\Big\} \cap \\ \cap \bigcap_{i=1}^{t}\Big\{\sum_{s\in\widehat{f_i}} I\{\xi_s \neq a_i\} \leqslant t - 3\Big\}. \quad (24)$$

Let $\mathcal{Q}_2(e,F)$ denote the union of the latter events, where $F=\{f_1,\dots,f_t\}$ is an element of $\binom{E(e)}{t}$ satisfying $|e\cap\cup_{f\in F}f|=t$:

$$\mathcal{Q}_2(e,F)=\bigcup_{u=1}^{r}\bigcup_{\substack{a_1,\dots,a_t=1\\a_i\neq u}}^{r}\Big\{\bigcap_{s\in e}\{\xi_s=u\}\cap\bigcap_{i=1}^{t}\{\eta_{v_i}=a_i\}\cap$$
$$\cap\bigcap_{i=1}^{t}\Big\{\bigcap_{s\in\widehat{f_i}}\Big(\{\xi_s\neq a_i,\ \eta_s=a_i\}\cup\{\xi_s=a_i\}\Big)\Big\}\cap$$
$$\cap\bigcap_{i=1}^{t}\Big\{\sum_{s\in\widehat{f_i}}I\{\xi_s\neq a_i\}\leqslant t-3\Big\}\Big\}. \tag{25}$$

The relations (23) and (24) imply

$$\bigcup_{u=1}^{r}\mathcal{S}_1(e,u)\subset\bigcup_{\substack{F\in\binom{E(e)}{t}:\\ \left|e\cap\{\cup_{f\in F}f\}\right|=t}}\mathcal{Q}_2(e,F). \tag{26}$$

Let us estimate the probability of $\mathcal{Q}_2(e,F)$:

$$\mathsf{P}\big(\mathcal{Q}_2(e,F)\big)=\sum_{u=1}^{r}\sum_{\substack{a_1,\dots,a_t=1\\a_i\neq u}}^{r}r^{-n}p^t\prod_{i=1}^{t}\sum_{j=0}^{t-3}\binom{|\widehat{f_i}|}{j}\left(\frac{r-1}{r}\right)^j p^j\left(\frac{1}{r}\right)^{|\widehat{f_i}|-j}=$$
$$=r(r-1)^t r^{-n}p^t r^{-\sum_{i=1}^{t}|\widehat{f_i}|}\prod_{i=1}^{t}\sum_{j=0}^{t-3}\binom{|\widehat{f_i}|}{j}q^j=$$
$$=r(r-1)^t r^{-n}p^t r^{-t(n-1)}\prod_{i=1}^{t}\sum_{j=0}^{t-3}\binom{n-1}{j}q^j\leqslant$$
$$\leqslant r^{-(t+1)(n-1)}q^t\prod_{i=1}^{t}\sum_{j=0}^{t-3}n^jq^j\leqslant r^{-(t+1)(n-1)}q^t(nq)^{t(t-2)}. \tag{27}$$

3.2.5. Third part of $\mathcal{F}$: event $\mathcal{C}_2(e,u)$

We shall show that if the event $\mathcal{C}_2(e,u)$ happens, then the sum $\sum_{s\in e} I\{\xi_s \neq u\}$ cannot be too small. We shall establish the equality

$$\mathcal{C}_2(e,u) = \mathcal{C}_2(e,u) \cap \left\{ \sum_{s\in e} I\{\xi_s \neq u\} \geqslant t-1 \right\}. \tag{28}$$

Indeed, let us consider the intersection of three events (see the definition of the event $\mathcal{C}_2(e,u)$ in (16)):

$$\mathcal{C}_2(e,u) \cap \left\{ \sum_{s\in e} I\{\xi_s \neq u\} \leqslant t-2 \right\} =$$
$$= \bigcap_{s\in e}\{\zeta_s = u\} \cap \bigcap_{a=1}^{r} \overline{\mathcal{M}(e,a)} \cap \left\{ \sum_{s\in e} I\{\xi_s \neq u\} \leqslant t-2 \right\}.$$

The second and the third events imply the event $\mathcal{AM}(e,u)$ (see (14)). The first one implies that for every $s \in e$ satisfying $\xi_s \neq u$ we have $\zeta_s = \eta_s = u$. Moreover, since the event $\mathcal{AM}(e,u)$ holds, the set of such vertices in not empty. Consider a vertex $v \in e$ satisfying $\xi_v \neq u$ and $\xi_s = u$ for every $s \in e,\ s > v$. It is clear that the event $\mathcal{A}_v$ holds. So, $\zeta_v = \xi_v \neq u$, and we get a contradiction with the first event in the intersection. Thus, these three events are inconsistent, which proves (28).

Let us estimate the probability of $\mathcal{C}_2(e,u)$. Consider the random set

$$T = \{s \in e:\ \xi_s \neq u\}.$$

The event $\mathcal{C}_2(e,u)$ implies that all $v \in T$ satisfy $\zeta_v = \eta_v = u$, and also that $|T| \geqslant t-1$ (see (28)). If $\zeta_v \neq \xi_v$ for some vertex v, then at least two events should happen: the event D_v and the event

$$\mathcal{B}(e,f_v,v,u,a_v) = \left\{ \mathcal{M}(f_v,a_v) \cap \bigcap_{s\in f_v: s<v} \{\zeta_s = a_v\} \cap \{\zeta_v = \eta_v = u\} \right\}, \tag{29}$$

where f_v is some edge satisfying $v = e \cap f_v$, and $a_v \neq u$ is some color. Since $girth(H) > 3$, the edges f_v are different for different v.

Let Y be an arbitrary subset of the edge e, satisfying $|Y| \geqslant t-1$. Then for every

$$S \subset Y, \quad S = \{v_1, \dots, v_{t-1}\},$$

we have the inclusion

$$\mathcal{C}_2(e,u)\cap\{T=Y\}\subset\bigcap_{s\in e\setminus Y}\{\xi_s=u\}\cap\bigcap_{s\in Y\setminus S}\{\xi_s\neq u,\ \eta_s=u\}\cap$$
$$\cap\bigcup_{\substack{a_1,\ldots,a_{t-1}=1\\ a_i\neq u}}^{r}\ \bigcup_{\substack{f_1,\ldots,f_{t-1}\in E(e)\\ v_i=f_i\cap e}}\ \bigcap_{i=1}^{t-1}\mathcal{B}(e,f_i,v_i,u,a_i). \tag{30}$$

The relation (30) is true for all $S\in\binom{Y}{t-1}$, so the following inclusion holds:

$$\mathcal{C}_2(e,u)\cap\{T=Y\}\subset\bigcup_{\substack{S\in\binom{Y}{t-1},\\ S=\{v_1,\ldots,v_{t-1}\}}}\Bigg\{\{\Theta(e,Y)=S\}\cap\bigcap_{s\in e\setminus Y}\{\xi_s=u\}\cap$$
$$\cap\bigcap_{s\in Y\setminus S}\{\xi_s\neq u,\ \eta_s=u\}\cap\bigcup_{\substack{a_1,\ldots,a_{t-1}=1\\ a_i\neq u}}^{r}\ \bigcup_{\substack{f_1,\ldots,f_{t-1}\in E\\ v_i=f_i\cap e}}\ \bigcap_{i=1}^{t-1}\mathcal{B}(e,f_i,v_i,u,a_i)\Bigg\}. \tag{31}$$

Let $S_0=\{v_1,\ldots,v_{t-1}\}$ be an arbitrary subset of the edge e. Recall that we denote $E(e)=\{f\in E\setminus\{e\}:\ |f\cap e|=1\}$. We say that $F\in\binom{E(e)}{t-1}$ *defines* the set S_0, if for every vertex $v_i\in S_0$ there exists an edge $f_i\in F$ such that $\{v_i\}=f_i\cap e$. We shall denote the set of all $F\in\binom{E(e)}{t-1}$ defining S_0 by $\widehat{F}(S_0)$.

Note that if $\bigcap_{i=1}^{t-1}\mathcal{B}(e,f_i,v_i,u,a_i)$ holds, then the set of edges $F=\{f_1,\ldots,f_{t-1}\}$ defines the set $S=\{v_1,\ldots,v_{t-1}\}$. So, the union on the right-hand side of (31) is taken only over the sets $F=\{f_1,\ldots,f_{t-1}\}$ from $\widehat{F}(S)$. Then the event from the right-hand side of (31) can be written in the following way:

$$\mathcal{C}_2(e,u)\cap\{T=Y\}\subset\bigcup_{\substack{S\in\binom{Y}{t-1},\\ S=\{v_1,\ldots,v_{t-1}\}}}\ \bigcup_{\substack{F\in\widehat{F}(S),\\ F=\{f_1,\ldots,f_{t-1}\}}}\Bigg\{\{\Theta(e,Y)=S\}\cap\bigcap_{s\in e\setminus Y}\{\xi_s=u\}\cap$$
$$\cap\bigcap_{s\in Y\setminus S}\{\xi_s\neq u,\eta_s=u\}\cap\bigcup_{\substack{a_1,\ldots,a_{t-1}=1\\ a_i\neq u}}^{r}\ \bigcap_{i=1}^{t-1}\mathcal{B}(e,f_i,v_i,u,a_i)\Bigg\}. \tag{32}$$

In view of (29) it is obvious that $\mathcal{B}(e, f_i, v_i, u, a_i)$ is contained in $\bigcap_{s\in f_i}\{\xi_s = a_i\}\cap$ $\cap\{\eta_{v_i} = u\}$. Hence, by (32), we get the relation

$$\mathcal{C}_2(e,u)\cap\{T=Y\}\subset \bigcup_{\substack{S\in\binom{Y}{t-1},\\ S=\{v_1,\dots,v_{t-1}\}}} \bigcup_{\substack{F\in\widehat{F}(S),\\ F=\{f_1,\dots,f_{t-1}\}}} \Bigg\{\{\Theta(e,Y)=S\}\cap\bigcap_{s\in e\setminus Y}\{\xi_s=u\}\cap$$
$$\cap\bigcap_{s\in Y\setminus S}\{\xi_s\neq u,\ \eta_s=u\}\cap\bigcup_{\substack{a_1,\dots,a_{t-1}=1\\ a_i\neq u}}^{r}\bigcap_{i=1}^{t-1}\bigcap_{s\in f_i}\{\xi_s=a_i\}\cap\bigcap_{s\in S}\{\eta_s=u\}\Bigg\}.$$

Let us take the union of both sides over u and Y:

$$\bigcup_{u=1}^{r}\mathcal{C}_2(e,u)\subset\bigcup_{Y\subset e}\bigcup_{\substack{S\in\binom{Y}{t-1},\\ S=\{v_1,\dots,v_{t-1}\}}}\bigcup_{\substack{F\in\widehat{F}(S),\\ F=\{f_1,\dots,f_{t-1}\}}}\bigcup_{u=1}^{r}\Bigg\{\{\Theta(e,Y)=S\}\cap\bigcap_{s\in e\setminus Y}\{\xi_s=u\}\cap$$
$$\cap\bigcap_{s\in Y\setminus S}\{\xi_s\neq u,\ \eta_s=u\}\cap\bigcup_{\substack{a_1,\dots,a_{t-1}=1\\ a_i\neq u}}^{r}\bigcap_{i=1}^{t-1}\bigcap_{s\in f_i}\{\xi_s=a_i\}\cap\bigcap_{s\in S}\{\eta_s=u\}\Bigg\}=\mathcal{S}(e). \qquad (33)$$

Rewrite the right-hand side of (33) in the following way:

$$\mathcal{S}(e)=\bigcup_{\substack{F\in\binom{E(e)}{t-1},\ F \text{ defines } S\in\binom{e}{t-1}\\ F=\{f_1,\dots,f_{t-1}\}}}\ \bigcup_{Y\subset e:\ S\subset Y}\ \bigcup_{u=1}^{r}\Bigg\{\{\Theta(e,Y)=S\}\cap\bigcap_{s\in e\setminus Y}\{\xi_s=u\}\cap$$
$$\cap\bigcap_{s\in Y\setminus S}\{\xi_s\neq u,\ \eta_s=u\}\cap\bigcup_{\substack{a_1,\dots,a_{t-1}=1\\ a_i\neq u}}^{r}\bigcap_{i=1}^{t-1}\bigcap_{s\in f_i}\{\xi_s=a_i\}\cap\bigcap_{s\in S}\{\eta_s=u\}\Bigg\}. \qquad (34)$$

Let us explain this equality. The triple union over Y, S and $F\in\widehat{F}(S)$ in the definition of $\mathcal{S}(e)$ (see (33)) can be replaced by the external union over all $F\in\binom{E(e)}{t-1}$ defining some $S\in\binom{e}{t-1}$, and the internal union over all $Y\subset e$ containing S.

Let us introduce the following event:

$$\mathcal{Q}_3(e,F)=\bigcup_{Y\subset e:\ S\subset Y}\ \bigcup_{u=1}^{r}\Bigg\{\{\Theta(e,Y)=S\}\cap\bigcap_{s\in e\setminus Y}\{\xi_s=u\}\cap$$
$$\cap\bigcap_{s\in Y\setminus S}\{\xi_s\neq u,\ \eta_s=u\}\cap\bigcup_{\substack{a_1,\ldots,a_{t-1}=1\\ a_i\neq u}}^{r}\ \bigcap_{i=1}^{t-1}\bigcap_{s\in f_i}\{\xi_s=a_i\}\cap\bigcap_{s\in S}\{\eta_s=u\}\Bigg\},\qquad(35)$$

where e is an edge of H, and $F\in\binom{E(e)}{t-1}$ ($F=\{f_1,\ldots,f_{t-1}\}$, the edges are written according to the original ordering) defines some $S\in\binom{e}{t-1}$. By (33) and (34) we get the relation

$$\bigcup_{u=1}^{r}\mathcal{C}_2(e,u)\subset\bigcup_{\substack{F\in\binom{E(e)}{t-1},\ F\text{ defines }S\in\binom{e}{t-1}\\ F=\{f_1,\ldots,f_{t-1}\}}}\mathcal{Q}_3(e,F).\qquad(36)$$

We shall estimate the probability of $\mathcal{Q}_3(e,F)$. Since H does not have 3-cycles, the sets $f_i\setminus e$ and $f_j\setminus e$ have empty intersection for different edges $f_i,f_j\in F$. Thus, the following equality holds:

$$\mathsf{P}\Bigg(\{\Theta(e,Y)=S\}\cap\bigcap_{s\in e\setminus Y}\{\xi_s=u\}\cap\bigcap_{s\in Y\setminus S}\{\xi_s\neq u,\eta_s=u\}\cap\bigcup_{\substack{a_1,\ldots,a_{t-1}=1\\ a_i\neq u}}^{r}\ \bigcap_{i=1}^{t-1}\bigcap_{s\in f_i}\{\xi_s=a_i\}\cap$$
$$\cap\bigcap_{s\in S}\{\eta_s=u\}\Bigg)=\frac{1}{\binom{|Y|}{t-1}}r^{|Y|-n}\left(\frac{q}{r}\right)^{|Y|-t+1}(r-1)^{t-1}r^{-n(t-1)}p^{t-1}.\qquad(37)$$

From (37) and (35) we get a bound for the probability of $\mathcal{Q}_3(e,F)$:

$$\mathsf{P}\big(\mathcal{Q}_3(e,F)\big)\leqslant\sum_{u=1}^{r}\ \sum_{Y\subset e:\ S\subset Y}\frac{1}{\binom{|Y|}{t-1}}r^{|Y|-n}\left(\frac{q}{r}\right)^{|Y|-t+1}(r-1)^{t-1}r^{-n(t-1)}p^{t-1}=$$
$$=r\sum_{y=t-1}^{n}\binom{n-t+1}{y-t+1}\frac{1}{\binom{y}{t-1}}r^{y-n}\left(\frac{q}{r}\right)^{y-t+1}(r-1)^{t-1}r^{-n(t-1)}p^{t-1}.\qquad(38)$$

Let us explain the transition to the sum over y. The sum over Y containing S (recall, that F defines S) can be replaced by a double sum: the first one is over $y = |Y|$, and the second one is over Y containing S and having the cardinality y. It is clear that the number of such sets is $\binom{n-t+1}{y-t+1}$ when S is fixed ($|S| = t-1$).

We shall reorganize the sum on the right-hand side of (38) with the help of the equality $\binom{n}{t-1}\binom{n-t+1}{y-t+1} = \binom{n}{y}\binom{y}{t-1}$. The following relations hold:

$$\mathsf{P}\big(\mathcal{Q}_3(e,F)\big) \leqslant r\sum_{y=t-1}^{n}\binom{n-t+1}{y-t+1}\frac{1}{\binom{y}{t-1}}r^{y-n}\left(\frac{q}{r}\right)^{y-t+1}(r-1)^{t-1}r^{-n(t-1)}p^{t-1} =$$

$$= r\sum_{y=t-1}^{n}\binom{n}{y}\frac{1}{\binom{n}{t-1}}r^{y-n}\left(\frac{q}{r}\right)^{y-t+1}(r-1)^{t-1}r^{-n(t-1)}p^{t-1} =$$

$$= r^{-n+t-n(t-1)}\frac{1}{\binom{n}{t-1}}\sum_{y=t-1}^{n}\binom{n}{y}q^{y-t+1}q^{t-1} =$$

$$r^{-(n-1)t}\frac{1}{\binom{n}{t-1}}\sum_{y=t-1}^{n}\binom{n}{y}q^{y} \leqslant r^{-(n-1)t}\frac{1}{\binom{n}{t-1}}(1+q)^{n}. \tag{39}$$

The bound (39) completes the estimation of different parts of $\mathcal{F}$. Now we shall prove that the probability of $\mathcal{F}$ is less than 1 under the hypothesis of Theorem 5.

3.2.6. Application of the Local Lemma to estimate the probability of $\mathcal{F}$

Recall that by the definitions (15) and (16) of the events $\mathcal{F}$ and $\mathcal{C}_i(e,u)$, $i = 1, 2, 3$, $e \in E$, $u = 1, \ldots, r$, we have the equality

$$\mathcal{F} = \bigcup_{e\in E}\bigcup_{u=1}^{r}\big(\mathcal{C}_1(e,u)\cup\mathcal{C}_2(e,u)\cup\mathcal{C}_3(e,u)\big).$$

It follows from the obtained relations (18), (20), (21), (26) and (36) that

$$\mathcal{F} \subset \bigcup_{e\in E}\{\mathcal{Q}_0(e)\cup\mathcal{Q}_1(e)\}\cup\bigcup_{e\in E}\bigcup_{F\in\binom{E(e)}{t}}\mathcal{Q}_2(e,F)\cup\bigcup_{e\in E}\bigcup_{F\in\binom{E(e)}{t-1}}\mathcal{Q}_3(e,F). \tag{40}$$

Later on we shall use a classical claim called the Local Lemma. This statement was first proved in the paper [3] by P. Erdős and L. Lovász. Here we formulate it in a special case.

THEOREM 6. *Given events $\mathcal{B}_1, \ldots, \mathcal{B}_N$ on some probability space, suppose $S_1, \ldots, S_N$ are subsets of $\mathcal{R}_N = \{1, \ldots, N\}$ such that for each $i = 1, \ldots, N$ the event $\mathcal{B}_i$ is independent of the algebra generated by the events $\{\mathcal{B}_j, j \in \mathcal{R}_N \setminus S_i\}$. Suppose also that for each $i = 1, \ldots, N$ the following inequalities hold*:

$$1)\ \mathsf{P}(\mathcal{B}_i) < 1/2, \quad 2) \sum_{j \in S_i \setminus \{i\}} \mathsf{P}(\mathcal{B}_j) \leqslant 1/4. \tag{41}$$

Then $\mathsf{P}\Big(\bigcap_{j=1}^{N} \overline{\mathcal{B}_j}\Big) \geqslant \prod_{j=1}^{N} \big(1 - 2\mathsf{P}(\mathcal{B}_i)\big) > 0.$

The proof of the Local Lemma can be found in the book [1]. Consider the system of events Ψ consisting of all the events $\mathcal{Q}_i(e)$, $i = 0, 1$, $e \in E$, the events $\mathcal{Q}_2(e, F)$, $e \in E$, $F \in \binom{E(e)}{t}$, and also of the events $\mathcal{Q}_3(e, F)$, $e \in E$, $F \in \binom{E(e)}{t-1}$. By (40) we have

$$\mathsf{P}(\mathcal{F}) \leqslant \mathsf{P}\Big(\bigcup_{\mathcal{B} \in \Psi} \mathcal{B}\Big) = 1 - \mathsf{P}\Big(\bigcap_{\mathcal{B} \in \Psi} \overline{\mathcal{B}}\Big). \tag{42}$$

We shall show that the probability of $\bigcap_{\mathcal{B} \in \Psi} \overline{\mathcal{B}}$ is greater than zero. Due to the Local Lemma (see Theorem 6), it suffices to find for every $\mathcal{B} \in \Psi$ a system of events $\Psi_{\mathcal{B}} \subset \Psi$ such that $\mathcal{B}$ and the algebra generated by $\{\mathcal{Q} \in \Psi \setminus \Psi_{\mathcal{B}}\}$ are independent, and moreover, such that the following inequality holds:

$$\sum_{\mathcal{Q} \in \Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{Q}) \leqslant 1/4. \tag{43}$$

The event $\mathcal{B} \in \Psi$ can be of three types. We shall consider them successively.

1. $\mathcal{B} = \mathcal{Q}_i(e)$ for some $e \in E$ and $i \in \{0, 1\}$. By (18) and (21) such $\mathcal{B}$ belongs to the algebra generated by the random variables $\{\xi_s, \eta_s :\ s \in e\}$. Then this event is independent of the algebra generated by the random variables

$\{\xi_s, \eta_s : s \in V\backslash e\}$ and the random sets $\{\Theta(f, Y) : f \cap e = \varnothing,\ Y \subset f\}$. Let $\Psi_{\mathcal{B}}$ be the system consisting of all the events $\mathcal{Q}_0(f)$ and $\mathcal{Q}_1(f)$ such that $f \cap e \neq \varnothing$, and of all the events $\mathcal{Q}_2(f, F)$ and $\mathcal{Q}_3(f, F)$ such that either $f \cap e \neq \varnothing$, or $g \cap e \neq \varnothing$ for some $g \in F$. It follows from the definitions of the events $\mathcal{Q}_0(f)$, $\mathcal{Q}_1(f)$, $\mathcal{Q}_2(f, F)$, $\mathcal{Q}_3(f, F)$ (see (18), (21), (25), (35)), that $\mathcal{B}$ is independent of the algebra generated by $\{\mathcal{Q} \in \Psi\backslash\Psi_{\mathcal{B}}\}$. We have to check the inequalities (43). Our choice of $\Psi_{\mathcal{B}}$ implies the relation

$$\sum_{\mathcal{Q}\in\Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{Q}) \leqslant \sum_{f\in E:\ f\cap e\neq\varnothing} \left(\mathsf{P}\left(\mathcal{Q}_0(f)\right) + \mathsf{P}\left(\mathcal{Q}_1(f)\right)\right) +$$
$$+ \sum_{f\in E:\ f\cap e\neq\varnothing}\ \sum_{F\in\binom{E(f)}{t}} \mathsf{P}\left(\mathcal{Q}_2(f,F)\right) + \sum_{f\in E}\ \sum_{\substack{F\in\binom{E(f)}{t}:\\ \exists\, g\in F:\ g\cap e\neq\varnothing}} \mathsf{P}\left(\mathcal{Q}_2(f,F)\right) +$$
$$+ \sum_{f\in E:\ f\cap e\neq\varnothing}\ \sum_{F\in\binom{E(f)}{t-1}} \mathsf{P}\left(\mathcal{Q}_3(f,F)\right) + \sum_{f\in E}\ \sum_{\substack{F\in\binom{E(f)}{t-1}:\\ \exists\, g\in F:\ g\cap e\neq\varnothing}} \mathsf{P}\left(\mathcal{Q}_3(f,F)\right). \qquad (44)$$

By the hypothesis of the Theorem, there exist at most d other edges intersecting an arbitrary $f \in E$. So, the first sum contains at most $d+1$ summands, the first double sum — at most $(d+1)\binom{d}{t}$, the second double sum — at most $(d+1)d\binom{d-1}{t-1}$, the third double sum — at most $(d+1)\binom{d}{t-1}$, and the fourth double sum — at most $(d+1)d\binom{d-1}{t-2}$. Thus, from the relation (44) and the bounds (19), (22), (27), (39), we get the inequality

$$\sum_{\mathcal{Q}\in\Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{Q}) \leqslant (d+1)\left(r(r-1)\left(\frac{p}{r}\right)^n + r^{1-n}(1-q)^{n-t}(nq)^t\right) +$$
$$+ \left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right) q^t r^{-(n-1)(t+1)}(nq)^{t(t-2)} +$$
$$+ \left((d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}\right) r^{-(n-1)t}\frac{1}{\binom{n}{t-1}}(1+q)^n = W_0. \qquad (45)$$

2. $\mathcal{B} = \mathcal{Q}_2(e, F)$ for some $e \in E$ and $F \in \binom{E(e)}{t}$. Let us denote $U(e, F) = e \cup \bigcup_{f \in F} f$. By the definition (25) such $\mathcal{B}$ belongs to the algebra generated by the random variables $\{\xi_s, \eta_s :\ s \in U(e, F)\}$. Then this event is obviously independent of the algebra generated by the random variables $\{\xi_s, \eta_s :\ s \in V \backslash U(e, F)\}$ and the random sets $\{\Theta(f, Y) :\ f \cap\ U(e, F) = \varnothing,\ Y \subset f\}$. Let $\Psi_{\mathcal{B}}$ be the system consisting of all the events $\mathcal{Q}_0(f)$ and $\mathcal{Q}_1(f)$ such that $f \cap U(e, F) \neq \varnothing$, and of all the events $\mathcal{Q}_2(f, G)$ and $\mathcal{Q}_3(f, G)$ such that either $f \cap U(e, F) \neq \varnothing$, or $g \cap U(e, F) \neq \varnothing$ for some $g \in G$. It is clear that $\mathcal{B}$ is independent of the algebra generated by $\{\mathcal{Q} \in \Psi \backslash \Psi_{\mathcal{B}}\}$. We have to check the inequalities (43). Our choice of $\Psi_{\mathcal{B}}$ implies the relation

$$\sum_{\mathcal{Q} \in \Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{Q}) \leqslant \sum_{f \in E:\, f \cap U(e,F) \neq \varnothing} (\mathsf{P}(\mathcal{Q}_0(f)) + \mathsf{P}(\mathcal{Q}_1(f))) +$$

$$+ \sum_{f \in E:\, f \cap U(e,F) \neq \varnothing} \sum_{G \in \binom{E(f)}{t}} \mathsf{P}(\mathcal{Q}_2(f, G)) + \sum_{f \in E} \sum_{\substack{G \in \binom{E(f)}{t}: \\ \exists g \in G:\, g \cap U(e,F) \neq \varnothing}} \mathsf{P}(\mathcal{Q}_2(f, G)) +$$

$$+ \sum_{f \in E:\, f \cap U(e,F) \neq \varnothing} \sum_{G \in \binom{E(f)}{t-1}} \mathsf{P}(\mathcal{Q}_3(f, G)) + \sum_{f \in E} \sum_{\substack{G \in \binom{E(f)}{t-1}: \\ \exists g \in G:\, g \cap U(e,F) \neq \varnothing}} \mathsf{P}(\mathcal{Q}_3(f, G)). \quad (46)$$

By the hypothesis of the Theorem, there exist at most d other edges intersecting an arbitrary $f \in E$. So, the first sum contains at most $(t + 1)(d + 1)$ summands, the first double sum — at most $(t + 1)(d + 1)\binom{d}{t}$, the second double sum — at most $(t + 1)(d + 1)d\binom{d-1}{t-1}$, the third double sum — at most $(t + 1)(d + 1)\binom{d}{t-1}$, and the fourth double sum — at most $(d + 1)(t + 1)d\binom{d-1}{t-2}$. Thus, from (46) and from the previously obtained bounds (19), (22), (27), (39) we get the inequality

$$\sum_{\mathcal{Q} \in \Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{Q}) \leqslant (t + 1)(d + 1) \left(r(r - 1) \left(\frac{p}{r}\right)^n + r^{1-n}(1 - q)^{n-t}(nq)^t \right) +$$

$$+ (t+1)\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right) q^t r^{-(n-1)(t+1)} (nq)^{t(t-2)} +$$

$$+ (t+1)\left((d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}\right) r^{-(n-1)t} \frac{(1+q)^n}{\binom{n}{t-1}} =$$

$$= (t+1)W_0 = W_1. \tag{47}$$

3. It remains to consider the case $\mathcal{B} = \mathcal{Q}_3(e, F)$ for some $e \in E$ and $F \in \binom{E(e)}{t-1}$. As in the previous case, let us denote $U(e, F) = e \cup \bigcup_{f \in F} f$. By (35), such $\mathcal{B}$ belongs to the algebra generated by the random variables $\{\xi_s, \eta_s : s \in U(e, F)\}$ and the random subsets $\{\Theta(e, Y) : Y \subset e\}$. It is clear that this event is independent of the algebra generated by the random variables $\{\xi_s, \eta_s : s \in V \backslash U(e, F)\}$ and the random sets $\{\Theta(f, Y) : f \cap U(e, F) = \varnothing, Y \subset f\}$. Let $\Psi_{\mathcal{B}}$ be the system consisting of all the events $\mathcal{Q}_0(f)$ and $\mathcal{Q}_1(f)$ such that $f \cap U(e, F) \neq \varnothing$, and of all the events $\mathcal{Q}_2(f, G)$ and $\mathcal{Q}_3(f, G)$ such that either $f \cap U(e, F) \neq \varnothing$, or $g \cap U(e, F) \neq \varnothing$ for some $g \in G$. Then $\mathcal{B}$ is independent of the algebra generated by $\{\mathcal{Q} \in \Psi \backslash \Psi_{\mathcal{B}}\}$. Let us check the inequalities (43). Our choice of $\Psi_{\mathcal{B}}$ implies the relation

$$\sum_{\mathcal{Q} \in \Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{Q}) \leqslant \sum_{f \in E:\, f \cap U(e,F) \neq \varnothing} (\mathsf{P}(\mathcal{Q}_0(f)) + \mathsf{P}(\mathcal{Q}_1(f))) +$$

$$+ \sum_{f \in E:\, f \cap U(e,F) \neq \varnothing} \sum_{G \in \binom{E(f)}{t}} \mathsf{P}(\mathcal{Q}_2(f,G)) + \sum_{f \in E} \sum_{\substack{G \in \binom{E(f)}{t}: \\ \exists g \in G:\, g \cap U(e,F) \neq \varnothing}} \mathsf{P}(\mathcal{Q}_2(f,G)) +$$

$$+ \sum_{f \in E:\, f \cap U(e,F) \neq \varnothing} \sum_{G \in \binom{E(f)}{t-1}} \mathsf{P}(\mathcal{Q}_3(f,G)) + \sum_{f \in E} \sum_{\substack{G \in \binom{E(f)}{t-1}: \\ \exists g \in G:\, g \cap U(e,F) \neq \varnothing}} \mathsf{P}(\mathcal{Q}_3(f,G)). \tag{48}$$

It follows from the hypothesis of the Theorem concerning the edge intersections, that the first sum contains at most $t(d+1)$ summands, the first double sum — at most $t(d+1)\binom{d}{t}$, the second double sum — at most $t(d+1)d\binom{d-1}{t-1}$, the third double sum — at most $t(d+1)\binom{d}{t-1}$, and the fourth double sum — at most $t(d+1)d\binom{d-1}{t-2}$. From (48) and (19), (22), (27), (39) we obtain the following upper bound for the sum of probabilities

$$\sum_{\mathcal{Q}\in\Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{Q}) \leqslant t(d+1)\left(r(r-1)\left(\frac{p}{r}\right)^n + r^{1-n}(1-q)^{n-t}(nq)^t\right) +$$
$$+t\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right) q^t r^{-(n-1)(t+1)}(nq)^{t(t-2)} +$$
$$+t\left((d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}\right) r^{-(n-1)t}\frac{1}{\binom{n}{t-1}}(1+q)^n =$$
$$= tW_0 = W_2. \tag{49}$$

Let us sum up the intermediate results. It follows from the estimates (45), (47), (49) that to prove the inequality (43) it suffices to show that

$$W_1 = (t+1)W_0 \leqslant \frac{1}{4}. \tag{50}$$

We shall need some additional estimates contained in the next section.

3.2.7. Auxiliary analytics

The quantity W_1 (see (47)) consists of the four summands

$$(t+1)(d+1)r(r-1)\left(p/r\right)^n, \quad (t+1)(d+1)r^{1-n}(1-q)^{n-t}(nq)^t,$$
$$(t+1)\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right) q^t r^{-(n-1)(t+1)}(nq)^{t(t-2)},$$
$$(t+1)\left((d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}\right) r^{-(n-1)t}\frac{1}{\binom{n}{t-1}}(1+q)^n.$$

Let us consider and estimate them separately.

1. The first summand is $(t+1)(d+1)r(r-1)\,(p/r)^n$. Using the restriction (10), the conditions (11) and (12), we obtain an upper bound for the first summand:

$$(t+1)(d+1)r(r-1)\left(\frac{p}{r}\right)^n \leqslant (t+1)nr^{n-1}r^{1-n}(r-1)\left(\frac{q}{r-1}\right)^n =$$
$$= (t+1)n(r-1)^{1-n}q^n \leqslant n^2q^n \leqslant n^2 2^{-n}. \tag{51}$$

2. The second summand is $(t+1)(d+1)r^{1-n}(1-q)^{n-t}(nq)^t$. Due to the choice of q in (9) we get the relations

$$(t+1)(d+1)r^{1-n}(1-q)^{n-t}(nq)^t \leqslant (t+1)nr^{n-1}r^{1-n}(1-q)^{n-t}(nq)^t =$$
$$= (t+1)n(1-q)^{n-t}\,(\alpha\ln n)^t \leqslant (t+1)ne^{qt-qn}\,(\alpha\ln n)^t =$$
$$= (t+1)n^{1-\alpha}e^{\alpha(\ln n)t/n}\,(\alpha\ln n)^t\,. \tag{52}$$

3. Let us consider the third summand

$$(t+1)\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right) q^t r^{-(n-1)(t+1)}(nq)^{t(t-2)}. \tag{53}$$

We shall need some preliminary estimates.

First, the following inequalities hold:

$$(t+1)\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right) = (t+1)\binom{d}{t}(d+1)(t+1) \leqslant$$
$$\leqslant (t+1)^2(d+1)\frac{d^t}{t!} \leqslant (t+1)^2\frac{(d+1)^t}{t!}. \tag{54}$$

Second, the choice of t and q (see (9)) implies the relations

$$q^t(nq)^{t(t-2)} = n^{-t}(nq)^{t(t-1)} \leqslant n^{-t}(nq)^{t^2} = n^{-t}\exp\left\{t^2\ln\left(\alpha\ln n\right)\right\} \leqslant$$
$$\leqslant n^{-t}\exp\{\ln n\} = n^{1-t}. \tag{55}$$

Finally, from (54), (55) and from the original restriction (10), we obtain an upper bound for the expression (53):

$$(t+1)\left((d+1)\binom{d}{t}+(d+1)d\binom{d-1}{t-1}\right)q^t r^{-(n-1)(t+1)}(nq)^{t(t-2)} \leqslant$$
$$\leqslant \frac{(t+1)^2}{t!}(d+1)^{t+1}r^{-(n-1)(t+1)}n^{1-t} \leqslant \frac{(t+1)^2}{t!}n^{(t+1)(1-k/t)}n^{1-t} \leqslant$$
$$\leqslant \frac{(t+1)^2}{t!}\,n^{t+1-(k(t+1)/t)}n^{1-t} = \frac{(t+1)^2}{t!}\,n^{2-k-(k/t)} \leqslant \frac{(t+1)^2}{t!}\,n^{2-k}. \quad (56)$$

4. It remains to estimate the fourth summand

$$(t+1)\left((d+1)\binom{d}{t-1}+(d+1)d\binom{d-1}{t-2}\right)r^{1-nt}\frac{1}{\binom{n}{t-1}}(1+q)^n. \quad (57)$$

By an analogy with (54), we get

$$(t+1)\left((d+1)\binom{d}{t-1}+(d+1)d\binom{d-1}{t-2}\right) = (t+1)\binom{d}{t-1}(d+1)t \leqslant$$
$$\leqslant (t+1)t\left(\frac{de}{t-1}\right)^{t-1}(d+1) \leqslant (d+1)^t\left(\frac{e}{t-1}\right)^{t-1}(t+1)t. \quad (58)$$

Further, by (9) we have

$$\frac{1}{\binom{n}{t-1}}(1+q)^n \leqslant \left(\frac{t-1}{n}\right)^{t-1}e^{qn} = \left(\frac{t-1}{n}\right)^{t-1}n^{\alpha}. \quad (59)$$

Finally, from (58), (59) and (10) we obtain an upper bound for the expression (57):

$$(t+1)\left((d+1)\binom{d}{t-1}+(d+1)d\binom{d-1}{t-2}\right)r^{-(n-1)t}\frac{1}{\binom{n}{t-1}}(1+q)^n \leqslant$$
$$\leqslant (t+1)t\left(\frac{e}{t-1}\right)^{t-1}(d+1)^t r^{-(n-1)t}\left(\frac{t-1}{n}\right)^{t-1}n^{\alpha} \leqslant$$
$$\leqslant (t+1)te^{t-1}r^{(n-1)t}n^{t(1-k/t)}r^{-(n-1)t}n^{\alpha+1-t} = (t+1)te^{t-1}n^{1+\alpha-k}. \quad (60)$$

The inequality (60) completes estimation of the parts of W_1.

3.2.8. The completion of the proof of Theorem 5

Let us gather the obtained bounds for the summands in the expression (47) for W_1. The relations (51), (52), (56) and (60) imply the inequalities

$$W_1 \leqslant \frac{n^2}{2^n} + (t+1)n^{1-\alpha}e^{\alpha(\ln n)t/n}\,(\alpha \ln n)^t + \frac{(t+1)^2}{t!}\,n^{2-k} + (t+1)te^{t-1}n^{1+\alpha-k} < \frac{1}{4},$$

the latter of which holds due to the condition (13) of Theorem 5. Thus, the required relation (50) is established. It implies the inequality (43), which is necessary for application of the Local Lemma. It follows from the Local Lemma that the probability of simultaneous happening of all the events $\overline{\mathcal{B}}$, where $\mathcal{B} \in \Psi$, is greater than zero. Taking (42) into account, we get

$$\mathsf{P}\,(\mathcal{F}) < 1.$$

We are ready to complete the proof. Indeed, we have proved that the probability of the event that the random coloring $\vec{\zeta}$ is not a proper coloring of H is less than one. So, $\vec{\zeta}$ is a proper coloring with positive probability, and $\chi(H) \leqslant r$. Theorem 5 is proved.

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